EXPANSION OF THE DISTURBING FUNCTION BY FACTORIZATION

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Abstract

We present an expansion of the disturbing function for the third-body perturbations in the form

$$R = \frac{\mu_3}{a_3} \sum_{n=2}^{\infty} \sum_{m=0}^{n} W_{nm}\, \varepsilon^n [X_{nm} X_{nm}^{(3)} + Y_{nm} Y_{nm}^{(3)}] ,$$

where ε is the ratio of semi-major axes (a/a') and X_{nm} and Y_{nm} depend only on the five elements e , i , M , ω , Ω of the satellite while $X_{nm}^{(3)}$ and $Y_{nm}^{(3)}$ depend on the corresponding elements of the perturbing body. Each one of the four functions X and Y is represented as a product of two factors as opposed to three factors in our previous publication of this subject (Broucke, 1981). We develop several relations for the construction of the X and Y-series. These series are relatively short and can be computed in advance and stored on magnetic tape, for instance.

I. Introduction

The expansion of the disturbing function still remains one of the most important problems in celestial mechanics, both in planetary

O. Calame (ed.), High-Precision Earth Rotation and Earth-Moon Dynamics, 337–348.

applications and artificial satellite problems. The classically known expansions come in two categories: with the Laplace coefficients (Broucke and Smith, 1971) and with the Legendre polynomials (Cefola and Broucke, 1975).

In the present article we develop a new form of expansion of the third-body disturbing function which is in the category of the Legendre expansions and which is especially efficient and easy to carry out to a high order with the use of a package of Poisson series programs for algebraic operations on a computer.

The expansion is of the following general form:

$$R = \frac{\mu_3}{a_3} \sum_{n=2}^{\infty} \sum_{m=0}^{n} W_{nm} \, \varepsilon^n [X_{nm} X_{nm}^{(3)} + Y_{nm} Y_{nm}^{(3)}] ,$$

with the remarkable property that it is separated in factors depending each only on one body. In other words X_{nm} and Y_{nm} depend only on the elements (e, i, M, ω, Ω) while $X_{nm}^{(3)}$ and $Y_{nm}^{(3)}$ depend only on the elements (e_3, i_3, M_3, ω_3, Ω_3) of the third-body (perturbing body). These factors are therefore Poisson series in five variables only. They are relatively short series, in comparison with the resulting series R which is an expansion in ten variables. Here ε represents the ratio of semi-major axes (a/a').

In our previous publication (Broucke, 1981), the X and Y-functions were presented as products of <u>three</u> factors each. We show here that they can also be represented as products of <u>two</u> factors only. This results in a much more efficient computer generation of the corresponding series in literal form.

https://doi.org/10.1017/S0252921100002761 Published online by Cambridge University Press

2. Expansion of the Third-Body Disturbing Function

The classical expression that is known for the disturbing function is as follows:

$$R = \mu_3 \left[\frac{1}{\Delta} - \frac{\vec{r} \cdot \vec{r}_3}{r_3^3}\right] .$$

Here, μ_3 is the constant $Gm_3 = k^2 m_3$ related to the mass m_3 of the third-body or disturbing body. Also, Δ is the distance between the satellite and the disturbing body:

$$\Delta^2 = r^2 + r_3^2 - 2r\, r_3 \cos \psi = |\vec{r}_3 - \vec{r}|^2 .$$

The vectors $\vec{r}_3$ and $\vec{r}$ are the position vectors of the third-body and the satellite relative to the central body (the earth in most artificial satellite applications), ψ being the angle between these two vectors, so that we may write

$$\vec{r} \cdot \vec{r}_3 = r\, r_3 \cos \psi .$$

Therefore, the disturbing function can be written in a slightly different form:

$$R = \mu_3 \left[\frac{1}{\Delta} - \frac{r \cos \psi}{r_3^2}\right] = \frac{\mu_3}{r_3} \left[\frac{r_3}{\Delta} - \frac{r}{r_3} \cos \psi\right] .$$

In Broucke (1981), it has been shown with the use of the Legendre addition theorem that the disturbing function can be represented in the form

$$R = \frac{\mu_3}{a_3} \sum_{n=2}^{\infty} \sum_{m=0}^{n} W_{nm}\, \varepsilon^n [X_{nm} X_{nm}^{(3)} + Y_{nm} Y_{nm}^{(3)}] ,$$

https://doi.org/10.1017/S0252921100002761 Published online by Cambridge University Press

where the X and Y-functions, called third-body harmonics, have been defined:

$$X_{nm} = \left(\frac{r}{a}\right)^n Q_{nm}(\sin\ \delta)\ C_m\ (\alpha, \delta) \quad ,$$

$$X_{nm}^{(3)} = \left(\frac{a_3}{r_3}\right)^{n+1} Q_{nm}(\sin\ \delta_3)\ C_m\ (\alpha_3, \delta_3) \quad .$$

The Y-factors are similar, except that they use the S_m-functions. This is what we call the separated form of the disturbing function: all the terms have been decomposed in two factors, each factor depending on only one of the two bodies, either the satellite or the perturbing body. These factors can be expanded separately in Poisson series. These series are relatively short, considering that they contain only a small number of orbit elements and angles.

The Q_{nm}-functions are essentially associated Legendre polynomials, depending on the declination δ (or latitude) only. The C_m and S_m-functions depend only on two angles α and δ (right ascension and declination). They are computed with very simple recurrence relations. The W_{nm}- factors are constants.

3. Expression of the Third-Body Harmonics in the Eccentric Anomaly

We consider again the third-body harmonics here:

$$X_{nm} = \left(\frac{r}{a}\right)^n Q_{nm}(\sin\ \delta)\ C_m\ (\alpha, \delta) \quad ,$$

$$Y_{nm} = \left(\frac{r}{a}\right)^n Q_{nm}(\sin\ \delta)\ S_m\ (\alpha, \delta) \quad .$$

where $n = 1, 2, \ldots, \infty$ and where $0 \leq m \leq n$.

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We will show that they can be written as finite polynomials in the rectangular coordinates x, y, z and the radius-vector r, without negative powers. First recall that C_m and S_m are homogeneous polynomials of degree m in x/r and y/r. Consequently, the factors $(r/a)^n C_m$ and $(r/a)^n S_m$ are homogeneous polynomials of degree $n-m$ in r.

On the other hand, the polynomials Q_{nm} are m^{th} derivatives of Legendre polynomials P_n. They are thus of degree $n-m$ in the argument $\sin\delta = z/r$. Therefore, the third-body harmonics X_{nm} and Y_{nm} contain only non-negative powers, 0 to $n-m$, of the radius vector r. They are also homogeneous polynomials of degree n in the set of four variables x, y, z, r. They are explicitly given for all values up to $n = m = 3$ in Table 1, except for the factor a^{-n} which is omitted. In other words, the X_{nm} and Y_{nm} can both be written in the form $P(x, y, z, r)/a^n$, where P represents a homogeneous polynomial of degree n. This also shows that the third-body harmonic functions can be written as finite Poisson series in the four polynomial variables e, $\sqrt{1-e^2}$, $\sin i$, $\cos i$, and three angles E, ω, Ω. Here, E represents the eccentric anomaly. This is a consequence of the equations

$$r = a(1 - e\cos E)$$

$$x,y,z = a\,\vec{P}(\cos E - e) + a\sqrt{1-e^2}\,\vec{Q}\sin E ,$$

where the unit vectors $\vec{P}$ and $\vec{Q}$ are closed-form expressions in two polynomial variables $\sin i$, $\cos i$, and two angles ω and Ω.

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This means that if the above formulas are used to substitute x , y , z , r in the X_{nm} and Y_{nm} functions, we obtain finite Fourier series in the eccentric anomaly E as well as the two other angles, the coefficients of these Fourier series being polynomials in four variables e , $\sqrt{1-e^2}$, $\sin i$ and $\cos i$.

As a general conclusion, we can now make the following statements about the third-body harmonic functions X_{nm} , Y_{nm} :

1. They can be expressed as finite homogeneous polynomials in x , y , z , r .
2. They can be expressed as finite Poisson series in e , $\sqrt{1-e^2}$, $\sin i$, $\cos i$ and the angles E , ω , Ω .
3. Their average, with respect to the satellite period, is a finite expression in six variables e , $\sqrt{1-e^2}$, $\sin i$, $\cos i$, ω and Ω .

A similar analysis in terms of the true anomaly shows that the $X^{(3)}_{nm}$ and $Y^{(3)}_{nm}$-functions can also be expressed in closed-form polynomials, in x , y , z and $1/r$.

4. The X_{nm} and Y_{nm}-Functions as Products of Two Factors

In the previous section, it was shown that X_{nm} and Y_{nm} are closed-form polynomials in the four variables x , y , z and r . We will here take maximum advantage of this property in order to write the functions X_{nm} and Y_{nm} as a product of two rather than three factors.

We will first show that, from the known explicit formulas for the Legendre polynomials P_n , we can obtain a simple explicit formula for the Q_{nm}-polynomials. First of all, the explicit expression for the Legendre polynomial is, according to the Rodriguez formula:

$$P_n(t) = \frac{1}{2^n n!} D^n (t^2 - 1)^n$$

$$= \frac{1}{2^n n!} D^n [\sum_0^n \binom{n}{k} (-1)^k t^{2n-2k}]$$

$$= \frac{1}{2^n n!} \sum_0^{[n/2]} (-1)^k \binom{n}{k} \frac{(2n-2k)!}{(n-2k)!} t^{n-2k}$$

$$= \frac{1}{2^n} \sum_0^{[n/2]} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} t^{n-2k} .$$

Here $[n/2]$ represents the integer part of $n/2$.

Now, to get the expression for $Q_{nm}(t)$ we have to differentiate m times:

$$Q_{nm}(t) = 2^{-n} \sum_0^L (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!} t^{n-m-2k} .$$

$$= 2^{-n} \sum_0^L Q_{nmk} t^{n-m-2k} ,$$

where $L = [\frac{n-m}{2}]$ = integer part of $(n-m)/2$, and

$$Q_{nmk} = (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-m-2k)!} .$$

To simplify the writings, we define a new homogeneous polynomial in two variables:

$$K_{nm}(t, u) = 2^{-n} \sum_0^{[\frac{n-m}{2}]} Q_{nmk} t^{n-m-2k} u^{2k} .$$

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This allows us to make a reduction of the product:

$$\left(\frac{r}{a}\right)^{n-m} Q_{nm} \left(\frac{z}{r}\right) .$$

This quantity is equal to

$$\left(\frac{r}{a}\right)^{n-m} \sum_{0}^{[\frac{n-m}{2}]} 2^{-n} Q_{nmk} \left(\frac{z}{r}\right)^{n-m-2k}$$

$$= 2^{-n} \sum_{0}^{[\frac{n-m}{2}]} Q_{nmk} \left(\frac{z}{a}\right)^{n-m-2k} \left(\frac{r}{a}\right)^{2k}$$

$$= K_{nm} \left(\frac{z}{a}, \frac{r}{a}\right) .$$

As a conclusion, we see that K_{nm} is a homogeneous polynomial, of degree $n-m$, having non-negative powers only; the powers of the second argument are even.

We also define new functions C_m^* and S_m^* to replace the previously defined functions C_m and S_m. The C_m and S_m are homogeneous polynomials in $\frac{x}{r}$, $\frac{y}{r}$, of degree m. However, the following are homogeneous polynomials in $\frac{x}{a}$ and $\frac{y}{a}$:

$$C_m^* \left(\frac{x}{a}, \frac{y}{a}\right) = \left(\frac{r}{a}\right)^m C_m (\alpha, \delta) ,$$

$$S_m^* \left(\frac{x}{a}, \frac{y}{a}\right) = \left(\frac{r}{a}\right)^m S_m (\alpha, \delta) .$$

This finally allows us to write X_{nm} and Y_{nm} in the form of a product of two factors:

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$$X_{nm} = (\frac{r}{a})^n Q_{nm} (\sin \delta) C_m (\alpha, \delta)$$

$$= (\frac{r}{a})^{n-m} Q_{nm} (\frac{z}{r}) (\frac{r}{a})^m C_m(\alpha, \delta)$$

$$= K_{nm} (\frac{z}{a}, \frac{r}{a}) C_m^* (\frac{x}{a}, \frac{y}{a}) .$$

We find in the same way:

$$Y_{nm} = K_{nm} (\frac{z}{a}, \frac{r}{a}) S_m^* (\frac{x}{a}, \frac{y}{a}) .$$

The C_m^* and S_m^* are homogeneous, of degree m, with non-negative powers only. For C_m^* the second argument has even powers only, while for S_m^*, the first argument has even powers only.

The use of the new functions allows a more simple construction of the X_{nm} and Y_{nm}-functions, in the form of a product of 2 rather than 3 factors. Also the closed form expressions in terms of the eccentric anomaly are computed in a faster way and the corresponding averaging operations are performed in a more efficient way.

Table 2 below gives some of the most simple functions K_{nm}, with n, m up to 4. The factors a in the denominators have been neglected for simplicity.

5. Conclusions

We are at the moment using the present formulation of the third-body effects for the purpose of averaging and estimating the long-term effects of the third-body perturbations. It should be recognized

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Table 1. The X and Y-Functions

$X_{10} = z$ $X_{20} = -\frac{1}{2}r^2 + \frac{3}{2}z^2$ $X_{30} = -\frac{3}{2}zr^2 + \frac{5}{2}z^3$	$X_{11} = x$ $X_{21} = 3xz$ $X_{31} = -\frac{3}{2}xr^2 + \frac{15}{2}xz^2$	$X_{33} = -45xy^2 + 15x^3$ $X_{22} = -3y^2 + 3x^2$ $X_{32} = -15y^2z + 15x^2z$
	$Y_{11} = y$ $Y_{21} = 3yz$ $Y_{31} = -\frac{3}{2}yr^2 + \frac{15}{2}yz^2$	$Y_{33} = -15y^3 + 45x^2y$ $Y_{22} = 6xy$ $Y_{32} = 30xyz$

Table 2. The Factors K_{nm}

$K_{11} = 1$ $K_{21} = 3z$ $K_{22} = 3$	$K_{31} = \frac{15}{2}z^2 - \frac{3}{2}r^2$ $K_{32} = 15z$ $K_{33} = 15$	$K_{41} = \frac{35}{2}z^3 - \frac{15}{2}r^2z$ $K_{42} = \frac{105}{2}z^2 - \frac{15}{2}r^2$ $K_{43} = 105z$ $K_{44} = 105$

that the difficulty of slow convergence in ε of all the Legendre type expansions is still present here. Therefore, it would be useful to investigate a similar factorization process for the Laplace-type expansions of the third-body disturbing function or even for some of the more modern expansions designed especially for intersecting orbits (Petrovskaya, 1970) (Yuasa and Hori, 1979).

Acknowledgements

The present work was performed with the support of the U. S. Air Force (Space Division, Los Angeles) as well as the Aerospace Corporation in El Segundo, California. Several exchanges of ideas with Bruce Baxter are especially appreciated.

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https://doi.org/10.1017/S0252921100002761 Published online by Cambridge University Press