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ON THE CONVERGENCE OF DISCRETE PROCESSES WITH MULTIPLE INDEPENDENT VARIABLES

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Abstract

We discuss discrete stochastic processes with two independent variables: one is the standard symmetric random walk, and the other is the Poisson process. Convergence of discrete stochastic processes is analysed, such that the symmetric random walk tends to the standard Brownian motion. We show that a discrete analogue of Ito's formula converges to the corresponding continuous formula.

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1. Introduction

The theory and method of discrete stochastic processes are important subjects for researchers from both a theoretical and a practical point of view. For example, Cox et al. [2] presented a well-known binomial model which provided a powerful discrete method for the analysis and computation of option pricing. We also note that the random walk gives a very good approximation to Brownian motion in general.

Here, we deal with the convergence of discrete stochastic processes with two independent variables. One random process is the standard symmetric random walk and the other is the Poisson process. The convergence is investigated, such that the symmetric random walk tends to the standard Brownian motion. Although the relation between the random walk and Brownian motion has been widely studied (see, for instance, [3, 7]), the case of two or multiple independent random processes is not so popular, despite its importance in mathematical finance (see [6], for example).

We prove that a discrete analogue of Ito's formula converges to the corresponding continuous formula. A discrete analogue of this formula has been investigated by several authors [4, 8]. On the other hand, for the latter case, the continuous one,

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the exact formula is somewhat complicated [1, 9], since the process involves a jump discontinuity. Our findings will shed light on the relation between these formulas from a different perspective, which seems to be new in the literature. One of the implications of our result is that our discrete process gives a good approximation for the continuous process.

The rest of the paper is organized as follows. We recall some basic tools such as our discrete processes and a discrete analogue of Ito's formula in Section 2. Section 3 provides our main result on convergence of discrete processes. A sketch of its proof is given in Section 4. The paper concludes with a discussion in Section 5.

2. A discrete analogue of Ito's formula

We begin by recalling our basic tools: a discrete analogue of Ito's formula and its applications. For more details we refer to the literature [3, 4, 8] and the references cited therein.

Let $t = 0, 1, 2, \ldots$ denote discrete time points and let $\{B_t\}_{t=0,1,2,\ldots}$ with $B_0 = 0$ be the one-dimensional symmetric random walk,

$$B_t = \sum_{n=1}^{t} Z_n, \tag{2.1}$$

where $\{Z_n\}_{n=1,2,\ldots}$ are independent and identically distributed (i.i.d.) random variables such that

$$P(Z_n = +1) = P(Z_n = -1) = \tfrac{1}{2}, \quad n = 1, 2, \ldots. \tag{2.2}$$

The process $\{B_t\}_{t=0,1,2,\ldots}$ may be regarded as a discrete version of the one-dimensional standard Brownian motion.

Another stochastic process we consider is the Poisson process. Let $\{N_t\}_{t=0,1,2,\ldots}$ with $N_0 = 0$ be such that

$$N_t = \sum_{n=1}^{t} D_n, \tag{2.3}$$

where $\{D_n\}_{n=1,2,\ldots}$ denote i.i.d. random variables with

$$D_n = \begin{cases} 1 & \text{with probability } \lambda, \\ 0 & \text{with probability } 1 - \lambda, \end{cases} \quad n = 1, 2, \ldots, \tag{2.4}$$

for some $0 < \lambda < 1$.

Next, we introduce a discrete price process $\{X_t\}_{t=0,1,2,\ldots}$. This is our basic underlying process, for which a discrete analogue of Ito's formula is formulated. Let $\{B_t\}_{t=0,1,2,\ldots}$ and $\{N_t\}_{t=0,1,2,\ldots}$ be defined by (2.1) and (2.3), respectively, and assumed to be independent. Then our price process $\{X_t\}_{t=0,1,2,\ldots}$ is governed by the stochastic difference equation,

$$X_{t+1} - X_t = \mu + \sigma(B_{t+1} - B_t) + \alpha(N_{t+1} - N_t),$$

where, just for simplicity, μ, σ, and α are given constants. The assumption that μ, σ, and α are constants can be generalized, so that they are allowed to be predictable processes.

Yoshida and Ishimura [10] established the following theorem, which may be interpreted as a discrete analogue of Ito's formula with jump process.

THEOREM 2.1.

(a) *For any function* $f : \mathbf{Z} \to \mathbf{R}$, *we have*

$$\begin{aligned} f(X_{t+1}) - f(X_t) = {} & f(X_t + \mu) - f(X_t) + \tfrac{1}{2}\{f(X_t + \mu + \sigma) - f(X_t + \mu - \sigma)\}Z_{t+1} \\ & + \tfrac{1}{2}\{f(X_t + \mu + \sigma) - 2f(X_t + \mu) + f(X_t + \mu - \sigma)\} \\ & + \tfrac{1}{2}\{f(X_t + \mu + \sigma + \alpha) - f(X_t + \mu + \sigma) - f(X_t + \mu - \sigma + \alpha) \\ & + f(X_t + \mu - \sigma)\}Z_{t+1}D_{t+1} \\ & + \tfrac{1}{2}\{f(X_t + \mu + \sigma + \alpha) - f(X_t + \mu + \sigma) + f(X_t + \mu - \sigma + \alpha) \\ & - f(X_t + \mu - \sigma)\}D_{t+1}. \end{aligned}$$

(b) *For any* $f : \mathbf{Z} \times \mathbf{N} \to \mathbf{R}$, *we have*

$$\begin{aligned} & f(X_{t+1}, t+1) - f(X_t, t) \\ & \quad = f(X_t, t+1) - f(X_t, t) + f(X_t + \mu, t+1) - f(X_t, t+1) \\ & \qquad + \tfrac{1}{2}\{f(X_t + \mu + \sigma, t+1) - f(X_t + \mu - \sigma, t+1)\}Z_{t+1} \\ & \qquad + \tfrac{1}{2}(f(X_t + \mu + \sigma, t+1) - 2f(X_t + \mu, t+1) + f(X_t + \mu - \sigma, t+1)) \\ & \qquad + \tfrac{1}{2}(f(X_t + \mu + \sigma + \alpha, t+1) - f(X_t + \mu + \sigma, t+1) \\ & \qquad - f(X_t + \mu - \sigma + \alpha, t+1) + f(X_t + \mu - \sigma, t+1))Z_{t+1}D_{t+1} \\ & \qquad + \tfrac{1}{2}(f(X_t + \mu + \sigma + \alpha, t+1) - f(X_t + \mu + \sigma, t+1) \\ & \qquad + f(X_t + \mu - \sigma + \alpha, t+1) - f(X_t + \mu - \sigma, t+1))D_{t+1}. \end{aligned}$$

The proof is elementary in the sense that we just check both sides of the equation for each possible case.

3. Convergence result

We now turn our attention to the problem of convergence of discrete to continuous processes. In particular, we discuss the convergence of a discrete analogue of Ito's formula.

For fixed $T > 0$ and $N \gg 1$, we put

$$\Delta t = \frac{T}{N}, \quad t_n = \frac{n}{N}T, \quad n = 0, 1, 2, \ldots, N.$$

We note that $N \to \infty$ means $\Delta t \to 0$ and vice versa. The discrete processes we consider

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should be modified and take the following form:

$$B_{t_k} = \sum_{n=1}^{k} Z_{t_n} \quad \text{with } P(Z_{t_k} = +1) = P(Z_{t_k} = -1) = \frac{1}{2},$$

$$N_{t_k} = \sum_{n=1}^{k} D_{t_n} \quad \text{with } \begin{cases} P(D_{t_n} = 1) = \lambda \Delta t, \\ P(D_{t_n} = 0) = 1 - \lambda \Delta t, \end{cases}$$

where $k = 1, 2, \ldots, N$, and we keep the same notation as in (2.1)–(2.4).

The price process is similarly modified to

$$X_{t_{n+1}} - X_{t_n} = \mu \Delta t + \sigma \sqrt{\Delta t}(B_{t_{n+1}} - B_{t_n}) + \alpha(N_{t_{n+1}} - N_{t_n}). \tag{3.1}$$

The discrete analogue of Ito's formula with respect to equation (3.1), which corresponds to Theorem 2.1(a), then becomes

$$\begin{aligned}
&f(X_T) - f(X_0) \\
&\quad = \sum_{n=0}^{N-1} (f(X_{t_{n+1}}) - f(X_{t_n})) \\
&\quad = \sum_{n=0}^{N-1} \Big[\{f(X_{t_n} + \mu \Delta t) - f(X_{t_n})\} \\
&\qquad + \frac{1}{2}\{f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t}) - f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t})\}(B_{t_{n+1}} - B_{t_n}) \\
&\qquad + \frac{1}{2}\{f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t}) - 2f(X_{t_n} + \mu \Delta t) + f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t})\} \\
&\qquad + \frac{1}{2}\{f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t}) \\
&\qquad - f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t} + \alpha) + f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t})\}(B_{t_{n+1}} - B_{t_n})(N_{t_{n+1}} - N_{t_n}) \\
&\qquad + \frac{1}{2}\{f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu \Delta t + \sigma \sqrt{\Delta t}) \\
&\qquad + f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu \Delta t - \sigma \sqrt{\Delta t})\}(N_{t_{n+1}} - N_{t_n}) \Big].
\end{aligned} \tag{3.2}$$

In the next section, we estimate the right-hand side of (3.2) term by term.

Now the main result of this paper is the following, where we examine the case when $N \to \infty$.

THEOREM 3.1. *As $N \to \infty$, the right-hand side of (3.2) converges in the sense of distribution, and we have the limit*

$$f(X_T) - f(X_0) = \int_0^T \mu f'(X_t)\, dt + \int_0^T \sigma f'(X_t)\, dW_t + \int_0^T \frac{\sigma^2}{2} f''(X_t)\, dt + \sum_{0 \le t \le T} (f(X_t) - f(X_{t\_})),$$

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for any smooth function $f : \mathbf{R} \to \mathbf{R}$, where $\{W_t\}_{t\geq 0}$ denotes the standard Brownian motion.

This theorem depicts Ito's formula with jumps [10].

4. Proof of Theorem 3.1

Here we give a sketch of the proof of Theorem 3.1. As $\Delta t \to 0$ we infer that

$$B_{t_{n+1}} - B_{t_n} = B_{t_n+\Delta t} - B_{t_n} \sim \sqrt{\Delta t}\, dW_t,$$
$$N_{t_{n+1}} - N_{t_n} = N_{t_n+\Delta t} - N_{t_n} \sim \tilde{N}_{t_n+\Delta t} - \tilde{N}_{t_n},$$

where $\{W_t\}_{t\geq 0}$ denotes the standard Brownian motion, and $\{\tilde{N}\}_{t\geq 0}$ is the homogeneous Poisson process with intensity λ.

We treat the right-hand side of (3.2) term by term. Note that the following convergences are quite standard:

$$\mathrm{I} := \sum_{n=0}^{N-1}\{f(X_{t_n} + \mu\Delta t) - f(X_{t_n})\} \to \int_0^T \mu f'(X_t)\, dt,$$

$$\mathrm{II} := \frac{1}{2}\sum_{n=0}^{N-1}\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}) - f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}(B_{t_{n+1}} - B_{t_n})$$
$$\to \int_0^T \sigma f'(X_t)\, dW_t,$$

$$\mathrm{III} := \frac{1}{2}\sum_{n=0}^{N-1}\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}) - 2f(X_{t_n} + \mu\Delta t) + f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}$$
$$\to \int_0^T \frac{\sigma^2}{2} f''(X_t)\, dt.$$

It thus suffices to show that

$$\mathrm{IV} := \frac{1}{2}\sum_{n=0}^{N-1}\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t})$$
$$- f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t} + \alpha) + f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}(B_{t_{n+1}} - B_{t_n})(N_{t_{n+1}} - N_{t_n})$$
$$\to 0,$$

and

$$\mathrm{V} := \frac{1}{2}\sum_{n=0}^{N-1}\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t})$$
$$+ f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}(N_{t_{n+1}} - N_{t_n})$$
$$\to \sum_{0\leq t\leq T}\{f(X_t) - f(X_{t-})\}.$$

To estimate IV, we infer that

$$
\begin{aligned}
&E[|\mathrm{IV}|] \\
&\quad \le \frac{1}{2}\sum_{n=0}^{N-1} E[|\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}) \\
&\qquad - f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t} + \alpha) + f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}(B_{t_{n+1}} - B_{t_n})(N_{t_{n+1}} - N_{t_n})|] \\
&\quad \le \frac{1}{2}\sum_{n=0}^{N-1} E[|\{f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t} + \alpha) - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}) \\
&\qquad - f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t} + \alpha) + f(X_{t_n} + \mu\Delta t - \sigma\sqrt{\Delta t})\}|]\lambda\Delta t \\
&\quad = \frac{1}{2}\sum_{n=0}^{N-1} 2\sigma\sqrt{\Delta t}E[|f'(X_{t_n} + \mu\Delta t + 2\theta_n^{(1)}\sigma\sqrt{\Delta t} + \alpha) \\
&\qquad - f'(X_{t_n} + \mu\Delta t - 2\theta_n^{(2)}\sigma\sqrt{\Delta t} + \alpha)|] \\
&\quad \to 0,
\end{aligned}
$$

where $0 < \theta_n^{(1)}, \theta_n^{(2)} < 1$ with $n = 0, 1, 2, \dots, N-1$.

On the other hand, we calculate V as follows:

$$
\begin{aligned}
\mathrm{V} &= \frac{1}{2}\sum_{n=0}^{N-1} E[f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{t_{n+1}} + \alpha) \\
&\quad - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{t_{n+1}})|\mathcal{F}_{t_n}](N_{t_{n+1}} - N_{t_n}) \\
&= \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} E[f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{t_{n+1}} + \alpha) - f(X_{t_n} + \mu\Delta t + \sigma\sqrt{\Delta t}Z_{t_{n+1}})|\mathcal{F}_{t_n}]\,d\tilde{N}_t \\
&\to \int_0^T E[f(X_{t-} + \alpha) - f(X_{t-})|\mathcal{F}_{t-}]\,d\tilde{N}_t = \int_0^T \{f(X_{t-} + \alpha) - f(X_{t-})\}\,d\tilde{N}_t,
\end{aligned}
$$

where $\{\mathcal{F}_t\}_{t\ge 0}$ denotes the minimum augmented filtration to which $\{X_t\}$ is adapted. Invoking Yoshida [9, Theorem 6], we deduce that

$$
\int_0^T \{f(X_{t-} + \alpha) - f(X_{t-})\}\,d\tilde{N}_t = \sum_{X_t \ne X_{t-}} \{f(X_{t-} + \alpha) - f(X_{t-})\} = \sum_{0\le t\le T} \{f(X_t) - f(X_{t-})\}.
$$

The proof is therefore complete. □

5. Conclusions

We have established the convergence of a discrete analogue of Ito's formula when underlying stochastic processes involve two independent random processes. Our result gives another characterization of Ito's formula with jump processes.

The problem of the relation between the discrete and continuous settings has attracted the interest of researches in many areas. There still remain points to be

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discussed further even in our case. One example is the optimal portfolio problem. We have derived a discrete Hamilton–Jacobi–Bellman equation for the value function to characterize the extremals (see [5, 10]). The convergence of discrete to continuous versions is worth further investigation. Another example to be examined is the applicability of our method to wider classes of stochastic processes such as Lévy processes. Both are challenging problems, and will be revisited in future work.

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