Canad. Math. Bull. Vol. **28** (2), 1985

EQUAL INTEGRALS OF FUNCTIONS

BY
F. S. CATER

ABSTRACT. Let $f_1, \ldots, f_k$ be finitely many L_1-functions on a measurable set E, and let d and r be numbers such that $\int_E f_j = d > r > 0$ for all j. Then there is a measurable subset S of E such that $\int_S f_j = r$ for all j.

1. In [1], Klamkin, McGregor and Meir observed that if f_1 and f_2 are L_1-functions on the real line, R, and if $\int_R f_1 = \int_R f_2 = 1$, then for each real number r $(0 < r < 1)$, there is a measurable set $S_r \subset R$ such that

$$\int_{S_r} f_1 = \int_{S_r} f_2 = r.$$

In the present note, we prove the (apparently harder) statement that this works for any finite number of functions.

THEOREM 1. *If $f_1, \ldots, f_k$ are L_1-functions on a measurable subset E of R such that*

$$\int_E f_1 = \ldots = \int_E f_k > 0.$$

Then for each real number r $(0 < r < \int_E f_j)$, there is a measurable set $S_r \subset E$ such that

$$\int_{S_r} f_1 = \ldots = \int_{S_r} f_k = r.$$

We show by example that this will not work for countably infinitely many functions f_j in general. To prove Theorem 1 we will construct a nest of measurable sets much like the nest of open sets constructed in the proof of Urysohn's Lemma in topology. When $k = 2$ this construction can be easily avoided.

Slight modifications of our arguments will show that Theorem 1 holds when R is replaced by a measure space that contains no atoms, but we will not do that here. The main difference is that the absence of atoms is used to prove the case $k = 1$. It can even be expressed in terms of finite signed measures on a σ-algebra of subsets of E. Let

Received by the editors January 4, 1984.
AMS Subject Classification: 28A25.
Key words: integral, L_1-function, measurable set.

$u_1, \ldots, u_k$ be such measures where $\Sigma_j |u_j|$ has no atoms and $0 < r < u_1(E) = \ldots = u_k(E)$. Then there is a set $S_r \subset E$ such that $u_j(S_r) = r$ for $j = 1, \ldots, k$.

2. The proof of Theorem 1 will be by induction on k. We begin with a lemma whose hypothesis appears excessive and requires positive functions, but it is precisely what we need in the induction argument. Notice the resemblance to the proof of Urysohn's Lemma.

LEMMA 1. *Let $f_1, \ldots, f_k$ be positive L_1-functions on a measurable set $E \subset R$ such that $\int_E f_1 = \ldots = \int_E f_k > 0$. Suppose that whenever $A \subset E$ is measurable and d is a number such that*

$$0 < d < \int_A f_1 = \ldots = \int_A f_k,$$

there exists a measurable set $B \subset A$ such that

$$\int_B f_1 = \ldots = \int_B f_k = d.$$

Then for each real number r ($0 \leq r \leq \int_E f_j$) there is a measurable set V_r such that $V_0 = \emptyset$, $V_{\int_E f_j} = E$, $\int_{V_r} f_j = r$ for all j and such that $V_r \subset V_{r'}$ if and only if $r < r'$.

PROOF. By replacing f_j with $f_j / \int_E f_j$ we can (and do) assume, without loss of generality, that $\int_E f_j = 1$ for all j. We first define V_r for dyadic rational numbers r between 0 and 1.

We define $V_{i2^{-p}}$ ($0 \leq i \leq 2^p$) by induction on p. For $p = 0$, put $V_0 = \phi$ and $V_1 = E$. Now suppose V_r has been chosen for $r = i2^{-p}$ ($0 \leq i \leq 2^p$, $0 \leq p \leq P - 1$) such that the conclusion of Lemma 1 holds for these numbers r. We define $V_{i2^{-P}}$ ($0 \leq i \leq 2^P$) as follows. For i odd, note that

$$\int_{A_i} f_1 = \ldots = \int_{A_i} f_k = 2^{1-P}$$

where

$$A_i = V_{1/2(i+1)2^{1-P}} \backslash V_{1/2(i-1)2^{1-P}}.$$

By hypothesis there is a measurable set $B_i \subset A_i$ such that

$$\int_{B_i} f_1 = \ldots = \int_{B_i} f_k = 2^{-P}.$$

Put

$$V_{i2^{-P}} = (V_{1/2(i-1)2^{1-P}}) \cup B_i.$$

Then

$$\int_{V_{i2^{-P}}} f_j = \int_{V_{1/2(i-1)2^{1-P}}} f_j + \int_{B_i} f_j$$
$$= 1/2(i-1)2^{1-P} + 2^{-P} = i2^{-P}$$

for each i and j. It follows that the conclusion of Lemma 1 holds for $r = i2^{-p}$ ($0 \leq i \leq 2^p$, $0 \leq p \leq P$). Of course V_r ($r = i2^{-P}$) was already defined when i is even. By induction it follows that the desired measurable set $V_{i2^{-p}}$ has been constructed for $0 \leq i \leq 2^p$, $p \geq 0$.

For $0 \leq r \leq 1$, let $V_r = \bigcup_{i2^{-p} \leq r} V_{i2^{-p}}$. Then

$$\int_{V_r} f_j = \sup_{i2^{-p} \leq r} \int_{V_{i2^{-p}}} f_j = r,$$

and the rest is straight-forward. □

If $k = 1$ the conclusion can be obtained more easily. Note that $G(t) = \int_{E \cap (-t,t)} f_1$ is a continuous function of t where $G(0) = 0$ and $\lim_{t \to \infty} G(t) = \int_E f_1$. For each r ($0 < r < \int_E f_1$) there is some value $t > 0$ such that $G(t) = r$. Let $V_r = E \cap (-t, t)$ for this t.

In our next lemma, the function g need not be positive, though of course the functions f_j must be positive.

LEMMA 2. *Let the hypothesis of Lemma* 1 *hold. Let g by any L_1-function. Then the function $G(r) = \int_{V_r} g$ is a continuous function of r for* $0 \leq r \leq \int_E f_j$ *where V_r is the set in the conclusion of Lemma* 1.

PROOF. Take any $c > 0$. Let S be a measurable set such that $\int_{R \setminus S} |g| < \frac{1}{2}c$ and $m(S) < \infty$. There is a $q > 0$ such that if $A \subset S$ and $m(A) < q$, then $\int_A |g| < \frac{1}{2}c$.

Because f_1 is positive on E, there is a number $d > 0$ such that if $A \subset S \cap E$ and $\int_A f_1 < d$, then $m(A) < q$.

Now suppose that $0 \leq r < r' \leq 1$ and $r' - r < d$. Then

$$\int_{(V_{r'} \setminus V_r) \cap S} f_1 \leq \int_{V_{r'} \setminus V_r} f_1 = r' - r < d, \qquad \int_{(V_{r'} \setminus V_r) \cap S} |g| < \frac{1}{2}c$$

and

$$|G(r') - G(r)| = \left| \int_{V_{r'} \setminus V_r} g \right| \leq \int_{(V_{r'} \setminus V_r) \cap S} |g| + \int_{R \setminus S} |g| < \frac{1}{2}c + \frac{1}{2}c = c.$$

□

LEMMA 3. *Let the hypothesis be as in Lemmas* 1 *and* 2, *and let $\int_E g = \int_E f_j$. Let $0 < r < \int_E g$. Then there is a measurable set $S \subset E$ such that*

$$\int_S f_1 = \ldots = \int_S f_k = \int_S g = r.$$

PROOF. We first consider the case in which $r = (1/n) \int_E f_j$ for some positive integer n. By hypothesis, we can partition E into mutually disjoint sets $E_1, \ldots, E_n$ such that

$$\int_{E_i} f_j = r \ (1 \leq i \leq n,\ 1 \leq j \leq k).$$

We assume without loss of generality that for each i, $\int_{E_i} g \neq r$. Reindex so that $\int_{E_1} g < r < \int_{E_2} g$.

By Lemma 1, we construct sets V_t and W_t $(0 \leq t \leq r)$ such that $V_t \subset E_1$, $W_t \subset E_2$, $V_0 = W_0 = \phi$, $V_r = E_1$, $W_r = E_2$, $V_t \subset V_{t'}$ and $W_t \subset W_{t'}$ if and only if $t < t'$, and $\int_{V_t} f_j = \int_{W_t} f_j = t$ for all $j = 1, \ldots, k$. Put

$$G(t) = \int_{V_t \cup W_{r-t}} g = \int_{V_t} g + \int_{W_{r-t}} g \ (0 \leq t \leq r).$$

By Lemma 2, G is continuous and $G(0) = \int_{W_r} g = \int_{E_2} g > r$, $G(r) = \int_{V_r} g = \int_{E_1} g < r$. There is a t_0 $(0 < t_0 < r)$ with $G(t_0) = r$. But then $r = \int_{V_{t_0} \cup W_{r-t_0}} g = t_0 + (r - t_0) = \int_{V_{t_0} \cup W_{r-t_0}} f_j$ $(j = 1, \ldots, k)$.

In the general case, let n_1 be the smallest integer such that $0 < (1/n_1) \int_E g < r$. Let $X_1 \subset E$ be a measurable set such that $\int_{X_1} g = \int_{X_1} f_j = (1/n_1) \int_E g$ for all j. Then $\int_{E \backslash X_1} g = \int_{E \backslash X_1} f_j$ for all j. Let n_2 be the smallest integer such that $0 < (1/n_2) \int_{E \backslash X_1} g < r - \int_{X_1} g$. Let $X_2 \subset E \backslash X_1$ such that $\int_{X_2} g = \int_{X_2} f_j = (1/n_2) \int_{E \backslash X_1} g$ for all j. Note that $r - \int_{X_1} g \leq \frac{1}{2} r$ and $r - \int_{X_1 \cup X_2} g \leq \frac{1}{2}(r - \int_{X_1} g) \leq \frac{1}{4} r$. We continue in the obvious way to construct a sequence of mutually disjoint measurable sets $X_1, X_2, \ldots, X_i, \ldots$ such that for each i,

$$0 < r - \int_{X_1 \cup \ldots \cup X_i} g = r - \int_{X_1 \cup \ldots \cup X_i} f_j \leq 2^{-i} r.$$

Finally $S = \bigcup_{i=1}^{\infty} X_i$ satisfies

$$\int_S g = \int_S f_j = r \qquad (j = 1, \ldots, k).$$ □

We are ready to prove Theorem 1 for positive f_j.

LEMMA 4. *Theorem* 1 *holds when all the functions* f_j *are positive on* E.

The proof is by induction on k. For $k = 1$, note that $G(t) = \int_{(-t,t) \cap E} f_1$ is a continuous function of t for $0 \leq t < \infty$. Also $\lim_{t \to \infty} G(t) = \int_E f_1$ and $G(0) = 0$. For some $s > 0$, $G(s) = r$. Let $S_r = (-s, s) \cap E$.

Now suppose that the conclusion holds for k such functions, $f_1, \ldots, f_k$. Let $\int_E f_1 = \ldots = \int_E f_k = \int_E f_{k+1} > 0$ where $f_1, \ldots, f_k, f_{k+1}$ are positive L_1-functions on E. By Lemma 3, the required set S_r exists. This concludes the induction on k. □

We use a trick to remove positivity.

PROOF OF THEOREM 1. Let $H(x) = |f_1(x)| + \ldots + |f_k(x)| + e^{-x^2}$. Then H is a positive L_1-function on E. Let $F_i = f_i + H$ $(i = 1, \ldots, k)$. Then each F_i is a positive L_1-function and

$$\int_E F_i = \int_E f_i + \int_E H = \int_E f_1 + \int_E H > 0 \qquad (i = 1, \ldots, k).$$

By Lemma 4, the functions F_i satisfy the hypotheses of Lemmas 1 and 2. Let V_t be the measurable set in the conclusion of Lemma 1 where $\int_{V_t} F_i = t$ for $i = 1, \ldots, k$. By Lemma 2, $G(t) = \int_{V_t} f_1$ is a continuous function of t for $0 \leq t \leq \int_E F_i$. Also $G(0) = 0$ and $G(\int_E F_i) = \int_E f_1$.

Now let r be any number such that $0 < r < \int_E f_1$. Then by continuity of G, there is a t_0, $0 < t_0 < \int_E F_i$, such that $G(t_0) = r = \int_{V_{t_0}} f_1$. But for $i = 1, \ldots, k$,

$$\int_{V_{t_0}} f_i + \int_{V_{t_0}} H = \int_{V_{t_0}} F_i = \int_{V_{t_0}} F_1 = \int_{V_{t_0}} f_1 + \int_{V_{t_0}} H = r + \int_{V_{t_0}} H.$$

Thus $\int_{V_{t_0}} f_i = r$ for $i = 1, \ldots, k$. □

3. In this section we find that Theorem 1 does not hold in general for infinitely many functions f_n. In Example 1, it will not matter which number r in the open interval (0, 1) is used.

EXAMPLE 1. *Let $I_1, I_2, I_3, \ldots$ be the closed subintervals of the unit interval* (0, 1) *with rational endpoints enumerated. For each $n > 0$, let $f_n(x) = 1/m(I_n)$ for x in I_n and $f_n(x) = 0$ otherwise. Then $\int_R f_n = 1$ for each $n > 0$.*

Choose any real number r with $0 < r < 1$. We claim that there is no measurable set E such that $\int_E f_n = r$ for all $n > 0$. Suppose that there were. Then $m(I_1 \cap E) > 0$, so there is a nonvoid open set $U \subset (0, 1)$ such that $m(U \cap E) > rm(U)$. Now U can be covered by countably many nonoverlapping intervals I_j from the sequence $(I_n)_{n=1}^{\infty}$. It follows that some one of the intervals I_j − call it I_i − satisfies $m(I_i \cap E) > rm(I_i)$. So

$$\int_E f_i = m(E \cap I_i)/m(I_i) > rm(I_i)/m(I_i) = r.$$

REFERENCES

1. M. S. Klamkin, J. McGregor and A. Meir, *Problem 6440*, American Math. Monthly, **90**, 8 (1983), p. 569.

DEPARTMENT OF MATHEMATICS
PORTLAND STATE UNIVERSITY
PORTLAND, OREGON 97207