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On the Size of an Expression in the Nyman–Beurling-Báez–Duarte Criterion for the Riemann Hypothesis

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Abstract. A crucial role in the Nyman–Beurling–Báez-Duarte approach to the Riemann Hypothesis is played by the distance

$$d_N^2 := \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \zeta A_N\left(\frac{1}{2} + it\right)\right|^2 \frac{dt}{\frac{1}{4} + t^2},$$

where the infimum is over all Dirichlet polynomials

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$$

of length N. In this paper we investigate d_N^2 under the assumption that the Riemann zeta function has four nontrivial zeros off the critical line.

1 Introduction

The Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis asserts that the Riemann hypothesis is true if and only if $\lim_{N\to\infty} d_N^2 = 0$, where

$$d_N^2 := \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|1 - \zeta A_N\left(\frac{1}{2} + it\right)\right|^2 \frac{dt}{\frac{1}{4} + t^2}, \tag{1.1}$$

and the infimum is over all Dirichlet polynomials

$$A_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}$$

of length N (see [3]).

Burnol [4], improving on work of Báez-Duarte, Balazard, Landreau, and Saias [1,2], showed that

$$\liminf_{N\to\infty} d_N^2 \log N \geq \sum_{\mathrm{Re}(\rho)=\frac{1}{2}} \frac{m(\rho)^2}{|\rho|^2},$$

where $m(\rho)$ denotes the multiplicity of the zero ρ.

This lower bound is believed to be optimal, and one expects that

$$d_N^2 \sim \frac{1}{\log N} \sum_{\mathrm{Re}(\rho)=\frac{1}{2}} \frac{m(\rho)^2}{|\rho|^2}. \tag{$*$}$$

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Under the Riemann hypothesis, one has

$$\sum_{\mathrm{Re}(\rho)=\frac{1}{2}} \frac{m(\rho)}{|\rho|^2} = 2 + \gamma - \log 4\pi,$$

where γ is the Euler–Mascheroni constant.

S. Bettin, J. B. Conrey, and D. W. Farmer [3] prove $(*)$ under an additional assumption and also identify the Dirichlet polynomials A_N, for which the expected infimum in (1.1) is assumed. They prove the following theorem.

Theorem ([3, Theorem 1]) *Let*

$$V_N(s) := \sum_{n=1}^{N} \Big(1 - \frac{\log n}{\log N}\Big) \frac{\mu(n)}{n^s}.$$

If the Riemann hypothesis is true and if

$$\sum_{|\,\mathrm{Im}(\rho)|\le T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{\frac{3}{2}-\delta}$$

for some $\delta > 0$, then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \Big|1 - \zeta V_N\Big(\frac{1}{2} + it\Big)\Big|^2 \frac{dt}{\frac{1}{4} + t^2} \sim \frac{2 + \gamma - \log 4\pi}{\log N}. \tag{1.2}$$

In this paper we investigate expression (1.2) under an assumption contrary to the Riemann hypothesis: There are exactly four nontrivial zeros off the critical line. We observe that nontrivial zeros off the critical line always appear as quadruplets. Indeed, if $\zeta(\rho) = 0$ for $\rho = \sigma + i\gamma$ with $1 > \sigma > \frac{1}{2}$, $\gamma > 0$, then from the functional equation

$$\Lambda(s) = \Lambda(1 - s), \tag{1.3}$$

where

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s)$$

and the trivial relation $\zeta(\bar{s}) = \overline{\zeta(s)}$, we obtain that

$$\zeta(\sigma + i\gamma) = \zeta(1 - \sigma + i\gamma) = \zeta(\sigma - i\gamma) = \zeta(1 - \sigma - i\gamma) = 0.$$

We prove the following theorem.

Theorem 1.1 *Let $\sigma_0 > 1/2$, $\gamma_0 > 0$, $\zeta(\sigma_0 \pm i\gamma_0) = \zeta(1 - \sigma_0 \pm i\gamma_0) = 0$, and $\zeta(\sigma + i\gamma) \neq 0$ for all other $\sigma + i\gamma$ with $\sigma > 1/2$. Assume that*

$$\sum_{|\,\mathrm{Im}(\rho)|\le T} \frac{1}{|\zeta'(\rho)|^2} \ll T^{\frac{3}{2}-\delta} \quad (T \to \infty),$$

for some $\delta > 0$. *Then there are real constants* $A = A(\sigma_0, \gamma_0)$, $B = B(\sigma_0, \gamma_0)$, *and* $C = C(\sigma_0, \gamma_0)$ *such that for all* $\epsilon > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big|1-\zeta V_N\Big(\frac{1}{2}+it\Big)\Big|^2\frac{dt}{\frac{1}{4}+t^2}$$
$$=\frac{1}{(\log N)^2}\big(AN^{2\sigma_0-1}\cos(2\gamma_0\log N)+BN^{2\sigma_0-1}\sin(2\gamma_0\log N)+CN^{2\sigma_0-1}\big)\big(1+O(N^{\frac{1}{2}-\sigma_0+\epsilon})\big).$$

2 Preliminary Lemmas and Definitions

Lemma 2.1 *Let* $\epsilon > 0$ *be fixed but arbitrarily small. Under the assumptions of Theorem 1.1 we have*

$$\zeta(\sigma+it) \ll |t|^{3\epsilon} \quad (|t|\to\infty), \tag{2.1}$$

for $\frac{1}{2}-\epsilon \le \sigma \le \frac{1}{2}+\epsilon$.

Proof Estimate (2.1) is well known as the Lindelöf hypothesis, which is a consequence of the Riemann hypothesis. In [5], the Lindelöf hypothesis is proved on the assumption of the Riemann hypothesis. This proof can be adapted to the new situation with slight modification.

The function $\log\zeta(s)$ is holomorphic in the domain

$$\mathcal{G} := \Big\{\sigma+it : \sigma > \frac{1}{2}\Big\} \smallsetminus \Big\{\Big[\frac{1}{2},1\Big] \cup \Big[\frac{1}{2}+i\gamma_0, \sigma_0+i\gamma_0\Big] \cup \Big[\frac{1}{2}-i\gamma_0, \sigma_0-i\gamma_0\Big]\Big\}.$$

Now let $\frac{1}{2} < \sigma^* \le \sigma \le 1$. As in [5], let $z = \sigma + it$, but now $|t|$ is sufficiently large.

We apply the Borel–Carathéodory theorem to the function $\log\zeta(z)$ and the circles with centre $2+it$ and radii $\frac{3}{2}-\frac{1}{2}\delta$ and $\frac{3}{2}-\delta$, $(0<\delta<\frac{1}{2})$.

On the larger circle,

$$\mathrm{Re}(\log\zeta(z)) = \log|\zeta(z)| < A\log t$$

for a fixed positive constant A. Hence, on the smaller circle,

$$|\log\zeta(z)| \le \frac{3-2\delta}{\frac{1}{2}\delta}A\log t + \frac{3-\frac{3}{2}\delta}{\frac{1}{2}\delta}\big|\log|\zeta(2+it)|\big| < A\delta^{-1}\log t.$$

We now apply Hadamard's three circle theorem as in [5]. The proof there can be taken over without change to obtain

$$\zeta(z) = O(t^{\epsilon}), \text{ for every } \sigma > \frac{1}{2}, \tag{2.2}$$

which is [5, (14.2.5)].

By the functional equation (1.3), we obtain

$$\Big|\zeta\Big(\frac{1}{2}-\epsilon+it\Big)\Big| = O\big(|t|^{3\epsilon}\big). \tag{2.3}$$

Claim (2.1) now follows from (2.2), (2.3), and the Phragmén–Lindelöf theorem. ■

Definition 2.2 For ρ a nontrivial zero of $\zeta(s)$, let

$$R_N(\rho, s) := \mathrm{Res}_{z=\rho} \frac{N^{z-s}}{\zeta(z)(z-s)^2},$$
$$F_s(z) := \pi z^s \sum_{n=1}^{\infty} (-1)^n \frac{(2\pi)^{2n+1} z^{2n}}{(2n)! : \zeta(2n+1)(2n+s)^2}.$$

Lemma 2.3 *If* $0 < \mathrm{Re}(s) < 1$, *then*

$$V_N(s) = \frac{1}{\zeta(s)}\Big(1 - \frac{1}{\log N}\frac{\zeta'}{\zeta}(s)\Big) + \frac{1}{\log N}\sum_{\rho} R_N(\rho, s) + \frac{1}{\log N} F_s\Big(\frac{1}{N}\Big),$$

where the sum is over all distinct nontrivial zeros of $\zeta(s)$.

Proof This is [3, Lemma 2]. ∎

Lemma 2.4 *Let* $\epsilon > 0$. *Under the assumptions of Theorem 1.1, we have*

$$\sum_{\rho, \mathrm{Re}(\rho)=\frac{1}{2}} R_N(\rho, s) \ll N^{\mp\epsilon} |s|^{\frac{3}{4}-\frac{\delta}{2}+\epsilon}.$$

Proof The proof is identical to the proof of [3, Lemma 3]. There the summation condition $\mathrm{Re}(s) = 1/2$ is not needed, since the Riemann hypothesis is assumed. ∎

Lemma 2.5

$$N^{\pm\epsilon} \sum_{\rho, |\rho|=\frac{1}{2}} R_N(\rho, s) \ll \sum_{\substack{|\rho-s|<\frac{|\rho|}{2} \\ \mathrm{Re}(\rho)=\frac{1}{2}}} \frac{1}{|\zeta'(\rho)||\rho-s|^2} + 1.$$

Proof This is [3, (5)]. ∎

Definition 2.6 We set

$$\Sigma^{(1)}(N, s) := \frac{1}{\log N} \sum_{\rho:\mathrm{Re}(\rho)=\frac{1}{2}} R_N(\rho, s),$$
$$\Sigma^{(2)}(N, s) := \frac{1}{\log N} \sum_{\rho\in\{\sigma_0 \pm i\gamma_0\}} R_N(\rho, s)$$

3 Proof of Theorem 1.1

We closely follow the proof of [3, Theorem 1]. We have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} \Big|1 - \zeta V_N\Big(\frac{1}{2} + it\Big)\Big|^2 \frac{dt}{\frac{1}{4} + t^2} \\
&\quad = \frac{1}{2\pi i}\int_{(\frac{1}{2})} (1 - \zeta V_N(s))(1 - \zeta V_N(1-s)) \frac{ds}{s(1-s)} \\
&\quad = \frac{1}{2\pi i}\int_{(\frac{1}{2}-\epsilon)} (1 - \zeta V_N(s))(1 - \zeta V_N(1-s)) \frac{ds}{s(1-s)}.
\end{aligned}$$

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By Lemma 2.3 and Definition 2.6, this is

$$\text{(3.1)}\quad \frac{1}{\log^2 N}\frac{1}{2\pi i}\int_{(\frac{1}{2}-\epsilon)}\Big(\frac{\zeta'}{\zeta^2}(s)-\Sigma^{(1)}(N,s)-\Sigma^{(2)}(N,s)-F_s\Big(\frac{1}{N}\Big)\Big)\times \Big(\frac{\zeta'}{\zeta^2}(1-s)-\Sigma^{(1)}(N,1-s)-\Sigma^{(2)}(N,1-s)-F_{1-s}\Big(\frac{1}{N}\Big)\Big)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}ds.$$

We now expand the product in (3.1) and separately estimate the products that do not contain terms $\Sigma^{(2)}$ and the products consisting of a term $\Sigma^{(2)}$ and another term.

By Definition 2.6, we obtain

$$\begin{aligned}&\frac{1}{(\log N)^2}\frac{1}{2\pi i}\int_{(\frac{1}{2}-\epsilon)}\Sigma^{(2)}(N,s)\Sigma^{(2)}(N,1-s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds\\&=\sum_{\epsilon_1,\epsilon_2\in\{-1,1\}}\frac{1}{(\log N)^2}\frac{1}{2\pi i}\\&\quad\times\int_{(\frac{1}{2}-\epsilon)}\frac{N^{\sigma_0+\epsilon_1 i\gamma_0-s}N^{\sigma_0+\epsilon_2 i\gamma_0+s-1}}{\zeta'(\sigma_0+\epsilon_1 i\gamma_0)\zeta'(\sigma_0+\epsilon_2 i\gamma_0)}\frac{\zeta(s)\zeta(1-s)}{(\sigma_0+\epsilon_1 i\gamma_0-s)^2(\sigma_0+\epsilon_2 i\gamma_0-s)^2}\frac{ds}{s(1-s)}.\end{aligned}$$

After factoring out the terms $N^{2\sigma_0-1+(\epsilon_1+\epsilon_2)i\gamma_0}$, we obtain integrals that do not depend on N. The path of integration can be shifted to $\mathrm{Re}(s)=1/2$. We obtain the terms

$$A^*N^{2\sigma_0-1+2i\gamma_0}+B^*N^{2\sigma_0-1}+C^*N^{2\sigma_0-1-2i\gamma_0}$$

with $A^*, B^*, C^*\in\mathbb{C}$. Since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big|1-\zeta V_N\Big(\frac{1}{2}+it\Big)\Big|^2\frac{dt}{\frac{1}{4}+t^2}$$

is real, we obtain the main term of Theorem 1.1. We now proceed to the estimate of the other terms.

We closely follow [3]. It follows from Lemmas 2.1, 2.4, and 2.5 that

$$\begin{aligned}&\frac{1}{(\log N)^2}\frac{1}{2\pi i}\int_{(\frac{1}{2}-\epsilon)}\sum_{\rho_1,\rho_2}R_N(\rho_1,s)R_N(\rho_2,1-s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds\ll\\&\qquad\frac{1}{(\log N)^2}\int_{(\frac{1}{2}-\epsilon)}\sum_{|\rho-s|<\frac{|\rho|}{2}}\frac{1}{|\zeta'(\rho)||\rho-s|^2}\frac{|ds|}{|s|^{\frac{5}{4}+\frac{\delta}{2}-5\epsilon}}+O\Big(\frac{1}{\log^2 N}\Big).\end{aligned}$$

Now by Lemma 2.4 and the trivial estimate

$$F_s\Big(\frac{1}{N}\Big)=O(N^{-5/2}),$$

all the other terms in (3.1) not containing factors $\Sigma^{(2)}$ are trivially $O(1/\log^2 N)$, apart from

$$-\frac{1}{(\log N)^2}\frac{1}{2\pi i}\int_{(\frac{1}{2}-\epsilon)}\frac{\zeta'}{\zeta}(1-s)\Sigma^{(1)}(N,s)\frac{\zeta(s)\zeta(1-s)}{s(1-s)}\,ds$$
$$= \log N - \frac{1}{2}\frac{\zeta''}{\zeta'}(\rho) + \frac{\chi'}{\chi}(\rho) + \frac{1-2\rho}{|\rho|^2}$$
$$= \frac{\log N}{|\rho|^2} + O\Big(\frac{1}{|\rho|^{2-\epsilon}|\zeta'(\rho)|} + \frac{1}{|\rho|^2}\Big),$$

where we set

$$\chi(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)$$

and use the bound

$$\zeta''\Big(\frac{1}{2}+it\Big) \ll |t|^{\epsilon},$$

which follows from Lemma 2.1 by the well-known estimate for the derivatives of a holomorphic function. By moving the line of integration to $\mathrm{Re}(s) = \frac{1}{2}+\epsilon$, we get that the contribution from the products not containing $\Sigma^{(2)}$ is

$$\frac{1}{\log N}\sum_{\rho:\mathrm{Re}(\rho)=\frac{1}{2}}\frac{1}{|\rho|^2} + O\Big(\frac{1}{\log^2 N}\Big).$$

We now come to the products that contain the factor $\Sigma^{(2)}(N,s)$.

They can be handled by adding the factor $N^{\sigma_0-\frac{1}{2}+\epsilon}$ stemming from N^{z-s} in Definition 2.2. These estimates yield the error-term in Theorem 1.1. ∎

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