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A CLASS OF FINITE GROUPS HAVING NILPOTENT INJECTORS

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Abstract

The purpose of this paper is to construct a class of groups which properly contains the class of $\mathscr{N}$-constrained groups, and which is such that all groups in this class have $\mathscr{N}$-injectors.

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All groups considered throughout this paper will be finite. We denote by $\mathscr{N}$ the class of nilpotent groups, by $\tilde{\mathscr{N}}$ the class of quasinilpotent groups, i.e. $\tilde{\mathscr{N}} = \{G \mid G = F(G)L(G) = F^*(G)\}$, and we put $\mathscr{L} = \{G \mid G = C_G(L(G))L(G)\}$, where $L(G)$ is the semisimple radical of G (the concept of semisimple group is taken from Gorenstein-Walter's paper [2]). The properties of the subgroups $F^*(G)$ and $L(G)$ which we shall use here are given in [3].

Let $\mathscr{M}$ be the class of all groups G such that for every subnormal subgroup N of G, we have $C_N(J) \leqslant J$, where J is any $\mathscr{N}$-injector of $F^*(N)$. By [4] we know that every $\mathscr{N}$-injector of $F^*(G)$ constitutes the product of $F(G)$ and an $\mathscr{N}$-injector of $L(G)$.

A group G is $\mathscr{N}$-constrained if $C_G(F(G)) \leqslant F(G)$. In 1971 A. Mann proved that an $\mathscr{N}$-constrained group has an unique conjugacy class of $\mathscr{N}$-injectors. In [4] we proved that all groups belonging to $\mathscr{L}$ have $\mathscr{N}$-injectors. Using Lausch's theorem [5] we can obtain the following

THEOREM. *Let G be an $\mathscr{M}$-group. Then G possesses $\mathscr{N}$-injectors, and these are exactly the $\mathscr{N}$-maximal subgroups of G that contain the product of $F(G)$ and an $\mathscr{N}$-injector of $L(G)$.*

Proof. We use induction on $|G|$. Put $Z = Z(L(G))$. Let G^* be a maximal normal subgroup of G. Then we have

$$L(G)/Z = \left(L(G) \cap G^*\right)Z/Z \cdot RZ/Z,$$

where $R \trianglelefteq G$ is semisimple. Moreover, we have

$$L(G) \cap G^* = \left(L(G) \cap G^*\right)Z\left(L(G) \cap G^*\right) = L(G^*)Z\left(L(G) \cap G^*\right).$$

On the other hand, $L(G) = (L(G) \cap G^*)R$, where $[L(G^*), R] \leqslant Z$, and by the three-subgroups lemma together with the perfectness of $L(G^*)$, it follows that $[L(G^*), R] = 1$. Therefore $R \leqslant C_{L(G)}(L(G^*))$, whence

$$[R, G^*] \leqslant \left[C_{L(G)}\left(L(G^*)\right), G^*\right] \leqslant C_{L(G) \cap G^*}\left(L(G^*)\right) = Z\left(L(G) \cap G^*\right),$$

and again by the three-subgroups lemma we obtain $[R, G^*] = 1$. Now let I be an $\mathcal{N}$-injector of $L(G)$. Then I/Z is an $\mathcal{N}$-injector of $L(G)/Z$, so that $I/Z \cap RZ/Z$ is an $\mathcal{N}$-injector of RZ/Z, and $I/Z \cap L(G^*)Z/Z$ is an $\mathcal{N}$-injector of $L(G^*)Z/Z$. Thus we have

$$I/Z = \left(I \cap L(G^*)\right)Z/Z \times (I \cap R)Z/Z,$$

whence

$$I = \left(I \cap L(G^*)\right)(I \cap R)Z = \left(I \cap L(G^*)\right)(I \cap R),$$

because $Z = Z(L(G^*))Z(R)$.

Now let V be an $\mathcal{N}$-maximal subgroup of G such that $IF(G) \leqslant V$, and suppose that $V \cap G^* \leqslant W \leqslant G^*$, where $W \in \mathcal{N}$. It is clear that $F(G^*) \leqslant V \cap G^* \leqslant W \leqslant G^*$, and that $I \cap L(G^*) \leqslant V \cap G^* \leqslant W \leqslant G^*$. Thus

$$I = \left(I \cap L(G^*)\right)(I \cap R) \leqslant (V \cap G^*)(I \cap R) \leqslant W(I \cap R) \leqslant G^*(I \cap R).$$

Moreover, $W(I \cap R) \in \mathcal{N}$ because $[G^*, R] = 1$.

By the inductive hypothesis, we can assume that W is an $\mathcal{N}$-injector of G^*. We now have two cases.

Case 1. $F(G) \leqslant G^*$. In this case $F(G^*) = F(G) \leqslant W$. Let V_1 be an $\mathcal{N}$-maximal subgroup of G that contains $W(I \cap R)$. Then $IF(G) \leqslant V_1$. Since $C_G(IF(G)) \leqslant IF(G)$, an application of Lausch's theorem implies that there exists $g \in G$ such that $V_1 = V^g$. In consequence, we have

$$V \cap G^* \leqslant W \leqslant V_1 \cap G^* = V^g \cap G^* = (V \cap G^*)^g,$$

and so $V \cap G^* = W$ is an $\mathcal{N}$-injector of G^*.

Case 2. $F(G) \nleqslant G^*$. In this case $G = F(G)G^*$, whence $G/G^* \cong C_p$ for a prime number p. Thus, either $VG^*/G^* = 1$, or $VG^* = G$. In the first case, $VG^* = G^*$, and so $F(G) \leqslant V \leqslant G^*$, which is a contradiction. Therefore, $VG^* = G$. Now we have

$$G/G^* = VG^*/G^* \cong V/(V \cap G^*) \cong C_p.$$

Since $V \cap W = V \cap G^*$, it follows that $V/(V \cap W) \cong C_p$, and since $F(G) \leqslant V$ but $F(G) \nleqslant W$, we obtain

$$V = F(G)(V \cap W) = V \cap F(G)W \leqslant F(G)W.$$

Now, as $F(G)W$ is solvable, it possesses a unique conjugacy class of $\mathscr{N}$-injectors. If W_1 is one of them, then W_1 and V are $\mathscr{N}$-maximal subgroups of $F(G)W$ containing $F(G)$; moreover, as $F(G)W/F(G)$ is nilpotent, it follows [1, Hilsatz 1] that V and W_1 are conjugated in $F(G)W$. Hence V is an $\mathscr{N}$-injector of $F(G)W$. Thus $V \cap G^*$ is an $\mathscr{N}$-maximal subgroup of $F(G)W \cap G^* = W(F(G) \cap G^*) = W$, and so $V \cap G^* = W$.

Trivially, an $\mathscr{N}$-injector of G is an $\mathscr{N}$-maximal subgroup of G that contains the product of $F(G)$ and an $\mathscr{N}$-injector of $L(G)$.

Proposition 1. $\mathscr{L} \subset \mathscr{M}$.

Proof. Let I be an $\mathscr{N}$-injector of $L(G)$, and let $G \in \mathscr{L}$. Then we have

$$C_G(IF(G)) \leqslant C_G(I) \cap C_G(F(G)) = C_{C_G(F(G))}(I);$$

but $L(G) \leqslant C_G(F(G)) \leqslant L(G)C_G(L(G))$, and so

$$\begin{aligned} C_G(F(G)) &= L(G)\big(C_G(F(G)) \cap C_G(L(G))\big) \\ &= L(G)C_G\big(F^*(G)\big) = L(G)Z\big(F^*(G)\big). \end{aligned}$$

From $Z(F^*(G)) \leqslant C_{C_G(F(G))}(I) \leqslant L(G)Z(F^*(G))$ we get

$$C_{C_G(F(G))}(I) = Z\big(F^*(G)\big)C_{L(G)}(I) \leqslant Z\big(F^*(G)\big)I = Z\big(F(G)\big)I.$$

Now if N is a subnormal subgroup of G, then since $\mathscr{L}$ is an S_n-closed class, N is also an $\mathscr{L}$-group, and by the above argument we obtain $C_N(J) \leqslant J$, where J is an $\mathscr{N}$-injector of $F^*(N)$. Thus G is an $\mathscr{M}$-group.

Remark. The class $\mathscr{L}$ is properly contained in $\mathscr{M}$. In fact, if we take $G = A_5$ wr C_7, then $L(G) = A_5^\#$, i.e. the base group of G, and $C_G(L(G)) = 1$. Hence $G \notin \mathscr{L}$.

Moreover, we know that $A_5^\#$ is the unique minimal normal subgroup of G, whence all proper subnormal subgroups of G are contained in $A_5^\#$. Let N be one of them, so that $F^*(N) = N = L(N)$, and trivially $C_N(I) \leqslant I$ for every $\mathscr{N}$-injector I of N. So it remains to prove that if I is an $\mathscr{N}$-injector of $L(G)$, then

$$C_G(IF(G)) \leqslant IF(G), \text{ i.e. } C_G(I) \leqslant I.$$

This follows easily from the corresponding statement for $\mathscr{N}$-injectors of $A_5^\#$ (which gives $C_{L(G)}(I) \leqslant I$) and from $C_G(I) \leqslant L(G)$ (which is obvious from the regular action of $C_7 \leqslant G$ on $L(G)$ together with the fact that 7 does not divide $|L(G)|$).

If $S \leqslant G$, we put $[S]_G = \{S^g \mid g \in G\}$.

PROPOSITION 2. *If $G \in \mathcal{M}$, then there exists a bijection φ between $\{[\mathcal{I}]_G \mid \mathcal{I}$ is an $\mathcal{N}$-injector of $L(G)\}$ and $\{[\mathcal{V}]_G \mid \mathcal{V}$ is an $\mathcal{N}$-injector of $G\}$ given by*

$$\varphi([\mathcal{I}]_G) = [\mathcal{V}]_G,$$

where $\mathcal{V}$ is an $\mathcal{N}$-injector of G that contains $\mathcal{I}$.

PROOF. Let I_1, I_2 be $\mathcal{N}$-injectors of $L(G)$ which are conjugate in G, i.e. $I_2 = I_1^g$ for some $g \in G$. Let V_1, V_2 be $\mathcal{N}$-injectors of G that contain I_1, I_2, respectively. Then $I_2 = I_1^g \leqslant V_1^g$. Therefore $I_2F(G) \leqslant V_1^g \cap V_2$, and by Lausch's Theorem, V_1^g and V_2 are conjugate in G.

Obviously φ is surjective. Moreover, if I_1, I_2 are $\mathcal{N}$-injectors of $L(G)$ such that

$$\varphi([I_1]_G) = [V_1]_G = [V_2]_G = \varphi([I_2]_G),$$

then there exists $g \in G$ such that $V_2 = V_1^g$, $I_2 = V_2 \cap L(G) = V_1^g \cap L(G) = (V_1 \cap L(G))^g = I_1^g$.

Set $S = \{[V]_G \mid V$ is an $\mathcal{N}$-injector of $G\}$.

COROLLARY. *The following conditions are equivalent*:

(i) *G is an $\mathcal{M}$-group, and $|\mathcal{S}| = 1$;*

(ii) *G is an $\mathcal{N}$-constrained group.*

PROOF. (ii) $\Rightarrow$ (i). By Proposition 1, we know that all $\mathcal{N}$-constrained groups are $\mathcal{M}$-groups. A. Mann's theorem [6] then implies that $|S| = 1$.

(i) $\Rightarrow$ (ii). We must prove that $L(G) = 1$ [4]. Let P, Q be p-Sylow and q-Sylow subgroups of $L(G)$ ($p \neq q$), respectively, and let I_1, I_2 be $\mathcal{N}$-injectors of $L(G)$ that contain P, Q, respectively. Let V_1, V_2 be $\mathcal{N}$-injectors of G such that $I_1 \leqslant V_1$ and $I_2 \leqslant V_2$. Then there exists $g \in G$ such that $V_1^g = V_2$, and so $I_2 = I_1^g$. Therefore $P^g \leqslant I_1^g = I_2$. This argument shows that $|I_2| = |L(G)|$, and so $L(G)$ is nilpotent. Hence $L(G) = 1$.

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