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FAMILIES OF PARTIAL FUNCTIONS

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The degree of disjunction, $\delta(F)$, of a family F of functions is the least cardinal τ such that every pair of functions in F agree on a set of cardinality less than τ.

Suppose $\theta, \mu, \lambda, \kappa$ are non-zero cardinals with $\theta \le \mu \le \lambda$. This paper is concerned with functions which map μ-sized subsets of λ into κ. We first show there is always a 'large' family F of such functions satisfying $\delta(F) \le \theta$. Next we determine the cardinalities of families F of such functions that are maximal with respect to $\delta(F) \le \theta$.

1. Introduction

Suppose μ, λ, κ are non-zero cardinals with $\mu \le \lambda$. Let $[\mu,\lambda]_\kappa$ denote the set of all functions which map a μ-sized subset of λ into κ. Given functions f, g; we use $E(f; g)$ to denote

$$\{x \in \mathrm{dom}(f) \cap \mathrm{dom}(g); f(x) = g(x)\} .$$

The *degree of disjunction*, $\delta(F)$, of a family F of functions is the least cardinal τ such that $|E(f; g)| < \tau$ for all pairs f, g of functions in F. More generally, the *degree of disjunction*, $\delta(S)$, of a family S of sets is the least cardinal τ such that $|S \cap S'| < \tau$ for all pairs S, S' of sets in S.

This paper is concerned with two problems about families of partial

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functions. Suppose θ is another non-zero cardinal and $\theta \le \mu$. We first determine the 'maximum' cardinality of a subset F of $[\mu,\lambda]_\kappa$ satisfying $\delta(F) \le \theta$. Secondly, we determine the cardinalities of subsets F of $[\mu,\lambda]_\kappa$ that are maximal with respect to $\delta(F) \le \theta$. The following two definitions will be useful.

DEFINITIONS. Let

(i) $F_\theta(\mu, \lambda, \kappa) = \sup\{|F|;\ F \subseteq [\mu,\lambda]_\kappa \text{ and } \delta(F) \le \theta\}$,

(ii) $\max_\theta F(\mu, \lambda, \kappa) = \{\zeta;\ \zeta$ is a cardinal and there is an ζ-sized subset F of $[\mu,\lambda]_\kappa$ that is maximal with respect to $\delta(F) \le \theta\}$.

The Generalized Continuum Hypothesis is assumed throughout the general discussion. We also assume that λ is infinite.

In Section 2 we show that there is always a 'large' subset F of $[\mu,\lambda]_\kappa$ satisfying $\delta(F) \le \theta$. $F_\theta(\mu, \lambda, \kappa)$ is as large as possible in the following sense. Suppose θ, μ, Σ are non-zero cardinals such that $\theta \le \mu \le \Sigma$. Let

$$S_\theta(\mu, \Sigma) = \sup\{|S|;\ S \subseteq [\Sigma]^\mu \text{ and } \delta(S) \le \theta\} .$$

The cofinality λ' of a cardinal λ is the least cardinal τ such that λ can be expressed as the sum of τ cardinals each less than λ. If Σ is infinite then the values of $S_\theta(\mu, \Sigma)$ are known under the Generalized Continuum Hypothesis. If $\theta < \mu$ or if $\mu' \ne \Sigma'$, then $S_\theta(\mu, \Sigma) = \Sigma$; otherwise $S_\theta(\mu, \Sigma) = \Sigma^+$ (see Baumgartner [1], Theorem 3.4). A comparison of these results and Proposition 1 shows that $F_\theta(\mu, \lambda, \kappa) = S_\theta(\mu, \lambda.\kappa)$ always.

Section 3 contains the substantial part of this paper: the description of the sets $\max_\theta F(\mu, \lambda, \kappa)$ of cardinals. We prove in Theorem 4 that if $\lambda \le \kappa$ or if $\mu < \lambda$, then all maximal families have the same cardinality; namely $F_\theta(\mu, \lambda, \kappa)$. When $\mu = \kappa$ and $\lambda > \kappa$, however, maximal families of differing cardinalities exist. The cardinalities of these maximal families are given in Theorem 7 below.

Our set notation is standard. An ordinal is identified with the set of its predecessors and cardinals are identified with initial ordinals. We use $\alpha, \beta, \gamma, \delta, \ldots$ to denote ordinals and $\zeta, \lambda, \kappa, \mu, \ldots$ to denote cardinals. $\mathrm{Cn}(\kappa)$ denotes the set of non-zero cardinals less than or equal to κ. The symbol $[S]^{\mu}$ denotes $\{S'; S' \subseteq S \text{ and } |S'| = \mu\}$. A (λ, κ) *family* is an indexed family $(S_i; i \in I)$ of sets where $|I| = \lambda$ and $|S_i| = \kappa$ for each i in I. A family S of sets is said to be *almost disjoint* if $|S \cap S'| < \min(|S|, |S'|)$ for all pairs S, S' of sets in S. Note that a subset F of ${}^{[\mu,\lambda]}\kappa$ is almost disjoint if and only if $|E(f; g)| < \mu$ for all pairs f, g of functions in F. An almost disjoint family X of λ-sized sets is said to be λ-*maximally almost disjoint* if $|\cup X| = \lambda$ and every λ-sized subset of $\cup X$ intersects some member of X in a set of cardinality λ. For sets S, T the symbol ${}^{S}T$ denotes $\{f; f : S \to T\}$. If $X \subseteq S$ and $g \in {}^{S}T$ then g/X denotes the *restriction of* g *to* X. The *cofinality* λ' of non-zero λ is the least cardinal τ such that λ can be expressed as the sum of τ cardinals all less than λ. We say λ is *regular* if $\lambda' = \lambda$; otherwise λ is *singular* in which case $\lambda' < \lambda$. A λ-*sequence* is a sequence $\langle \lambda_\sigma; \sigma < \lambda' \rangle$ of cardinals all less than λ such that $\lambda = \sum (\lambda_\sigma; \sigma < \lambda')$. If λ is singular then strictly increasing λ-sequences exist. We refer the reader to Williams [4] for any further set theoretical background.

For the remainder of the paper we assume that $\theta, \mu, \lambda, \kappa$ are non-zero cardinals such that λ is infinite and $\theta \le \mu \le \lambda$. Neither μ nor κ is necessarily infinite.

2. Values of $F_\theta(\mu, \lambda, \kappa)$

We show that $F_\theta(\mu, \lambda, \kappa) = S_\theta(\mu, \lambda.\kappa)$ always.

PROPOSITION 1 (Generalized Continuum Hypothesis). *(i) If* $\theta < \mu$ *or if* $\mu' \ne (\lambda.\kappa)'$, *then* $F_\theta(\mu, \lambda, \kappa) = \lambda.\kappa$.

(ii) If $\mu' = (\lambda.\kappa)'$ *then* $F_\mu(\mu, \lambda, \kappa) = (\lambda.\kappa)^+$.

Proof. The proof is not difficult.

Suppose $F \subseteq {}^{[\mu,\lambda]}\kappa$ and $\delta(F) \leq \theta$. Since $F \subseteq [\lambda \times \kappa]^{\mu}$ and $|\lambda \times \kappa| = \lambda.\kappa$, it follows that $|F| \leq S_{\theta}(\mu, \lambda.\kappa)$. Therefore

(i) if $\theta < \mu$ or if $\mu' \neq (\lambda.\kappa)'$, then $F_{\theta}(\mu, \lambda, \kappa) \leq \lambda.\kappa$,

(ii) if $\mu' = (\lambda.\kappa)'$ then $F_{\mu}(\mu, \lambda, \kappa) \leq (\lambda, \kappa)^{+}$.

To show that these upper bounds are the values of $F_{\theta}(\mu, \lambda, \kappa)$ we construct, in each case, a 'suitably large' subset F of ${}^{[\mu,\lambda]}\kappa$ with $\delta(F) \leq \theta$.

(i) Suppose that either $\theta < \mu$ or $\mu' \neq (\lambda.\kappa)'$. Let $(B_{\alpha}; \alpha < \lambda)$ be a pairwise disjoint (λ, μ) decomposition of λ and, for each ordered pair $\langle \alpha, \beta \rangle$ in $\lambda \times \kappa$, let $f_{\alpha,\beta}$ denote the constant function defined on B_{α} which maps each ordinal in B_{α} to β. Put
$F = \{f_{\alpha,\beta}; \langle \alpha, \beta \rangle \in \lambda \times \kappa\}$. The family F is a pairwise disjoint subset of ${}^{[\mu,\lambda]}\kappa$ and $|F| = \lambda.\kappa$.

(ii) Next suppose $\mu' = (\lambda.\kappa)'$. We consider the cases $\lambda \leq \kappa$ and $\lambda > \kappa$ separately.

CASE 1. $\lambda \leq \kappa$. In this case κ is infinite and $\lambda.\kappa = \kappa$. Since $\mu' = \kappa'$ it follows from Williams [4], Theorem 1.2.7, that there is an almost disjoint subset F of ${}^{\mu}\kappa$ with $|F| = \kappa^{+} = (\lambda.\kappa)^{+}$. Since ${}^{\mu}\kappa \subseteq {}^{[\mu,\lambda]}\kappa$ the family F suffices.

CASE 2. $\lambda > \kappa$. In this case $\lambda.\kappa = \lambda$ and we appeal to the results on $S_{\theta}(\mu, \lambda)$. Let $B = (B_{\alpha}; \alpha < \lambda^{+})$ be an almost disjoint (λ^{+}, μ) decomposition of λ and set $F = \{f_{\alpha,\beta}; \langle \alpha, \beta \rangle \in \lambda^{+} \times \kappa\}$ (where $f_{\alpha,\beta}$ is defined as above). Certainly $F \subseteq {}^{[\mu,\lambda]}\kappa$. To show that F is almost disjoint, suppose $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ are distinct members of $\lambda^{+} \times \kappa$. If $\beta \neq \delta$ then $E(f_{\alpha,\beta}; f_{\gamma,\delta}) = \emptyset$. If $\beta = \delta$ then $\alpha \neq \gamma$ and $E(f_{\alpha,\beta}; f_{\gamma,\delta}) \subseteq B_{\alpha} \cap B_{\gamma}$. It follows that $|E(f_{\alpha,\beta}; f_{\gamma,\delta})| \leq |B_{\alpha} \cap B_{\gamma}| < \mu$ since B is almost disjoint. Hence F is an almost disjoint subset of

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$[\mu,\lambda]_\kappa$ and $|F| = (\lambda.\kappa)^+$.

This completes the proof of Proposition 1. □

We remark that there is always a subset F of $[\mu,\lambda]_\kappa$ such that $\delta(F) \le \theta$ and $|F| = F_\theta(\mu, \lambda, \kappa)$; the supremum in the definition of $F_\theta(\mu, \lambda, \kappa)$ is a maximum and not a strict supremum.

3. Cardinalities of maximal families of partial functions

In this section we describe the cardinalities of subsets F of $[\mu,\lambda]_\kappa$ that are maximal with respect to $\delta(F) \le \theta$. We first make a few simple observations about maximal families of partial functions.

LEMMA 2. *Suppose* $F \subseteq [\mu,\lambda]_\kappa$ *and* F *is maximal with respect to* $\delta(F) \le \theta$. *Then*

(A) $|F| \ge \kappa$,

(B) $|\lambda - \bigcup\{\mathrm{dom}(f);\ f \in F\}| < \mu$,

(C) $\kappa = \bigcup\{\mathrm{ran}(f);\ f \in F\}$.

Proof. (A) For a contradiction suppose $|F| < \kappa$. Choose a function g from ${}^\mu\kappa$ such that $g(\alpha) \notin \{f(\alpha);\ f \in F\}$ for each α less than μ . This is possible since $|\{f(\alpha);\ f \in F\}| \le |F| < \kappa$ by assumption. Then $E(f;\ g) = \emptyset$ for each f in F ; contradicting the maximality of F .

(B) If $X \in [\lambda - \bigcup\{\mathrm{dom}(f);\ f \in F\}]^\mu$ and $g \in {}^X\kappa$, then $E(f;\ g) = \emptyset$ for each f in F ; contradicting the maximality of F .

(C) If $\beta \in \kappa - \bigcup\{\mathrm{ran}(f);\ f \in F\}$ and g is any function in $[\mu,\lambda]_\kappa$ that is constant with value β , then $E(f, g) = \emptyset$ for each f in F ; contradicting the maximality of F . □

LEMMA 3. *Suppose* μ *is infinite and* $\mu' = \kappa'$. *If* F *is an almost disjoint subset of* $[\mu,\lambda]_\kappa$ *and* F *is maximal with respect to almost disjointness then* $|F| \ge \kappa^+$.

Proof. Suppose F is an almost disjoint subset of $[\mu,\lambda]_\kappa$ and $|F| \le \kappa$. We show that F is not maximal with respect to almost

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disjointness by constructing a function g in ${}^{\mu}\kappa$ such that $|E(f; g)| < \mu$ for each f in F. Write $F = (f_\nu; \nu < \kappa)$ where repetitions occur if $|F| < \kappa$. Let $\langle \delta_\tau, \tau < \mu' \rangle$, $\langle \gamma_\sigma; \sigma < \mu' \rangle$ be strictly increasing sequences of ordinals such that $\mu = \sup\{\delta_\tau; \tau < \mu'\}$ and $\kappa = \sup\{\gamma_\sigma; \sigma < \mu'\}$. Inductively define the function values $g(\alpha)$ as follows. Suppose that $\alpha < \mu$ and $g(\delta)$ has been defined for each δ less than α. Let $\tau(\alpha)$ be the least τ less than μ' such that $\alpha < \delta_\tau$ and choose $g(\alpha)$ from $\kappa - \{f_\nu(\alpha); \nu < \gamma_{\tau(\alpha)}\}$. This is possible since $|\{f_\nu(\alpha); \nu < \gamma_{\tau(\alpha)}\}| \le |\gamma_{\tau(\alpha)}| < \kappa$. To show that g suffices, suppose that $\nu < \kappa$ and let $\sigma(\nu)$ be the least σ less than μ' such that $\nu < \gamma_\sigma$. If $\delta_{\sigma(\nu)} \le \alpha < \mu$ then $\nu < \gamma_{\sigma(\nu)} < \gamma_{\tau(\alpha)}$, and it follows from the choice of $g(\alpha)$ that $g(\alpha) \ne f_\nu(\alpha)$. Hence $E(f_\nu; g) \subseteq \delta_{\sigma(\nu)}$ and $|E(f_\nu, g)| \le |\delta_{\sigma(\nu)}| < \mu$ as required.

The family F, then, is not maximal and the result follows. □

With this lemma the sets $\max_\theta F(\mu, \lambda, \kappa)$ of cardinals can be described in the case when either $\lambda \le \kappa$ or $\mu < \lambda$.

THEOREM 4 (Generalized Continium Hypothesis). *Suppose that either* $\lambda \le \kappa$ *or* $\mu < \lambda$.

(i) If $\theta < \mu$ *or if* $\mu' \ne (\lambda.\kappa)'$, *then* $\max_\theta F(\mu, \lambda, \kappa) = \{\lambda.\kappa\}$.

(ii) If $\mu' = (\lambda.\kappa)'$ *then* $\max_\mu F(\mu, \lambda, \kappa) = \{(\lambda.\kappa)^+\}$.

Proof. Certainly $\max_\theta F(\mu, \lambda, \kappa) \subseteq \mathrm{Cn}(F_\theta(\mu, \lambda, \kappa))$. Since there is a subset F of ${}^{[\mu,\lambda]}\kappa$ with $\delta(F) \le \theta$ and $|F| = F_\theta(\mu, \lambda, \kappa)$, a simple application of Zorn's Lemma implies that $F_\theta(\mu, \lambda, \kappa) \in \max_\theta F(\mu, \lambda, \kappa)$. From these observations it follows that

(i) If $\theta < \mu$ or if $\mu' \ne (\lambda.\kappa)'$, then $\max_\theta F(\mu, \lambda, \kappa) \subseteq \mathrm{Cn}(\lambda.\kappa)$ and $\lambda.\kappa \in \max_\theta F(\mu, \lambda, \kappa)$.

(ii) If $\mu' = (\lambda.\kappa)'$ then $\max_\mu F(\mu, \lambda, \kappa) \subseteq \mathrm{Cn}((\lambda.\kappa)^+)$ and

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$(\lambda.\kappa)^+ \in \max_\mu F(\mu, \lambda, \kappa)$.

We show, in each case, that the cardinals above are the only members of $\max_\theta F(\mu, \lambda, \kappa)$. For this, suppose that $F \subseteq {}^{[\mu,\lambda]}\kappa$ and F is maximal with respect to $\delta(F) \le \theta$. We consider two cases.

CASE 1. $\lambda \le \kappa$.

Property (A) of Lemma 2 implies that $|F| \ge \kappa = \lambda.\kappa$. This is all that is needed if either $\theta < \mu$ or $\mu' \ne \kappa'$. If $\mu' = \kappa'$ then Lemma 3 implies that $|F| \ge \kappa^+ = (\lambda.\kappa)^+$.

CASE 2. $\mu < \lambda$.

Since the case when $\lambda \le \kappa$ has been settled we may further assume that $\lambda > \kappa$. Property (B) of Lemma 2 implies that $\lambda = |\bigcup\{\mathrm{dom}(f);\ f \in F\}| \le \mu.|F|$. Since $\mu < \lambda$ it follows that $|F| \ge \lambda = \lambda.\kappa$. This is all that is needed if either $\theta < \mu$ or $\mu' \ne \lambda'$.

If $\mu' = \lambda'$ and $\theta = \mu$, we claim that $|F| \ge \lambda^+ = (\lambda.\kappa)^+$. For suppose that $|F| \le \lambda$ and write $F = \{f_\nu;\ \nu < \lambda\}$. We define a function g such that $|E(f_\nu;\ g)| < \mu$ for all ν less than λ . Let $\langle \delta_\tau;\ \tau < \mu' \rangle$ and $\langle \gamma_\sigma;\ \sigma < \mu' \rangle$ be strictly increasing sequences of ordinals such that $\mu = \sup\{\delta_\tau;\ \tau < \mu'\}$ and $\lambda = \sup\{\gamma_\sigma;\ \sigma < \mu'\}$. Inductively define a sequence $\langle x_\alpha;\ \alpha < \mu \rangle$ of pairwise distinct elements of λ as follows. Suppose that $\alpha < \mu$ and x_δ has been defined for each δ less than α . Let $\tau(\alpha)$ be the least τ less than μ' such that $\delta_\tau > \alpha$ and choose x_α from

$$\lambda - \left(\bigcup\{\mathrm{dom}(f_\nu);\ \nu < \gamma_{\tau(\alpha)}\} \cup \{x_\delta;\ \delta < \alpha\}\right) .$$

This is possible since

$$|\bigcup\{\mathrm{dom}(f_\nu);\ \nu < \gamma_{\tau(\alpha)}\}| \le \mu.|\gamma_{\tau(\alpha)}| < \lambda .$$

Set $X = \{x_\alpha;\ \alpha < \mu\}$ and choose g from ${}^X\kappa$. Then $X \in [\lambda]^\mu$ and $g \in {}^{[\mu,\lambda]}\kappa$. To show that $|E(f_\nu;\ g)| < \mu$ for each ν less than λ , it

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suffices to show that $|X \cap \mathrm{dom}(f_\nu)| < \mu$ for each ν less than λ. To see this suppose ν is less than λ and let $\sigma(\nu)$ be the least σ less than μ' such that $\nu < \gamma_\sigma$. If $\delta_{\sigma(\nu)} \leq \alpha < \mu$ then $\nu < \gamma_{\sigma(\nu)} < \gamma_{\tau(\alpha)}$ and the choice of x_α implies that $x_\alpha \notin \mathrm{dom}(f_\nu)$. Therefore $X \cap \mathrm{dom}(f_\nu) \subseteq \{x_\alpha;\ \alpha < \delta_{\sigma(\nu)}\}$ and $|X \cap \mathrm{dom}(f_\nu)| \leq |\delta_{\sigma(\nu)}| < \mu$ as claimed. Hence $|E(f_\nu;\ g)| < \mu$ for each ν less than λ; the required contradiction.

The proof of Theorem 4 is now complete. □

For the remainder of the section suppose that $\lambda > \kappa$. To determine the nature of the set $\max_\theta F(\lambda, \lambda, \kappa)$ we follow a programme similar to one used in Erdös and Hechler [2] to determine the cardinalities of λ-maximally almost disjoint families.

The following lemma is essentially Theorem 2.3 from the above paper by Erdös and Hechler and provides a method of constructing λ-maximally almost disjoint families.

LEMMA 5 (Erdös and Hechler). *Suppose* λ *is singular,* $1 \leq \xi < \lambda$, *and* $\langle \lambda_\sigma;\ \sigma < \lambda' \rangle$ *is a strictly increasing* λ*-sequence of regular cardinals greater than* ζ. *Suppose that*

(i) $\left\{S^\sigma_\beta;\ \sigma < \lambda' \text{ and } \beta < \zeta\right\}$ *is a pairwise disjoint family of sets such that* $\left|S^\sigma_\beta\right| = \lambda_\sigma$ *for each* $\langle \sigma, \beta \rangle$ *in* $\lambda' \times \zeta$,

(ii) G *is an almost disjoint subset of* ${}^{[\lambda',\lambda']}\zeta$ *that is maximal with respect to almost disjointness,*

(iii) $S_g = \bigcup\left\{S^\sigma_{g(\sigma)};\ \sigma \in \mathrm{dom}(g)\right\}$ *for each* g *in* G.

Then the family $\left(S_g;\ g \in G\right)$ *is* λ*-maximally almost disjoint and has the same cardinality as* G. □

The next lemma asserts that $\max_\lambda F(\lambda, \lambda, \kappa)$ is closed under limits at singular cardinals. It is modification of Theorem 3.1 of Hechler [3] and its proof is similar.

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LEMMA 6. *If* ζ *is an infinite singular cardinal and* $\langle \zeta_\tau ; \tau < \zeta' \rangle$ *is a strictly increasing* ζ*-sequence such that* $\zeta_\tau \in \max_\lambda F(\lambda, \lambda, \kappa)$ *for each* τ *less than* ζ' *, then* $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. □

With these two lemmas it is possible to describe the sets $\max_\theta F(\lambda, \lambda, \kappa)$ when $\lambda > \kappa$.

THEOREM 7 (Generalized Continium Hypothesis). *Suppose that* $\lambda > \kappa$.

(i) If $\theta < \lambda$ *then* $\max_\theta F(\lambda, \lambda, \kappa) = \{\zeta \in \mathrm{Cn}(\lambda) ; \kappa \le \zeta\}$.

(ii) If $\lambda' \ne \kappa'$ *then* $\max_\lambda F(\lambda, \lambda, \kappa) = \{\zeta \in \mathrm{Cn}(\lambda^+) ; \kappa \le \zeta\} - \{\lambda'\}$.

(iii) If $\lambda' = \kappa'$ *then*

$$\max_\lambda F(\lambda, \lambda, \kappa) = \{\zeta \in \mathrm{Cn}(\lambda^+) ; \kappa^+ \le \zeta\} - \{\lambda'\} .$$

Proof. We deal with the three cases separately.

CASE *(i)*. $1 \le \theta < \lambda$.

Suppose $\zeta \in \max_\theta F(\lambda, \lambda, \kappa)$. Property (A) of Lemma 2 implies that $\zeta \ge \kappa$. On the other hand $\zeta \le \lambda$ since $F_\theta(\lambda, \lambda, \kappa) = \lambda$. Hence $\max_\theta F(\lambda, \lambda, \kappa) \subseteq \{\zeta \in \mathrm{Cn}(\lambda) ; \kappa \le \zeta\}$.

We now show that if $\kappa \le \zeta \le \lambda$ then $\zeta \in \max_\theta F(\lambda, \lambda, \kappa)$. Since $F_\theta(\lambda, \lambda, \kappa) = \lambda$ and there is a subset $\mathcal{F}$ of ${}^{[\lambda,\lambda]}\kappa$ with $|\mathcal{F}| = \lambda$ and $\delta(\mathcal{F}) \le \theta$, it follows from a simple application of Zorn's lemma that $\lambda \in \max_\theta F(\lambda, \lambda, \kappa)$. Next suppose $\kappa \le \zeta < \lambda$. We show $\zeta \in \max_\theta F(\lambda, \lambda, \kappa)$.

In this paragraph suppose that ζ is infinite. Let $\langle B_\alpha ; \alpha < \zeta\rangle$ be a pairwise disjoint (ζ, λ) decomposition of λ and, for each ordered pair $\langle \alpha, \beta \rangle$ in $\zeta \times \kappa$, let $f_{\alpha,\beta}$ be the constant function defined on B_α that takes value β . Put $\mathcal{F} = \{f_{\alpha,\beta} ; \alpha < \zeta \text{ and } \beta < \kappa\}$. The family $\mathcal{F}$ is a pairwise disjoint subset of ${}^{[\lambda,\lambda]}\kappa$ and $|\mathcal{F}| = \zeta$. Note that $\mathcal{F}$ decomposes $\lambda \times \kappa$. We claim that $\mathcal{F}$ is maximal with respect to

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$\delta(F) \le \theta$. For suppose $g \in {}^{[\lambda,\lambda]}\kappa$. Now $|g| = \lambda$, $g = \bigcup\{f_{\alpha,\beta} \cap g;\ \langle\alpha, \beta\rangle \in \zeta \times \kappa\}$, $|\zeta \times \kappa| < \lambda$ and $\theta < \lambda$. Hence there is an ordered pair $\langle\alpha, \beta\rangle$ in $\zeta \times \kappa$ such that $|f_{\alpha,\beta} \cap g| > \theta$. Thus $|E(f_{\alpha,\beta};\ g)| > \theta$ and F is maximal with respect to $\delta(F) \le \theta$ as claimed.

Now suppose ζ is finite. Let $(B_\alpha;\ \alpha < \zeta-\kappa+1)$ be a pairwise disjoint $(\zeta-\kappa+1,\ \lambda)$ decomposition of λ . For each α less than $\zeta - \kappa + 1$ let g_α denote the constant function defined on B_α that takes value 0 . For each β with $1 \le \beta < \kappa$ let h_β denote the constant function defined on λ that takes value β . Put

$$F = \{g_\alpha;\ \alpha < \zeta-\kappa+1\} \cup \{h_\beta;\ 1 \le \beta < \kappa\} .$$

Then F is a pairwise disjoint subset of ${}^{[\lambda,\lambda]}\kappa$ and $|F| = \zeta$. Since $\bigcup F = \lambda \times \kappa$ and $\zeta < \aleph_0 \le \lambda'$, it follows that F is λ-maximally almost disjoint and so is certainly maximal with respect to $\delta(F) \le \theta$.

In either case, $\zeta \in \max_\theta F(\lambda, \lambda, \kappa)$ as required; and the theorem is established in Case *(i)*.

Before dealing with Cases *(ii)* and *(iii)*, we make the following three observations.

(α) $\lambda' \notin \max_\lambda F(\lambda, \lambda, \kappa)$.

For a contradiction, suppose $\lambda' \in \max_\lambda F(\lambda, \lambda, \kappa)$ and let F be a λ'-sized almost disjoint subset of ${}^{[\lambda,\lambda]}\kappa$ that is maximal with respect to almost disjointness. Then F is an almost disjoint subset of $[\lambda \times \kappa]^\lambda$ where $|\lambda \times \kappa| = \lambda$. In fact F is λ-maximally almost disjoint. To see this suppose $X \in [\lambda \times \kappa]^\lambda$. Since $|X| = \lambda$ and $\lambda > \kappa$, it follows that there is a function g in ${}^{[\lambda,\lambda]}\kappa$ such that $g \subseteq X$. Since $g \in {}^{[\lambda,\lambda]}\kappa$ the maximality of F implies there is f in F such that $|f \cap g| = \lambda$. Therefore $|X \cap f| = \lambda$ and F is λ-maximally almost disjoint as claimed. But no λ-maximally almost disjoint family of cardinality λ' exists (see Erdös and Hechler [2]); the required contradiction.

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(β) If $\kappa < \lambda'$, $\kappa \le \zeta \le \lambda$ and $\zeta \ne \lambda'$, then $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$.

In this paragraph we assume that ζ is infinite. Let $\mathcal{B} = (B_\alpha;\ \alpha < \zeta)$ be a λ-maximally almost disjoint (ζ, λ) decomposition of λ (see Erdős and Hechler [2]) and set $F = \{f_{\alpha,\beta};\ \alpha < \zeta \text{ and } \beta < \kappa\}$ (where $f_{\alpha,\beta}$ is defined as above). Certainly $F \subseteq {}^{[\lambda,\lambda]}\kappa$, F is almost disjoint and $|F| = \zeta$. We claim F is maximal with respect to almost disjointness. For suppose $g \in {}^{[\lambda,\lambda]}\kappa$. Since $\kappa < \lambda'$ the function g is constant on a set of power λ : there is an ordinal β less than κ and a set X in $[\mathrm{dom}(g)]^\lambda$ such that $g(\nu) = \beta$ for all ν in X . Since $X \in [\lambda]^\lambda$, the λ-maximally almost disjointness of $\mathcal{B}$ implies there is an ordinal α less than ζ such that $|X \cap B_\alpha| = \lambda$. It follows that $X \cap B_\alpha \subseteq E(f_{\alpha,\beta};\ g)$ and $|E(f_{\alpha,\beta};\ g)| = \lambda$. The family F , then, witnesses that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$.

Now suppose ζ is finite. Let $(B_\alpha;\ \alpha < \zeta-\kappa+1)$ be a pairwise disjoint $(\zeta-\kappa+1, \lambda)$ decomposition of λ . For each α less than $\zeta - \kappa + 1$ let g_α denote the constant function defined on B_α that takes value 0 . For each β with $1 \le \beta < \kappa$ let h_β denote the constant function defined on λ that takes value β . Put

$$F = \{g_\alpha;\ \alpha < \zeta-\kappa+1\} \cup \{h_\beta;\ 1 \le \beta < \kappa\} .$$

Then F is a pairwise disjoint subset of ${}^{[\lambda,\lambda]}\kappa$ and $|F| = \zeta$. Since $\bigcup F = \lambda \times \kappa$ and $\zeta < \aleph_0 \le \lambda'$, it follows that F is λ-maximally almost disjoint and so witnesses that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$.

In either case, $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$ and observation (β) follows.

(γ) If $\lambda' \le \kappa$, $\kappa \le \zeta \le \lambda$ and $\zeta' \ne \lambda'$, then $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$.

We construct an ζ-sized almost disjoint subset F of ${}^{[\lambda,\lambda]}\kappa$ that is maximal with respect to almost disjointness. Since $\lambda' \le \kappa < \lambda$ and $\zeta < \lambda$, we have that λ is singular and there exists a strictly increasing

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λ-sequence $\langle \lambda_\sigma;\ \sigma < \lambda' \rangle$ of regular cardinals greater than ζ. Let $\left\{S^\sigma_\beta;\ \sigma < \lambda' \text{ and } \beta < \zeta\right\}$ be a pairwise disjoint decomposition of λ with $\left|S^\sigma_\beta\right| = \lambda_\sigma$ always. For each γ less than κ let f^σ_β denote the constant function defined on S^σ_β taking value γ. The following properties hold:

(a) $f^\sigma_{\beta,\gamma} \in {}^{[\lambda_\sigma,\lambda]}\kappa$ always;

(b) $\lambda \times \kappa = \bigcup\left\{f^\sigma_{\beta,\gamma};\ \sigma < \lambda',\ \beta < \zeta \text{ and } \gamma < \kappa\right\}$.

Next observe that since $\zeta' \neq \lambda'$ and $\lambda' \leq \zeta$, it follows that $F_{\lambda'}(\lambda', \lambda', \zeta) = \lambda'.\zeta = \zeta$ and there is an ζ-sized almost disjoint subset G of ${}^{[\lambda',\lambda']}\zeta \times \kappa$ maximal with respect to almost disjointness. Further, we can assume, without loss of generality, that $\bigcup G = \lambda' \times \zeta \times \kappa$. We now apply the construction of Lemma 5. For each g in G put $F_g = \bigcup\left\{f^\sigma_{g(\sigma)};\ \sigma \in \operatorname{dom}(g)\right\}$ and put $\mathcal{F} = \{F_g;\ g \in G\}$. Since $|\operatorname{dom}(g)| = \lambda'$ and property (a) holds, it follows that $\mathcal{F} \subseteq {}^{[\lambda,\lambda]}\kappa$. Lemma 5 guarantees that $\mathcal{F}$ is λ-maximally almost disjoint and $|\mathcal{F}| = \zeta$. Finally, since $\bigcup G = \lambda' \times \zeta \times \kappa$ and property (b) holds, it follows that $\bigcup \mathcal{F} = \lambda \times \kappa$. Hence $\mathcal{F}$ is a λ-maximally almost disjoint (ζ, λ) decomposition of $\lambda \times \kappa$ and so is certainly maximal with respect to almost disjointness as claimed. The family $\mathcal{F}$, then, witnesses that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$.

With these three observations we can now settle the theorem in Cases *(ii)* and *(iii)*. By the usual argument, $\lambda^+ = F_\lambda(\lambda, \lambda, \kappa) \in \max_\lambda F(\lambda, \lambda, \kappa)$.

CASE *(ii)*. $\lambda' \neq \kappa'$.

Suppose ζ is a cardinal and $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. Property (A) of Lemma 2 implies $\zeta \geq \kappa$. On the other hand $\zeta \leq \lambda^+$ and observation (α) implies $\zeta \neq \lambda'$. Hence $\max_\lambda F(\lambda, \lambda, \kappa) \subseteq \{\zeta \in \operatorname{Cn}(\lambda^+);\ \kappa \leq \zeta\} - \{\lambda'\}$. As above, $\lambda^+ \in \max_\lambda F(\lambda, \lambda, \kappa)$. Next suppose $\kappa \leq \zeta \leq \lambda$ and $\zeta \neq \lambda'$. We show that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. If $\kappa < \lambda'$ then observation (β)

establishes that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. If $\lambda' \le \kappa$ and $\zeta' \ne \lambda'$, then observation (γ) implies that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. The remaining case is when $\lambda' \le \kappa$ and $\zeta' = \lambda'$. Here we appeal to Lemma 6. Since $\lambda' \ne \kappa'$ it follows that $\zeta' = \lambda' < \kappa \le \zeta$ and ζ is singular. Let $\langle \zeta_\tau ; \tau < \zeta' \rangle$ be a strictly increasing ζ-sequence of regular cardinals all greater than κ . For each τ less than ζ' we have $\kappa \le \zeta_\tau \le \lambda$ and $\zeta'_\tau \ne \lambda'$; so observation (γ) implies that $\zeta_\tau \in \max_\lambda F(\lambda, \lambda, \kappa)$. Lemma 6 now gives that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$ and the proof is complete in Case *(ii)*.

CASE *(iii)*. κ is infinite and $\lambda' = \kappa'$ (so $\lambda' \le \kappa$).

Suppose ζ is a cardinal and $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. Since $\lambda' = \kappa'$, Lemma 3 implies that $\zeta \ge \kappa^+$. On the other hand $\zeta \le \lambda^+$ and observation (α) implies $\zeta \ne \lambda'$. Hence $\max_\lambda F(\lambda, \lambda, \kappa) \subseteq \{\zeta \in \mathrm{Cn}(\lambda^+) ; \kappa^+ \le \zeta\} - \{\lambda'\}$. As above, $\lambda^+ \in \max_\lambda F(\lambda, \lambda, \kappa)$. Next suppose $\kappa^+ \le \zeta \le \lambda$ and $\zeta \ne \lambda'$. We show that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. If $\zeta' \ne \lambda'$ then observation (γ) implies $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$. If $\zeta' = \lambda'$ then ζ is singular (since $\zeta' = \lambda' = \kappa' < \kappa^+ \le \zeta$) and the proof that $\zeta \in \max_\lambda F(\lambda, \lambda, \kappa)$ in this case is identical to the corresponding proof when $\lambda' \le \kappa$ and $\zeta' = \lambda'$ in Case *(ii)* above. Hence $\{\zeta \in \mathrm{Cn}(\lambda^+) ; \kappa^+ \le \zeta\} - \{\lambda'\} \subseteq \max_\lambda F(\lambda, \lambda, \kappa)$. This then proves the theorem in Case *(iii)* and establishes the result. □

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