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Size, Order, and Connected Domination

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Abstract. We give a sharp upper bound on the size of a triangle-free graph of a given order and connected domination. Our bound, apart from strengthening an old classical theorem of Mantel and of Turán improves on a theorem of Sanchis. Further, as corollaries, we settle a long standing conjecture of Graffiti on the leaf number and local independence for triangle-free graphs and answer a question of Griggs, Kleitman, and Shastri on a lower bound of the leaf number in triangle-free graphs.

1 Introduction

Let $G = (V, E)$ be a connected graph of order n and size m. We say that G is *triangle free* if it does not contain C_3, *i.e.,* the cycle on three vertices, as a subgraph. A *dominating set* of G is a set $S \subset V$ of vertices of G such that every vertex $v \in V$ is either in S or adjacent to a vertex of S. The *connected domination number* $\gamma_c(G)$ of G is the minimum order of a connected dominating set of G. On the other hand, the *leaf number* $L(G)$ of G is defined as the maximum number of leaf vertices contained in a spanning tree of G, a leaf vertex being a vertex of degree 1 in G. The leaf and the connected domination number, whose applications in the optimization of centralized terminal networks are legion [3], are much studied graph invariants that determine each other (see, for example, [2]):

$$L(G) = n - \gamma_c(G). \tag{1.1}$$

A subset S of V is *independent* if no two vertices in S are adjacent. The *independence number* of G is defined as the cardinality of a largest independent set in G. The *local independence* $\alpha(v)$ of a vertex v is the independence number of the subgraph induced by its neighborhood. The *average local independence* $\overline{\alpha}(G)$ of G is defined as $\frac{1}{n}\sum_{x\in V}\alpha(x)$.

Fajtlowicz and Waller's computer program, Graffiti (see, for example, [2]), which sorts through various graphs and looks for simple relations among parameters, posed the following conjecture, which, for a human mathematician, relates two seemingly unrelated quantities.

Conjecture 1.1 *Let G be a connected graph. Then* $L(G) \geq 2(\overline{\alpha}(G) - 1)$.

To date, no attempt on this long standing open conjecture of Graffiti has been reported. In [4], Griggs, Kleitman, and Shastri, concerned about lower bounds on the

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leaf number in triangle-free graphs, remarked "it could be that triangle-free graphs contain significantly more leaves".

In this note, we are particularly interested, among other things, in the maximum number of links of a network in which the connected domination number of the underlying graph is limited. Several upper and lower bounds on the size of a graph in terms of other graph parameters have been investigated. For instance, as early as 1907 Mantel [5], and subsequently Turán [9] in 1941, showed that the size m of a general triangle-free graph of order n is at most

$$m \le \left\lfloor \frac{n^2}{4} \right\rfloor, \tag{1.2}$$

with equality holding if and only if G is the Turán graph $T_2(n)$, *i.e.,* the complete bipartite graph whose classes are as nearly equal as possible. An upper bound on the size in terms of order and diameter was determined by Ore [7] in 1968, while Vizing [10] gave an upper bound in terms of order and radius. Recently, Dankelmann, and Volkmann [1] reported lower bounds in terms of order, radius, and minimum degree. In 2000, Sanchis [8] proved the bound

$$m \le \frac{(n-\gamma_c)^2}{2} + O(n) \tag{1.3}$$

for a general graph G of order n, size m, and connected domination number γ_c.

In this note, we present a strengthening of the bound (1.2) if connected domination is prescribed. Our result also improves on the bound (1.3) by Sanchis for triangle-free graphs. As corollaries, we settle Conjecture 1.1 for this class of graphs and confirm Griggs, Kleitman, and Shastri's speculation [4] that triangle-free graphs contain significantly more leaves.

We denote the degree of a vertex u in G by deg u. If H is a subgraph of G, we write $H \le G$. The following simple observation, which we use in this work, was proved in [6].

Lemma 1.2 *Let G be a connected graph and $T' \le G$ a tree. Then $L(G) \ge L(T')$.*

2 Results

We begin by reporting on a strengthening of the theorem by Mantel [5], and by Turán [9] if connected domination is prescribed.

Theorem 2.1 *Let G be a connected triangle-free graph of order n, size m and connected domination number γ_c. Then*

$$m \le \frac{(n-\gamma_c)^2}{4} + n - 1,$$

and this bound is tight.

Proof Suppose, to the contrary, that there exists a counterexample G for which

$$|E(G)| > \frac{(|V(G)| - \gamma_c(G))^2}{4} + |V(G)| - 1. \tag{2.1}$$

Of all such counterexamples, choose G to have the smallest order, n, maximizing size.

Claim *Let uv be any edge in G. Then* $\deg u + \deg v \leq n - \gamma_c + 2$.

Proof. Let T' be the subgraph of G with vertex set all vertices in the neighbourhood of u or v and edge set all edges incident with either u or v. Since G is triangle-free, T' is a tree with $\deg u + \deg v - 2$ end vertices. By Lemma 1.2, $L(G) \geq L(T') = \deg u + \deg v - 2$. Hence from (1.1) we have

$$n - \gamma_c(G) = L(G) \geq L(T') = \deg u + \deg v - 2.$$

It follows that $\deg u + \deg v \leq n - \gamma_c + 2$, and the claim is proved.

Now let uv be any edge in G such that $G' = G - \{u, v\}$ is connected. Then G' is triangle-free, has order $n - 2$, and $\gamma_c(G) \leq \gamma_c(G')$. By our choice of G, G' is not a counterexample. It follows that

$$|E(G')| \leq \frac{((n-2) - \gamma_c(G'))^2}{4} + (n-2) - 1 \leq \frac{(n - 2 - \gamma_c)^2}{4} + n - 3.$$

Hence, in conjunction with the above claim, we have

$$\begin{aligned} m &= |E(G')| + [\deg\ u + \deg\ v] - 1 \\ &\leq \frac{(n-2-\gamma_c)^2}{4} + n - 3 + [n - \gamma_c + 2] - 1 \\ &= \frac{(n-\gamma_c)^2}{4} + n - 1, \end{aligned}$$

which is a contradiction to (2.1), and so the bound in the theorem is proved.

To see that the bound is tight, for integers n and γ_c, where $n - \gamma_c$ is even, consider the graph G_{n,γ_c} obtained by taking the path $P_{\gamma_c} = v_1, v_2, \dots, v_{\gamma_c}$, a complete bipartite graph $K_{\frac{n-\gamma_c}{2}, \frac{n-\gamma_c}{2}}$ with partite sets V_1 and V_2, and joining v_1 to every vertex in V_1 and joining v_{γ_c} to every vertex in V_2. Then G_{n,γ_c} is triangle-free, has order n, and

$$m(G_{n,\gamma_c}) = \frac{[n - \gamma_c(G_{n,\gamma_c})]^2}{4} + n - 1,$$

as desired. ■

We now settle Conjecture 1.1 for triangle-free graphs.

Corollary 2.2 *Let G be a connected triangle-free graph. Then* $L(G) \geq 2(\overline{\alpha}(G) - 1)$.

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Proof Let v be a vertex in G. Since G is triangle free, $\alpha(v) = \deg v$. It follows that $\overline{\alpha}(G) = \frac{1}{n}\sum_{x\in V}\alpha(x) = \frac{1}{n}\sum_{x\in V}\deg x = \frac{2m}{n}$, where n is the order and m is the size of G. From (1.1) and Theorem 2.1, we have, for $\gamma_c \in [2, n-2]$,

$$m \leq \frac{(n-\gamma_c)^2}{4} + n - 1 \leq \frac{n(n-\gamma_c+2)}{4} = \frac{n(L+2)}{4}.$$

Hence, for $2 \leq \gamma_c \leq n-2$, we have

$$L \geq \frac{4m}{n} - 2 = 2\overline{\alpha}(G) - 2,$$

and the corollary is proven. Using (1.1), we see that $\gamma_c \leq n-2$. Finally, if $\gamma_c = 1$, then since G is triangle-free, it is a star. Thus, $L = n-1$ and $\overline{\alpha}(G) = 2 - \frac{2}{n}$. An easy calculation shows that the corollary holds. ■

Finally, we confirm Griggs, Kleitman, and Shastri's speculation [4] that triangle-free graphs contain significantly more leaves.

Corollary 2.3 *Let G be a connected triangle-free graph of order n and size m. Then*

$$L(G) \geq \frac{4m}{n} - 2.$$

Proof As in Corollary 2.2, since G is triangle free, $\overline{\alpha}(G) = \frac{2m}{n}$. Therefore, by Corollary 2.2,

$$L(G) \geq 2(\overline{\alpha}(G) - 1) = \frac{4m}{n} - 2.$$ ■

We mention that the bounds in the above corollaries are attained by the complete bipartite graph, $K_{\frac{n}{2},\frac{n}{2}}$.

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