AN INTEGRAL FORMULA FOR HYPERSURFACES IN SPACE FORMS

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1. Introduction. Let Q_c^{n+1} be an $n+1$-dimensional, complete simply connected Riemannian manifold of constant sectional curvature c and $P_0 \in Q_c^{n+1}$. We consider the function $r(\cdot) = d(\cdot, P_0)$ where d stands for the distance function in Q_c^{n+1} and we denote by $\operatorname{grad} r$ the gradient of r in Q_c^{n+1}. The *position vector* (see [**1**]) with origin P_0 is defined as $x = \varphi_c(r)\operatorname{grad} r$, where $\varphi_c(r)$ equals

$$\frac{\sin(\sqrt{c}r)}{\sqrt{c}} \quad \text{or} \quad \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}}$$

r, if $c = 0$, $c > 0$ or $c < 0$ respectively.

Let $f: M^n \to Q_c^{n+1}$ be an immersed, oriented, connected hypersurface. We decompose the position vector x, restricted to M^n, in a component normal to M^n, and a component x_T tangent to M^n:

$$x = x_T + pN, \tag{1.1}$$

where N is the given orientation. The function $p = \langle x, N\rangle$ is called the *support function* (cf. [**1**]) with respect to the origin P_0. We denote by Ric, τ and dv respectively the Ricci curvature, the scalar curvature and the associated volume element on M^n.

The main purpose of this paper is to establish the following theorem.

THEOREM 1. *Let $f: M^n \to Q_c^{n+1}$ be an oriented compact hypersurface. Then*

$$\int_{M^n} (n(n-1) - p^2\tau + \operatorname{Ric}(x_T) - c(n-1)(n+2)\,|x_T|^2)\, dv = 0.$$

The above integral formula yields the following characterizations of geodesic spheres in space forms.

THEOREM 2. *Let $f: M^n \to Q_c^{n+1}$ be an oriented, compact and connected hypersurface. If* $\operatorname{Ric}(X) > c(n-1)(n+2)\,|X|^2$ *for every non-zero tangent vector X and $p^2\tau \le n(n-1)$, then $f(M^n)$ is a geodesic sphere.*

THEOREM 3. *Let $f: M^n \to Q_c^{n+1}$ be an oriented, compact and connected hypersurface. If all sectional curvatures of M^n are greater than c and $p^2\tau \le n(n-1) - c(n-1)(n+1)\,|x_T|^2$ then $f(M^n)$ is a geodesic sphere.*

An immediate consequence of Theorem 3 is the following.

COROLLARY. *Let $f: M^n \to Q_c^{n+1} (c \le 0)$ be an oriented, compact and connected hypersurface. If all sectional curvatures of M^n are greater than c and $p^2\tau \le n(n-1)$, then $f(M^n)$ is a geodesic sphere.*

REMARK. Theorem 1 and Theorem 2 were obtained by S. Deshmukh [**2**] for $c = 0$. The spherical case was considered in [**4**]. Our approach shows that similar results are

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valid in any space form including the hyperbolic space. Moreover Theorem 2 extends the result of S. Deshmukh in [3].

2. Preliminaries. We need the following result.

LEMMA 2.1. *Let $\bar{\nabla}$ denotes the Riemannian connection of Q_c^{n+1}. Then the position vector with respect to an origin $P_0 \in Q_c^{n+1}$ satisfies*

$$\bar{\nabla}_X x = \varphi_c'(r)X, \tag{2.1}$$

for any tangent vector X in Q_c^{n+1}.

Proof. The case $c=0$ is trivial.

Case $c>0$. Assume that Q_c^{n+1} is the hypersphere of radius $\dfrac{1}{\sqrt{c}}$ in the Euclidean space $\mathbf{R}^{n+2}$. The position vector at $P \in Q_c^{n+1}$ is given by (cf. [1])

$$x(P)=\cos(\sqrt{c}r(P))P-P_0,$$

where $\langle\ ,\ \rangle$ stands for the usual inner product in $\mathbf{R}^{n+2}$. Differentiating covariantly in $\mathbf{R}^{n+2}$ in the direction of X and taking the tangential component we obtain (2.1).

Case $c<0$. Let L^{n+2} be the Euclidean space $\mathbf{R}^{n+2}$ endowed with the pseudo-Riemannian metric given by

$$\langle u, w\rangle=\sum_{i=1}^{n+1} u_i w_i-u_{n+2}w_{n+2},$$

where $u=(u_1,\ldots,u_{n+2})$ and $w=(w_1,\ldots,w_{n+2})$. It is well known that the hyperbolic space Q_c^{n+1} can be realized as

$$Q_c^{n+1}=\left\{u\in L^{n+2} \mid u_{n+2}>0 \quad\text{and}\quad \langle u,u\rangle=\frac{1}{c}\right\}.$$

The position vector at $P \in Q_c^{n+1}$ is given by (cf. [1])

$$x(P)=\cosh(\sqrt{-c}r(P))P-P_0.$$

Differentiating in the direction of X and using the second fundamental form of Q_c^{n+1} as a hypersurface of L^{n+2} we get (2.1).

Let $f: M^n \to Q_c^{n+1}$ be an oriented hypersurface with given orientation N. Denote by ∇ the Riemannian connection of M^n. For tangent vectors X and Y of M^n we have the Gauss formula

$$\bar{\nabla}_X Y=\nabla_X Y+\langle AX, Y\rangle N$$

and the Weingarten formula

$$\bar{\nabla}_X N=-AX,$$

where A is the Weingarten map. Using these, (1.1) and (2.1) we compute

$$\begin{aligned}\varphi_c' X &= \bar{\nabla}_X x_T+(Xp)N+p\bar{\nabla}_X N\\ &= \nabla_X x_T+\langle Ax_T, X\rangle N+(Xp)N-pAX.\end{aligned}$$

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Taking the tangential and the normal component of both sides of this equation we obtain (cf. [5])

$$\nabla_X x_T = \varphi_c'(r)X + pAX, \tag{2.2}$$

$$\operatorname{grad} p = -Ax_T. \tag{2.3}$$

LEMMA 2.2. *Let* $f: M^n \to Q_c^{n+1}$ *be an oriented hypersurface with mean curvature* H. *Then*

$$\frac{1}{2}\Delta |x|^2 = -c\,|x_T|^2 + \varphi_c'(n\varphi_c' + npH), \tag{2.4}$$

$$-\Delta p = n\langle \operatorname{grad} H, x_T\rangle + n\varphi_c' H + p\operatorname{tr} A^2 \tag{2.5}$$

where Δ *is the Laplace operator of* M^n.

Proof. Using (2.1) we easily find

$$\operatorname{grad} |x|^2 = 2\varphi_c' x_T. \tag{2.6}$$

From (2.2) we get (cf. [1])

$$\operatorname{div} x_T = n\varphi_c' + npH. \tag{2.7}$$

Equations (2.6) and (2.7) imply

$$\frac{1}{2}\Delta |x|^2 = \frac{\varphi_c''}{\varphi_c}|x_T|^2 + \varphi_c'(n\varphi_c' + npH).$$

By virtue of $\varphi_c'' = -c\varphi_c$ we obtain (2.4).

Let $X_1, \ldots, X_n$ be an orthonormal frame on M^n. On account of (2.3) we have

$$-\Delta p = \sum_{i=1}^{n} \langle (\nabla_{X_i} A)x_T, X_i\rangle + \sum_{i=1}^{n} \langle A(\nabla_{X_i} x_T), X_i\rangle.$$

Using Codazzi equation, the symmetry of A and (2.2) we get (2.5).

3. Proofs of the results.

Proof of Theorem 1. Using (2.3) and (2.5) we obtain

$$\frac{1}{2}\Delta p^2 = -np\langle \operatorname{grad} H, x_T\rangle - n\varphi_c' pH - p^2 \operatorname{tr} A^2 + |Ax_T|^2.$$

The Ricci curvature in the direction x_T is given by (cf. [6])

$$\operatorname{Ric}(x_T) = c(n-1)\,|x_T|^2 + nH\langle Ax_T, x_T\rangle - |Ax_T|^2.$$

Therefore, by means of (2.3) we have

$$-\frac{1}{2}\Delta p^2 = n\langle \operatorname{grad}(pH), x_T\rangle + n\varphi_c' pH + p^2 \operatorname{tr} A^2 + \operatorname{Ric}(x_T) - c(n-1)\,|x_T|^2. \tag{2.8}$$

From (2.7) we easily find

$$\langle \operatorname{grad}(pH), x_T\rangle = \operatorname{div}(pHx_T) - n\varphi_c' pH - np^2H^2. \tag{2.9}$$

Combining (2.8), (2.9) and the equation $\tau = n^2H^2 - \operatorname{tr} A^2 + n(n-1)c$ we get

$$-\frac{1}{2}\Delta p^2 - n\,\operatorname{div}(pHx_T) = n(1-n)\varphi_c' pH - p^2\tau + \operatorname{Ric}(x_T) + c(n-1)(np^2 - |x_T|^2).$$

By integration we have

$$\int_{M^n} (n(1-n)\varphi_c' pH - p^2\tau + \operatorname{Ric}(x_T) + c(n-1)n\,|x|^2 - c(n-1)(n+1)\,|x_T|^2)\,dv = 0.$$

Furthermore from (2.4) we obtain

$$\int_{M^n} n\varphi_c' pH\,dv = \int_{M^n} (c\,|x_T|^2 - n(\varphi_c')^2)\,dv.$$

Hence

$$\int_{M^n} (n(n-1)((\varphi_c')^2 + c\,|x|^2) - p^2\tau + \operatorname{Ric}(x_T) - c(n-1)(n+2)\,|x_T|^2)\,dv = 0.$$

Moreover it is easy to see that $|x|^2 = (\varphi_c)^2$ and so $(\varphi_c')^2 + c\,|x|^2 = 1$. This completes the proof of Theorem 1.

Proof of Theorem 2. Using the assumptions and Theorem 1 we get

$$\operatorname{Ric}(x_T) = c(n-1)(n+2)\,|x_T|^2.$$

Hence $x_T = 0$ on M^n. From (2.6) we conclude that $|x|^2 = \text{const}$. On account of $|x|^2 = (\varphi_c)^2$ we infer that $f(M^n)$ is a geodesic sphere.

Proof of Theorem 3. The integral formula stated in Theorem 1 can be rearranged as

$$\int_{M^n} (\operatorname{Ric}(x_T) - c(n-1)\,|x_T|^2 + n(n-1) - p^2\tau - c(n-1)(n+1)\,|x_T|^2)\,dv = 0.$$

Since all sectional curvatures are greater than c we have

$$\operatorname{Ric}(X) > c(n-1)\,|X|^2$$

for every non-zero tangent vector X. Our assumptions imply that $\operatorname{Ric}(x_T) = c(n-1)\,|x_T|^2$ and so $x_T = 0$ on M^n. Since $|x|^2 = (\varphi_c)^2$ from (2.6) we deduce that $f(M^n)$ is a geodesic sphere.

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