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ASYMPTOTIC BEHAVIOUR OF THE SPECTRA OF SYSTEMS OF MAXWELL EQUATIONS IN PERIODIC COMPOSITE MEDIA WITH HIGH CONTRAST

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Abstract. We analyse the behaviour of the spectrum of the system of Maxwell equations of electromagnetism, with rapidly oscillating periodic coefficients, subject to periodic boundary conditions on a "macroscopic" domain $(0, T)^3$, $T > 0$. We consider the case where the contrast between the values of the coefficients in different parts of their periodicity cell increases as the period of oscillations η goes to zero. We show that the limit of the spectrum as $\eta \to 0$ contains the spectrum of a "homogenized" system of equations that is solved by the limits of sequences of eigenfunctions of the original problem. We investigate the behaviour of this system and demonstrate phenomena not present in the scalar theory for polarized waves.

§1. *Introduction.* The behaviour of systems (of Maxwell equations) with periodic coefficients in the regime of "high contrast" or "large coupling", that is, when the ratio between material properties of some of the constituents within the composite is large, is understood to be of special interest in applications. This is due to the improved band-gap properties of the spectra for such materials compared to the usual moderate-contrast composites. A series of recent studies have analysed asymptotic limits of scalar high-contrast problems, either in the strong L^2 sense (see [**11**, **12**]) or in the norm-resolvent L^2 sense (see [**2**]). These have resulted in sharp operator convergence estimates in the homogenization of such problems (i.e. in the limit as the period tends to zero) and have provided a link between the study of effective properties of periodic media and the behaviour of waves in such media, in particular their scattering characteristics. The studies have also highlighted the need to extend the classical compactness techniques in homogenization to cases where the symbol of the operator involved is no longer uniformly positive definite, thus leading to "degenerate" problems. The work [**6**] has opened a way to one such extension procedure, based on a "generalized Weyl decomposition", from the perspective of the strong L^2 convergence.

The set of tools developed in the literature is now poised for the treatment of vector problems with degeneracies such as the linearized elasticity equations

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and the Maxwell equations; these examples are typically invoked in the physics and applications literature, and are prototypes for wider varieties of partial differential equations. The recent work [10] has studied the spectral behaviour of periodic operators with rapidly oscillating coefficients in the context of linearized elasticity. It shows that the related spectrum exhibits the phenomenon of "partial" wave propagation, depending on the number of eigenmodes available at each given frequency. This is close in spirit to the work [7], where "partial wave propagation" was studied for a wider class of vector problems, with a general high-contrast anisotropy.

The high-contrast system of Maxwell equations poses an analytic challenge in view of the special structure of the "space of microscopic variations" (using the terminology of [6]), which consists of functions that are curl-free on the "stiff" component, in the case of a two-component composite of a "stiff" matrix and "soft" inclusions. In the work [1] the authors analysed the two-scale structure of solutions to the high-contrast system of Maxwell equations in the low-frequency limit, and derived the corresponding system of homogenized equations, by developing an appropriate compactness argument on the basis of the general theory of [6]. In the present paper we consider the associated wave propagation problem for monochromatic waves of a given frequency by constructing two-scale asymptotic series for eigenfunctions. We justify these asymptotic series by demonstrating that for each element of the spectrum of the homogenized equations there exist convergent eigenvalues and eigenfunctions for the original heterogeneous problem. Our analysis is set in the context of a "supercell" spectral problem, that is, the problem of vibrations of a square-shaped domain with periodicity conditions on the boundary (equivalently seen as a torus). The problem of the "spectral completeness" of the homogenized description in question remains open: it is not known, for the full-space problem, whether there may exist sequences of eigenvalues converging to a point outside the spectrum of the homogenized problem. This will be addressed in a future publication, using the method developed in [2].

§2. *Problem formulation and main results.* In this paper we consider Maxwell equations for a three-dimensional two-component periodic dielectric composite when the dielectric properties of the constituent materials exhibit a high degree of contrast. We assume that the reference cell $Q := [0, 1)^3$ contains an inclusion Q_0, which is an open set with sufficiently smooth boundary. We also assume that the "matrix" $Q_1 := Q \backslash \overline{Q_0}$ is simply connected Lipschitz set.

We consider a composite with high contrast in the dielectric permittivity $\epsilon_\eta = \epsilon_\eta(x/\eta)$ at points $x \in \eta(Q_1 + m)$, $m \in \mathbb{Z}^3$, and $x \in \eta(Q_0 + m)$, $m \in \mathbb{Z}^3$, namely

$$\epsilon_\eta(y) = \begin{cases} \eta^{-2}\epsilon_0(y), & y \in Q_0, \\ \epsilon_1(y), & y \in Q_1, \end{cases}$$

where $\eta \in (0, 1)$ is the period and ϵ_0, ϵ_1 are continuously differentiable Q-periodic positive-definite scalar functions.

We also assume moderate contrast in the magnetic permeability, and for simplicity of exposition we shall set $\mu \equiv 1$. We consider the open cube $\mathbb{T} := (0, T)^3$ and those values of the parameter η for which $T/\eta \in \mathbb{N}$. By rescaling the spatial variable (which can also be viewed as non-dimensionalization) we assume that $T = 1$ and that $\eta^{-1} \in \mathbb{N}$. We shall study the behaviour of the magnetic component H^η of the electromagnetic wave of frequency ω propagating through the domain $\mathbb{T}$ occupied by a dielectric material with permittivity $\epsilon_\eta(x/\eta)$. More precisely, we consider pairs $(\omega_\eta, H^\eta) \in \mathbb{R}_+ \times [H^1_\#(\mathbb{T})]^3$ satisfying the system of equations

$$\operatorname{curl}\left(\epsilon_\eta^{-1}\left(\frac{\cdot}{\eta}\right)\operatorname{curl} H^\eta\right) = \omega_\eta^2 H^\eta. \tag{2.1}$$

Notice that solutions of (2.1) are automatically solenoidal, that is, div $H^\eta = 0$.

We seek solutions to the above problem in the form of an asymptotic expansion

$$\begin{gathered} H^\eta(x) = H^0\left(x, \frac{x}{\eta}\right) + \eta H^1\left(x, \frac{x}{\eta}\right) + \eta^2 H^2\left(x, \frac{x}{\eta}\right) + \cdots, \\ \omega_\eta = \omega + \eta\omega_1 + \eta^2\omega_2 + \cdots, \end{gathered} \tag{2.2}$$

where the vector functions $H^j(x, y)$, $j = 0, 1, 2, \ldots,$ are Q-periodic in the variable y. (Note that the terms of order $O(\eta)$ and higher in the expansion for ω_η will be of no importance in what follows.) Substituting (2.2) into (2.1) and gathering the coefficients for each power of the parameter η results in a system of recurrence relations for H^j, $j = 0, 1, 2, \ldots$; see §4. In particular, the function H^0 is an eigenfunction of a limit ("homogenized") system of partial differential equations, as described in the following theorem.

THEOREM 2.1. *Consider the constant matrix*

$$A^{\text{hom}} := \int_{Q_1} \epsilon_1^{-1}(y)(\operatorname{curl} N(y) + I)\, dy,$$

where the vector function N is a solution to the "unit-cell problem"

$$\begin{gathered} \operatorname{curl}(\epsilon_1^{-1}[\operatorname{curl} N + I]) = 0 \quad \text{in } Q_1, \\ \epsilon_1^{-1}(\operatorname{curl} N + I) \times n = 0 \quad \text{on } \partial Q_0,\ N \text{ is } Q\text{-periodic}, \end{gathered} \tag{2.3}$$

in which n is the exterior normal to ∂Q_0.

Suppose that $\omega \in \mathbb{R}_+$ and $H^0(x, y) = u(x) + \nabla_y v(x, y) + z(x, y)$, where the triplet[1] *$(u, v, z) \in [H^1_{\#\text{curl}}(\mathbb{T})]^3 \times L^2(\mathbb{T}; H^2_\#(Q)) \times [L^2(\mathbb{T}; H^1_0(Q_0))]^3$ satisfies*

[1] For a cube $\mathbb{T}$, we denote by $H^1_\#(\mathbb{T})$, $H^1_{\#\text{curl}}(\mathbb{T})$, the closures of the set of $\mathbb{T}$-periodic smooth functions with respect to the norm of $H^1(\mathbb{T})$ and the norm

$$\left(\int_{\mathbb{T}} |\cdot|^2 + \int_{\mathbb{T}} |\operatorname{curl} \cdot|^2\right)^{1/2},$$

respectively. Throughout the paper we only consider real-valued functions and spaces thereof.

the system of equations

$$\operatorname{curl}_x(A^{\text{hom}}\operatorname{curl}_x u(x)) = \omega^2\Big(u(x) + \int_{Q_0} z(x,y)\,dy\Big), \quad x \in \mathbb{T}, \tag{2.4}$$

$$\operatorname{div}_y(\nabla_y v(x,y) + z(x,y)) = 0, \quad (x,y) \in \mathbb{T}\times Q, \tag{2.5}$$

$$\operatorname{curl}_y(\epsilon_0^{-1}(y)\operatorname{curl}_y z(x,y)) = \omega^2(u(x) + \nabla_y v(x,y) + z(x,y)), \quad (x,y) \in \mathbb{T}\times Q_0. \tag{2.6}$$

Then:

(1) *There exists at least one eigenfrequency ω_η for (2.1) such that*

$$|\omega_\eta - \omega| < C\eta, \tag{2.7}$$

with an η-independent constant $C > 0$.

(2) *Consider the finite-dimensional vector space*

$$X_\eta := \operatorname{span}\{H^\eta : (2.1) \text{ holds, where } \omega_\eta \text{ satisfies } (2.7)\}.$$

There exists an η-independent constant $\widehat{C} > 0$ such that

$$\inf_{H\in X_\eta}\Big\|H^0\Big(\cdot, \frac{\cdot}{\eta}\Big) - H(\cdot)\Big\|_{L^2(\mathbb{T})} < \widehat{C}\eta.$$

The matrix A^{hom} is described by solutions to certain degenerate "cell problems", as follows. Consider the spaces

$$V := \{v \in [H^1_\#(Q)]^3 : \operatorname{curl} v = 0 \text{ in } Q_1\} \tag{2.8}$$

and $V^\perp$, the orthogonal complement of V in $[H^1_\#(Q)]^3$ with respect to the equivalent $H^1(Q)$-norm

$$\|v\|_{H^1(Q)} := \Big(\Big|\int_Q v\Big|^2 + \int_Q |\nabla v|^2\Big)^{1/2},$$

associated with the inner product

$$(v,w)_{H^1(Q)} := \Big(\int_Q v\Big)\cdot\Big(\int_Q w\Big) + \int_Q \nabla v\cdot\nabla w.$$

Then

$$A^{\text{hom}}\xi = \int_Q \epsilon_1^{-1}(\operatorname{curl} N_\xi + \xi), \quad \xi \in \mathbb{R}^3, \tag{2.9}$$

where N_ξ, $\xi \in \mathbb{R}^3$, is the unique (weak) solution in $V^\perp$ to the problem (2.3), that is,

$$\int_Q \epsilon_1^{-1}(\operatorname{curl} N_\xi + \xi)\cdot\operatorname{curl}\varphi = 0 \quad \text{for all } \varphi \in V^\perp. \tag{2.10}$$

Existence and uniqueness of N_ξ are discussed in §4.

Notice that

$$A^{\text{hom}}\xi\cdot\xi=\min_{U\in[H^1_\#(Q)]^3}\int_{Q_1}\epsilon_1^{-1}(\operatorname{curl}U+\xi)\cdot(\operatorname{curl}U+\xi),\qquad \xi\in\mathbb{R}^3. \tag{2.11}$$

Indeed, for the functional

$$F_\xi(U):=\int_{Q_1}\epsilon_1^{-1}(\operatorname{curl}U+\xi)\cdot(\operatorname{curl}U+\xi),$$

we find $F_\xi(U)=F_\xi(P_{V^\perp}U)$ for all $U\in[H^1_\#(Q)]^3$, where $P_{V^\perp}$ is the orthogonal projection onto $V^\perp$. Therefore, without loss of generality, F_ξ can be minimized on $V^\perp$ for which (2.10) is the corresponding Euler–Lagrange equation.

The variational formulation (2.11) allows one to obtain a representation for the matrix $\epsilon^{\text{hom}}_{\text{stiff}}$ such that

$$\epsilon^{\text{hom}}_{\text{stiff}}\xi\cdot\xi:=\inf_{\substack{u\in H^1_\#(Q),\\ \nabla u=-\xi\text{ in }Q_0}}\int_{Q_1}\epsilon_1(\nabla u+\xi)\cdot(\nabla u+\xi),\qquad \xi\in\mathbb{R}^3, \tag{2.12}$$

which arises in the homogenization of periodic problems with stiff inclusions; see [5, §3.2].[2]

Indeed, as shown in [5, p. 101], the following representation holds:

$$(\epsilon^{\text{hom}}_{\text{stiff}})^{-1}\xi\cdot\xi=\inf_{\substack{v\in[L^2(Q)]^3_{\text{sol}},\\ \langle v\rangle=0}}\int_{Q_1}\epsilon_1^{-1}(v+\xi)\cdot(v+\xi),\qquad \xi\in\mathbb{R}^3. \tag{2.13}$$

Notice that for each vector v in (2.13) there exists $U_v\in[H^1_\#(Q)]^3$ such that $v=\operatorname{curl}U_v$ (see [5, pp. 6–7]), and hence

$$\int_{Q_1}\epsilon_1^{-1}(v+\xi)\cdot(v+\xi)=\int_{Q_1}\epsilon_1^{-1}(\operatorname{curl}(P_{V^\perp}U_v)+\xi)\cdot(\operatorname{curl}(P_{V^\perp}U_v)+\xi).$$

It follows that for all $\xi\in\mathbb{R}^3$ one has

$$\begin{aligned}(\epsilon^{\text{hom}}_{\text{stiff}})^{-1}\xi\cdot\xi&=\inf_{U\in[H^1_\#(Q)]^3}\int_{Q_1}\epsilon_1^{-1}(\operatorname{curl}(P_{V^\perp}U)+\xi)\cdot(\operatorname{curl}(P_{V^\perp}U)+\xi)\\&=A^{\text{hom}}\xi\cdot\xi.\end{aligned}$$

§3. *On the spectrum of the limit problem.* In this section we study the set of values ω^2 such that there exists a non-trivial triple (u,v,z) solving the two-scale limit spectral problem (2.4)–(2.6).

[2] The Euler–Lagrange equation for (2.12) is as follows: find u such that $\nabla u=-\xi$ in Q_0 and

$$\int_{Q_1}\epsilon_1(\nabla u+\xi)\cdot\nabla\phi=0\quad\text{for all }\phi\in H^1_\#(Q),\ \nabla\phi=0\text{ in }Q_0.$$

The equivalent "strong" form of the same problem is to find a Q-periodic function u such that

$$\operatorname{div}(\epsilon_1(\nabla u+\xi))=0\quad\text{in }Q_1,\qquad\int_{\partial Q_0}\epsilon_1(\nabla u+\xi)\cdot n=0,$$
$$u\text{ is continuous across }\partial Q_0,\qquad\nabla u=-\xi,\ \text{in }Q_0.$$

3.1. *Equivalent formulation and spectral decomposition of the limit problem.* Let G be the Green function for the scalar periodic Laplacian, that is, for all $y \in Q$, one has

$$-\Delta G(y) = \delta_0(y) - 1, \quad y \in Q, G \text{ is } Q\text{-periodic},$$

where δ_0 is the Dirac delta function supported at zero, on Q considered as a torus. Then, as the functions v, z solve (2.5), we have $v(x, \cdot) = G * (\operatorname{div}_y z)(x, \cdot)$, and (2.6) takes the form

$$\begin{aligned} &\operatorname{curl}_y(\epsilon_0^{-1}(y)\operatorname{curl}_y z(x, y)) \\ &\quad = \omega^2\left(u(x) + \nabla_y \int_{Q_0} G(y - y')\operatorname{div}_{y'} z(x, y')\,dy' + z(x, y)\right), \\ &\quad (x, y) \in \mathbb{T} \times Q_0. \end{aligned} \tag{3.1}$$

For the case $\omega = 0$ the set of solutions z to (3.1) subject to the condition $z(x, y) = 0$, $x \in \mathbb{T}$, $y \in \partial Q_0$, is clearly given by $L^2(\mathbb{T}, \mathcal{H}_0)$, where $\mathcal{H}_0 := \{u \in [H_0^1(Q_0)]^3 : \operatorname{curl} u = 0\}$.

Further, for $\omega \neq 0$, as (3.1) is linear in $u(x)$ and $\operatorname{curl}_y \nabla_y = 0$, we set

$$\nabla_y \int_{Q_0} G(y - y')\operatorname{div}_{y'} z(x, y')\,dy' + z(x, y) = \omega^2 B(y)u(x), \tag{3.2}$$

where B is a 3×3 matrix function whose column vectors B^j, $j = 1, 2, 3$, are solutions in $[H_\#^1(Q)]^3$ to the system

$$\operatorname{curl}(\epsilon_0^{-1}\operatorname{curl} B^j) = e_j + \omega^2 B^j \quad \text{in } Q_0, \tag{3.3}$$

$$\operatorname{curl} B^j = 0 \quad \text{in } Q_1, \tag{3.4}$$

$$\operatorname{div} B^j = 0 \quad \text{in } Q, \tag{3.5}$$

$$a(B^j) = 0, \tag{3.6}$$

where e_j, $j = 1, 2, 3$, are the Euclidean basis vectors and $a(B^j)$ is the "circulation" of B^j, that is defined as the continuous extension, in the sense of the H^1 norm, of the map given by $a(\phi)_i = \int_0^1 \phi_i(te_i)\,dt$, $i = 1, 2, 3$, for $\phi \in [C^\infty(Q)]^3$. Note that, since $B^j \in [H^1(Q)]^3$, equation (3.4) implies $\epsilon_0^{-1}\operatorname{curl} B^j \times n|_- = 0$ on ∂Q_0. Furthermore, the system (3.3)–(3.6) implies the variational problem of finding $B^j \in [H_\#^1(Q)]^3$, subject to the constraints (3.4)–(3.6), such that the following identity holds:

$$\begin{aligned} &\int_{Q_0} \epsilon_0^{-1}\operatorname{curl} B^j \cdot \operatorname{curl}\varphi \\ &\quad = \int_Q e_j \cdot \varphi + \omega^2 \int_Q B^j \cdot \varphi \quad \text{for all } \varphi \in [H_\#^1(Q)]^3 \text{ satisfying (3.4)–(3.6).} \end{aligned} \tag{3.7}$$

Indeed, functions $\varphi \in [H_\#^1(Q)]^3$ which satisfy (3.4) and (3.6) admit (see Lemma 4.1 below) the representation $\varphi = \nabla p + \psi$, $p \in H_\#^2(Q)$, $\psi \in [H_0^1(Q_0)]^3$.

Therefore, it is straightforward to show (3.7) holds for $\varphi = \psi$ if and only if (3.3) holds. Similarly, one can show (3.7) holds for $\varphi = \nabla p$ if and only if (3.5) holds.

Substituting the representation (3.2) into (2.4) and using the fact that

$$\int_Q (\nabla_{y'} G * (\mathrm{div}_y z)(x, y') + z(x, y'))\, dy' = \int_{Q_0} z(x, y)\, dy,$$

leads to the operator-pencil spectral problem

$$\mathrm{curl}\,(A^{\mathrm{hom}}\, \mathrm{curl}\, u(x)) = \Gamma(\omega) u(x), \quad x \in \mathbb{T}, \tag{3.8}$$

where Γ is a matrix-valued function that vanishes at $\omega = 0$, and for $\omega \neq 0$ has elements

$$\Gamma_{ij}(\omega) = \omega^2 \left(\delta_{ij} + \omega^2 \int_Q B_i^j \right), \quad i, j = 1, 2, 3. \tag{3.9}$$

We denote by $\mathcal{H}_1$ the space of vector fields in $[H^1_\#(Q)]^3$ that satisfy conditions (3.4)–(3.6). It can be shown[3] that there exist countably many pairs $(\alpha^k, r^k) \in \mathbb{R} \times \mathcal{H}_1$ such that $\|r^k\|_{[L^2(Q)]^3} = 1$ and

$$\mathrm{curl}\,(\epsilon_0^{-1}\, \mathrm{curl}\, r^k) = \alpha^k r^k \quad \text{in } Q_0.$$

Moreover, the sequence $(r^k)_{k \in \mathbb{N}}$ can be chosen to form an orthonormal basis of the closure $\overline{\mathcal{H}}_1$ of $\mathcal{H}_1$ in $[L^2(Q)]^3$ and, upon a suitable rearrangement, one has

$$0 < \alpha^1 \leqslant \alpha^2 \leqslant \cdots \leqslant \alpha^k \leqslant \cdots \overset{k \to \infty}{\longrightarrow} \infty.$$

Performing a decomposition[4] of the functions B^j, $j = 1, 2, 3$, with respect to the above basis yields

$$B_i^j = \sum_{k=1}^{\infty} \frac{\int_Q r_j^k}{\alpha^k - \omega^2} r_i^k, \quad \omega^2 \notin \cup \{\alpha^k\}_{k=1}^{\infty},$$

where r_j^k, $j = 1, 2, 3$, are the components of the vector r^k, $k \in \mathbb{N}$.

Consider the functions $\phi^k \in [H^1_0(Q_0)]^3$, $k \in \mathbb{N}$, that solve the non-local problems

[3] Note that $|||\cdot||| := (\int_{Q_0} \epsilon_0^{-1} |\mathrm{curl}\,\cdot|^2)^{1/2}$ is a norm in $\mathcal{H}_1$ equivalent to the $[H^1(Q)]^3$ norm, due to the fact that $(|a(\cdot)|^2 + \|\mathrm{div}\,\cdot\|^2_{L^2(Q)} + \|\mathrm{curl}\,\cdot\|^2_{[L^2(Q)]^3})^{1/2}$ is an equivalent norm in the space $u \in [H^1_\#(Q)]^3$. Therefore, the equation $\mathrm{curl}\, \epsilon_0^{-1}\, \mathrm{curl}\, u = \lambda u$, $u \in \mathcal{H}_1$, can be written as $\lambda^{-1} u = K u$ in the sense of the "energy" inner product generated by the norm $|||\cdot|||$ and K is a compact self-adjoint operator in $(\mathcal{H}_1, |||\cdot|||)$. The claim then follows by a standard Hilbert–Schmidt argument.

[4] When applying the standard Fourier representation approach with respect to the basis $(r^k)_{k \in \mathbb{N}}$, the vector e_j in the right-hand side of (3.3) is treated as an element of the "dual" of $\overline{\mathcal{H}}_1$, the space of linear continuous functionals on $\mathcal{H}_1$.

$$\operatorname{curl}(\epsilon_0^{-1}(y)\operatorname{curl}\phi^k(y)) = \alpha^k\biggl(\nabla\int_{Q_0} G(y-y')\operatorname{div}\phi^k(y')\,dy' + \phi^k(y)\biggr), \quad y\in Q_0, \quad (3.10)$$

and satisfy the orthonormality conditions

$$\int_{Q_0}\int_{Q_0}(\nabla^2 G(y-y') + I)\phi^j(y)\cdot\overline{\phi^k(y')}\,dy\,dy' = \delta_{jk}, \quad j,k=1,2,\ldots,$$

where $\nabla^2 G$ is the Hessian matrix of G. Using the formula

$$r^k(y) = \nabla\int_{Q_0} G(y-y')\operatorname{div}\phi^k(y')\,dy' + \phi^k(y), \quad y\in Q,$$

we obtain the following representation for Γ:

$$\Gamma_{ij}(\omega) = \omega^2\delta_{ij} + \omega^4\sum_{k=1}^{\infty}\frac{(\int_{Q_0}\phi_i^k)(\int_{Q_0}\phi_j^k)}{\alpha^k-\omega^2}, \quad i,j=1,2,3, \quad \omega^2\notin\{0\}\cup\{\alpha^k\}_{k=1}^{\infty}. \quad (3.11)$$

3.2. *Analysis of the limit spectrum.* Consider the Fourier expansion for the function u in (3.8):

$$u(x) = \sum_{m\in\mathbb{Z}^3}\exp(2\pi\mathrm{i}m\cdot x)\hat{u}(m), \qquad \hat{u}(m) := \int_{\mathbb{T}}\exp(-2\pi\mathrm{i}m\cdot x)u(x)\,dx,$$

where the integral is taken componentwise. As u solves (3.8), the coefficients $\hat{u}(m)$ satisfy the equation

$$\mathcal{M}(m)\hat{u}(m) = \Gamma(\omega)\hat{u}(m), \quad m\in\mathbb{Z}^3, \quad (3.12)$$

with the matrix-valued function $\mathcal{M}$ given by

$$\mathcal{M}_{lp}(m) = 4\pi^2\sum_{i,j,s,t=1}^{3}\varepsilon_{ils}m_s A_{ij}^{\mathrm{hom}}\varepsilon_{jpt}m_t = 4\pi^2(e_l\times m)\cdot A^{\mathrm{hom}}(e_p\times m), \quad m\in\mathbb{Z}^3, l,p=1,2,3,$$

where e_j, $j=1,2,3$ are the Euclidean basis vectors. Here ε is the Levi-Civita symbol:

$$\varepsilon_{jkl} = \begin{cases} 1, & (jkl)=(123),(231),(312),\\ -1, & (jkl)=(132),(321),(213),\\ 0 & \text{otherwise.}\end{cases}$$

Notice that, for all $m\in\mathbb{Z}^3\setminus\{0\}$, zero is a simple eigenvalue of $\mathcal{M}(m)$ with eigenvector m, and since the matrix A^{hom} is symmetric and positive definite, the values of $\mathcal{M}$ are also symmetric and positive definite on vectors ξ such that $\xi \cdot m = 0$. In particular, for all $m \in \mathbb{Z}^3$, one has

$$\Gamma(\omega)\hat{u}(m) \cdot m = 0 \tag{3.13}$$

whenever $\hat{u}(m)$ is a solution to (3.12). Denote $\tilde{m} := |m|^{-1}m$ and notice that $\mathcal{M}(m) = |m|^2\mathcal{M}(\tilde{m})$. Further, we denote by $\tilde{e}_1(\tilde{m}) = (\tilde{e}_{11}(\tilde{m}), \tilde{e}_{12}(\tilde{m}), \tilde{e}_{13}(\tilde{m}))$ and $\tilde{e}_2(\tilde{m}) = (\tilde{e}_{21}(\tilde{m}), \tilde{e}_{22}(\tilde{m}), \tilde{e}_{23}(\tilde{m}))$ the normalized eigenvectors of the matrix $\mathcal{M}(\tilde{m})$ corresponding to its two positive eigenvalues $\lambda_1(\tilde{m})$ and $\lambda_2(\tilde{m})$, respectively.

We write $\hat{u}(m)$ in terms of the basis $(\tilde{e}_1(\tilde{m}), \tilde{e}_2(\tilde{m}), \tilde{m})$, as follows:

$$\hat{u}(m) = C(\tilde{m})^\top \tilde{u}(\tilde{m}) + \alpha(\tilde{m})\tilde{m}, \quad \tilde{u}(\tilde{m}) \in \mathbb{R}^2, \alpha(\tilde{m}) \in \mathbb{R},$$
$$C(\tilde{m}) = \begin{pmatrix} \tilde{e}_{11}(\tilde{m}) & \tilde{e}_{12}(\tilde{m}) & \tilde{e}_{13}(\tilde{m}) \\ \tilde{e}_{21}(\tilde{m}) & \tilde{e}_{22}(\tilde{m}) & \tilde{e}_{23}(\tilde{m}) \end{pmatrix}.$$

Finding a non-trivial solution to problem (3.12)–(3.13) is equivalent to determining $(\tilde{u}(\tilde{m}), \alpha(\tilde{m})) \in \mathbb{R}^3 \setminus \{0\}$ such that

$$\begin{gathered} |m|^2\Lambda(\tilde{m})\tilde{u}(\tilde{m}) = C(\tilde{m})\Gamma(\omega)C(\tilde{m})^\top\tilde{u}(\tilde{m}) + \alpha(\tilde{m})C(\tilde{m})\Gamma(\omega)\tilde{m}, \\ \Gamma(\omega)C(\tilde{m})^\top\tilde{u}(\tilde{m}) \cdot \tilde{m} = -\alpha(\tilde{m})\Gamma(\omega)\tilde{m} \cdot \tilde{m}, \end{gathered} \tag{3.14}$$

where

$$\Lambda(\tilde{m}) := \begin{pmatrix} \lambda_1(\tilde{m}) & 0 \\ 0 & \lambda_2(\tilde{m}) \end{pmatrix}.$$

We have thus proved the following statement.

PROPOSITION 3.1. *The spectrum of the problem (2.4)–(2.6) is the union of the following sets.*

(1) *The elements of $\{\alpha^k : k \in \mathbb{Z}\}$ such that the corresponding r^k has zero mean over Q. These are eigenvalues of infinite multiplicity and the corresponding eigenfunctions $H^0(x, y)$ are of the form $w(x)r^k(y)$ for an arbitrary $w \in L^2(\mathbb{T})$.*
(2) *The set $\{\omega^2 : \exists m \in \mathbb{Z}^3$ such that (3.14) holds$\}$, with the corresponding eigenfunctions $H^0(x, y)$ of (2.4)–(2.6) having the form $u(x) + \nabla_y v(x, y) + z(x, y)$, where $u(x) = \exp(2\pi i m \cdot x)\hat{u}(m)$ is an eigenfunction of macroscopic problem (3.8) and*

$$\nabla_y v(x, y) + z(x, y) = \omega^2 B(y)u(x) \quad \text{a.e. } (x, y) \in \mathbb{T} \times Q,$$

that is, $H^0(x, y) = (I + \omega^2 B(y)) \exp(2\pi i m \cdot x)\hat{u}(m)$.

An immediate consequence of the above analysis is the following result.

COROLLARY 3.1. *If the matrix $\Gamma(\omega)$ is negative definite, the value $\lambda = \omega^2$ does not belong to the spectrum of (2.4)–(2.6).*

Proof. Since $\mathcal{M}(\tilde{m})$ admits the spectral decomposition $C'(\tilde{m})\Lambda'(\tilde{m})C'(\tilde{m})^\top$, where

$$C'(\tilde{m}) := \begin{pmatrix} \tilde{e}_{11}(\tilde{m}) & \tilde{e}_{12}(\tilde{m}) & \tilde{e}_{13}(\tilde{m}) \\ \tilde{e}_{21}(\tilde{m}) & \tilde{e}_{22}(\tilde{m}) & \tilde{e}_{23}(\tilde{m}) \\ \tilde{m}_1 & \tilde{m}_2 & \tilde{m}_3 \end{pmatrix}, \qquad \Lambda'(\tilde{m}) := \begin{pmatrix} \lambda_1(\tilde{m}) & 0 & 0 \\ 0 & \lambda_2(\tilde{m}) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

a necessary condition for pairs (m, ω) such that (3.12) has a solution is as follows:

$$\det(|m|^2\Lambda'(\tilde{m}) - C'(\tilde{m})^\top\Gamma(\omega)C'(\tilde{m})) = 0.$$

This is not possible since $\Lambda'(\tilde{m})$ is positive semidefinite and, by assumption, the matrix $\Gamma(\omega)$ is negative definite, and consequently the matrix $C'(\tilde{m})^\top\Gamma(\omega)C'(\tilde{m})$ is also negative definite. □

3.3. *Examples of different admissible wave propagation regimes for the effective spectral problem.* In this section we explore the effective wave propagation properties of high-contrast electromagnetic media. We demonstrate that the sign-indefinite nature of the matrix-valued function Γ gives rise to phenomena not present in the case of polarized waves.

Suppose that the inclusion is symmetric under a rotation by π around at least two of the three coordinate axes. Then the matrices A^{hom} and $\Gamma(\omega)$ are diagonal (see Appendix): $4\pi^2 A^{\text{hom}} = \text{diag}(a_1, a_2, a_3)$, $\Gamma(\omega) = \text{diag}(\beta_1(\omega), \beta_2(\omega), \beta_3(\omega))$. Here a_i, $i = 1, 2, 3$, are positive constants and β_i, $i = 1, 2, 3$, are real-valued scalar functions. Notice that, since $|\tilde{m}| = 1$, the eigenvalues $\lambda_{1,2}(\tilde{m})$ of $\mathcal{M}(\tilde{m})$ are the solutions to the quadratic equation

$$\begin{aligned}\lambda^2 - \lambda\{(a_2 + a_3)\tilde{m}_1^2 + (a_1 + a_3)\tilde{m}_2^2 + (a_1 + a_2)\tilde{m}_3^2\} \\ + (a_1a_2\tilde{m}_3^2 + a_2a_3\tilde{m}_1^2 + a_1a_3\tilde{m}_2^2) = 0.\end{aligned} \tag{3.15}$$

We will now solve the eigenvalue problem (3.12), equivalently (3.14), for particular examples of such inclusions.

3.3.1. *Isotropic propagation (no "weak" band gaps).* If the inclusion Q_0 is symmetric by a $\pi/2$-rotation around at least two of the three axes, say x_1 and x_2, then $a = a_1 = a_2 = a_3$ and $\beta(\omega) = \beta_1(\omega) = \beta_2(\omega) = \beta_3(\omega)$. Equation (3.15) takes the form $(\lambda - a)^2 = 0$, and therefore $\lambda_1(\tilde{m}) = \lambda_2(\tilde{m}) = a$ is an eigenvalue of multiplicity two of $\mathcal{M}(\tilde{m})$, with orthonormal eigenvectors given by

$$\tilde{e}_1(\tilde{m}) = e_2, \qquad \tilde{e}_2(\tilde{m}) = e_3 \quad \text{if } |\tilde{m}_1| = 1, \tag{3.16}$$

and

$$\tilde{e}_1(\tilde{m}) = \frac{1}{\sqrt{1-\tilde{m}_1^2}}e_1 \times \tilde{m}, \; \tilde{e}_2(\tilde{m}) = \frac{1}{\sqrt{1-\tilde{m}_1^2}}(e_1 \times \tilde{m}) \times \tilde{m} \quad \text{if } |\tilde{m}_1| < 1. \tag{3.17}$$

As before, the e_j, $j = 1, 2, 3$, are the Euclidean basis vectors. The system (3.14) takes the form

$$a|m|^2\tilde{u}(\tilde{m}) = \beta(\omega)\tilde{u}(\tilde{m}), \qquad \alpha(\tilde{m})\beta(\omega) = 0.$$

Notice that if ω is a zero of β then necessarily $\tilde{u}(\tilde{m})$ is the zero vector. For such values of ω, the above system is satisfied for any $\alpha(\tilde{m})$, that is, the non-trivial eigenvectors to (3.12) are parallel to $\tilde{m}$. On the other hand, if $\beta(\omega) \neq 0$, then $\alpha(\tilde{m}) = 0$ and ω is an eigenvalue of (3.12) if and only if it solves the equation $\beta(\omega) = a|m|^2$. In this case $\tilde{u}(\tilde{m})$ is an arbitrary element of $\mathbb{R}^2$ and $\hat{u}(m) = C(\tilde{m})^\top\tilde{u}(\tilde{m})$ is an arbitrary vector of the (two-dimensional) eigenspace spanned by the vectors $\tilde{e}_1(\tilde{m})$ and $\tilde{e}_2(\tilde{m})$. Finally, there are no non-trivial solutions $\hat{u}$ when $\beta(\omega) < 0$.

3.3.2. *Directional propagation (existence of "weak" band gaps).* If the inclusion Q_0 is symmetric by a $\pi/2$-rotation around one of the three coordinate axis, say x_1, and by a π-rotation around another axis, say x_2, one has $a = a_1$, $b = a_2 = a_3$ and $\beta_2(\omega) = \beta_3(\omega)$. Here, recalling $|\tilde{m}| = 1$, (3.15) takes the form

$$(\lambda - b)(\lambda - a(1 - \tilde{m}_1^2) - b\tilde{m}_1^2) = 0,$$

whence $\lambda_1(\tilde{m}) = a(1 - \tilde{m}_1^2) + b\tilde{m}_1^2$, $\lambda_2(\tilde{m}) = b$. There are now two separate cases to consider.

Case 1. Assume that $|\tilde{m}_1| = 1$, that is, the vector $\tilde{m}$ is parallel to the axis of higher symmetry. Here, $\mathcal{M}(\tilde{m}) = \operatorname{diag}(0, b, b)$ and b is an eigenvalue of multiplicity two with the eigenspace spanned by the vectors (3.16). The system (3.14) takes the form

$$bm_1^2\tilde{u}(\tilde{m}) = \beta_2(\omega)\tilde{u}(\tilde{m}), \qquad \alpha(\tilde{m})\beta_1(\omega) = 0.$$

Here, if $\beta_2(\omega) < 0$, then necessarily $\tilde{u}(\tilde{m}) = 0$ and non-trivial solutions $\hat{u}(m) = \alpha(\tilde{m})\tilde{m}$ exist if and only if $\beta_1(\omega) = 0$. On the other hand, if $\beta_1(\omega) < 0$, then necessarily $\alpha(\tilde{m}) = 0$ and non-trivial solutions $\hat{u}(m) = C(\tilde{m})^\top\tilde{u}(\tilde{m})$ exist if and only if $\beta_2(\omega) > 0$. The first situation only occurs at a discrete set of values ω, while, unlike in the isotropic case, the second situation can give rise to intervals each of which contains a sequence of admissible ω, obtained from the condition $\sqrt{\beta_2(\omega)/b} \in \mathbb{N}$, with a reduced number of eigenmodes. In the case of the full-space problem these intervals form part of the continuous spectrum of the problem with a reduced number of propagating modes ("weak band gaps"; cf. [**7**, **9**, **10**]).

Case 2. Assume $|\tilde{m}_1| < 1$, that is, the vector $\tilde{m}$ is not parallel to the axis of higher symmetry. In this case the eigenvectors corresponding to $\lambda_1(\tilde{m})$, $\lambda_2(\tilde{m})$ are given by $\tilde{e}_1(\tilde{m})$, $\tilde{e}_2(\tilde{m})$ in (3.17). By setting

$$\Gamma(\omega) = \operatorname{diag}(\beta_1(\omega) - \beta_2(\omega), 0, 0) + \beta_2(\omega)I$$

it is easy to see that the system (3.14) takes the form

$$\begin{aligned} &|m|^2\lambda_1(\tilde m)\tilde u_1(\tilde m) = \beta_2(\omega)\tilde u_1(\tilde m),\\ &(\beta_1(\omega)-\beta_2(\omega))\tilde m_1\sqrt{1-\tilde m_1^2}\tilde u_2(\tilde m) = \alpha(\tilde m)((\beta_1(\omega)-\beta_2(\omega))\tilde m_1^2+\beta_2(\omega)),\\ &|\tilde m|^2\lambda_2(\tilde m)\tilde u_2(\tilde m) = (\beta_2(\omega)+(\beta_1(\omega)-\beta_2(\omega))(1-\tilde m_1^2))\tilde u_2(\tilde m)\\ &\qquad\qquad - \alpha(\tilde m)(\beta_1(\omega)-\beta_2(\omega))\tilde m_1\sqrt{1-\tilde m_1^2}. \end{aligned} \tag{3.18}$$

If $\tilde m_1 = 0$, that is, the vector $\tilde m$ is perpendicular to the direction of higher symmetry, then the system (3.18) fully decouples and reduces to

$$|m|^2 a\tilde u_1(\tilde m) = \beta_2(\omega)\tilde u_1(\tilde m), \qquad |m|^2 b\tilde u_2(\tilde m) = \beta_1(\omega)\tilde u_2(\tilde m),$$
$$\alpha(\tilde m)\beta_2(\omega) = 0.$$

Suppose $\beta_1(\omega)$ (respectively, $\beta_2(\omega)$) is negative for some ω. Then the above system implies that $\tilde u_2(\tilde m) = 0$ (respectively, $\tilde u_1(\tilde m) = 0$). In this case, we see that propagation is restricted solely to the direction of $\tilde e_1(\tilde m)$ (respectively, $\tilde e_2(\tilde m)$), which is orthogonal to the eigenvector(s) corresponding to the negative eigenvalue of $\Gamma(\omega)$. In both situations weak band gaps are present in the similar full-space problem.

Remark 3.1. Recently there have been several works on the analysis of problems with "partial" or "directional" wave propagation in the context of elasticity, where at some frequencies, propagation occurs for some but not for all values of the wave vector: the analysis of the vector problems for thin structures of critical thickness [**9**], the analysis of high-contrast [**10**], and partially high-contrast [**7**] periodic elastic composites. To our knowledge, the effect we describe here is the first example of a similar kind for Maxwell equations.

Remark 3.2. When the "size" T of the domain $\mathbb{T}$ increases to infinity (equivalently, for a given macroscopic domain, the parameter by which its size is scaled (see §2) tends to zero), the spectrum of (2.4)–(2.6) converges to a union of intervals ("bands") separated by intervals of those values ω^2 for which the matrix $\Gamma(\omega)$ is negative definite ("gaps" or "lacunae"). As above, we say that ω^2 belongs to a weak band gap (in the spectrum of (2.4)–(2.6)) if at least one eigenvalue of $\Gamma(\omega)$ is positive semidefinite and at least one eigenvalue of $\Gamma(\omega)$ is negative.

§4. *Two-scale asymptotic expansion of the eigenfunctions.* Here we give the details of the recurrent procedure for the construction of the series (2.2). We use $|_+$ and $|_-$ to denote the limit values of the expressions to which these symbols are attached, on the outside and on the inside of the boundary of the inclusion Q_0, respectively.

Substituting the expansions (2.2) into (2.1) and equating coefficients on η^{-2}, η^{-1} and η^0, we arrive at the following sets of equations, where $x \in \mathbb{T}$ is a parameter:

$$\operatorname{curl}_y(\epsilon_1^{-1}(y)\operatorname{curl}_y H^0(x,y)) = 0, \quad y \in Q_1, \tag{4.1}$$
$$\epsilon_1^{-1}(y)\operatorname{curl}_y H^0(x,y)\times n(y)|_+ = 0, \quad y \in \partial Q_0, \tag{4.2}$$
$$\begin{aligned}&\operatorname{curl}_y(\epsilon_1^{-1}(y)\operatorname{curl}_y H^1(x,y))\\ &\quad = -(\operatorname{curl}_y \epsilon_1^{-1}(y)\operatorname{curl}_x + \operatorname{curl}_x \epsilon_1^{-1}(y)\operatorname{curl}_y)H^0(x,y), \quad y \in Q_1,\end{aligned} \tag{4.3}$$
$$\begin{aligned}&(\epsilon_1^{-1}(y)\operatorname{curl}_y H^1(x,y)\times n(y) + \epsilon_1^{-1}(y)\operatorname{curl}_x H^0(x,y)\times n(y))|_+ = 0,\\ &\quad y \in \partial Q_0,\end{aligned} \tag{4.4}$$
$$\begin{aligned}&\operatorname{curl}_y(\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y))\\ &\quad = -(\operatorname{curl}_y \epsilon_1^{-1}(y)\operatorname{curl}_x + \operatorname{curl}_x \epsilon_1^{-1}(y)\operatorname{curl}_y)H^1(x,y)\\ &\qquad - \operatorname{curl}_x \epsilon_1^{-1}(y)\operatorname{curl}_x H^0(x,y) + \omega^2 H^0(x,y), \quad y \in Q_1,\end{aligned} \tag{4.5}$$
$$\begin{aligned}&(\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y) + \epsilon_1^{-1}(y)\operatorname{curl}_x H^1(x,y))\times n(y)|_+\\ &\quad = \epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)\times n(y)|_-, \quad y \in \partial Q_0,\end{aligned} \tag{4.6}$$

and

$$\operatorname{curl}_y(\epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)) = \omega^2 H^0(x,y), \quad y \in Q_0, \tag{4.7}$$
$$H^0(x,y)|_- = H^0(x,y)|_+, \quad y \in \partial Q_0. \tag{4.8}$$

Multiplying equation (4.1) by H^0, integrating by parts over Q_1 and using (4.2) shows that $\operatorname{curl}_y H^0(x,y) = 0$, $y \in Q_1$. More precisely, for all $x \in \mathbb{T}$, we seek $H^0(x,\cdot)$ from the space V; see (2.8). Before proceeding, we recall a characterization of the space V (see [**1**]) that proves useful in the analysis of the term H^0.

LEMMA 4.1 (Characterization of V). *A function $v \in [H^1_\#(Q)]^3$ is an element of the space V if and only if*

$$v(y) = a + \nabla b(y) + c(y) \quad a.e.\ y \in Q,$$

for some $a \in \mathbb{R}^3$, $b \in H^2_\#(Q)$, $c \in [H^1_0(Q_0)]^3$.

Taking into account, via Lemma 4.1, that the leading-order term H^0 is of the form

$$H^0(x,y) = u(x) + \nabla_y v(x,y) + z(x,y) \tag{4.9}$$

and substituting (4.9) into equations (4.3)–(4.4), we find that the coefficient H^1 has the representation $H^1(x,y) = N(y)\operatorname{curl} u(x) + \widetilde{H}^1(x,y)$, up to the addition of an element of V. Here the term $\widetilde{H}^1(x,y)$ satisfies

$$\operatorname{curl}_y(\epsilon_1^{-1}(y)\,(\operatorname{curl}_y \widetilde{H}^1(x,y) + \operatorname{curl}_x \nabla_y v(x,y))) = 0, \quad y \in Q_1, \tag{4.10}$$
$$\epsilon_1^{-1}(y)(\operatorname{curl}_y \widetilde{H}^1(x,y) + \operatorname{curl}_x \nabla_y v(x,y))\times n(y)|_+ = 0, \quad y \in \partial Q_0, \tag{4.11}$$

and $N = N(y)$ is a Q-periodic matrix-valued function whose columns $N^r = N^r(y)$, $r = 1, 2, 3$, are solutions to the problems

$$\begin{aligned} \operatorname{curl}(\epsilon_1^{-1}(y)(\operatorname{curl} N^r(y) + e^r)) = 0, \quad & y \in Q_1, \\ \epsilon_1^{-1}(y)(\operatorname{curl} N^r(y) + e^r) \times n(y) = 0, \quad & y \in \partial Q_0, \end{aligned} \tag{4.12}$$

where e^r is the rth Euclidean basis vector. It is shown [**4**, **6**] that (4.12) admits a unique solution in $V^\perp$, the orthogonal complement to V in the space $[H^1_\#(Q)]^3$.

Looking for $H^1(x, \cdot) \in [H^1_\#(Q)]^3$ and taking into account the identity $\operatorname{curl}_x \nabla_y = -\operatorname{curl}_y \nabla_x$ together with (4.10)–(4.11), we infer that for all $x \in \mathbb{T}$ the function $h(x, \cdot) := \widetilde{H}^1(x, \cdot) - \nabla_x v(x, \cdot)$ is a solution in $[H^1_\#(Q)]^3$ to

$$\begin{aligned} \operatorname{curl}_y(\epsilon_1^{-1}(y) \operatorname{curl}_y h(x, y)) = 0, \quad & y \in Q_1, \\ \epsilon_1^{-1}(y) \operatorname{curl}_y h(x, y) \times n(y)|_+ = 0, \quad & y \in \partial Q_0. \end{aligned}$$

In particular, the function h belongs to the space V. Therefore, one has

$$H^1(x, y) = N(y) \operatorname{curl}_x u(x) + \nabla_x v(x, y), \tag{4.13}$$

up to the addition of an element of V. (As we discuss in Remark 5.1 below, one can specify the divergence $\operatorname{div}_y H^1(x, y)$. This, along with the condition that the y-average of H^1 vanishes, defines this additional element of V in a unique way.)

Further, multiplying equation (4.5) by an arbitrary test function $\phi \in V$ and integrating over Q_1 yields

$$\begin{aligned} &\int_{Q_1} \operatorname{curl}_y(\epsilon_1^{-1}(y) \operatorname{curl}_y H^2(x, y)) \cdot \phi(y)\, dy \\ &\quad = \int_{Q_1} \omega^2 H^0(x, y) \cdot \phi(y)\, dy - \int_{Q_1} \operatorname{curl}_y(\epsilon_1^{-1}(y) \operatorname{curl}_x H^1(x, y)) \cdot \phi(y)\, dy \\ &\quad\quad - \left(\int_{Q_1} \operatorname{curl}_x(\epsilon_1^{-1}(y) \operatorname{curl}_x H^0(x, y)) \cdot \phi(y) \right. \\ &\quad\quad \left. + \operatorname{curl}_x(\epsilon_1^{-1}(y) \operatorname{curl}_y H^1(x, y)) \cdot \phi(y)\, dy \right). \end{aligned} \tag{4.14}$$

We integrate in the left-hand side of (4.14) by parts to determine that

$$\begin{aligned} &\int_{Q_1} \operatorname{curl}_y(\epsilon_1^{-1}(y) \operatorname{curl}_y H^2(x, y)) \cdot \phi(y)\, dy \\ &\quad = \int_{\partial Q_0} \epsilon_1^{-1}(y)(\operatorname{curl}_y H^2(x, y) \times n(y)|_+) \cdot \phi(y)\, dS(y). \end{aligned} \tag{4.15}$$

Now we perform integration by parts on the individual terms on the right-hand side of (4.14).

$$
\begin{aligned}
&-\int_{Q_1} \operatorname{curl}_y(\epsilon_1^{-1}(y)\operatorname{curl}_x H^1(x,y))\phi(y)\,dy \\
&\quad= -\int_{\partial Q_0} (\epsilon_1^{-1}(y)\operatorname{curl}_x H^1(x,y)\times n(y)|_+)\cdot\phi(y)\,dS(y) \\
&\quad\overset{\text{by (4.6)}}{=} \int_{\partial Q_0} \{(\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y)\times n(y)|_+)\cdot\phi(y) \\
&\qquad\quad - (\epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)\times n(y)|_-)\cdot\phi(y)\}\,dS(y) \\
&\quad= \int_{\partial Q_0} (\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y)\times n(y)|_+)\cdot\phi(y)\,dS(y) \\
&\qquad\quad + \int_{Q_0} \operatorname{curl}_y(\epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y))\cdot\phi(y)\,dy \\
&\qquad\quad - \int_{Q_0} \epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)\cdot\operatorname{curl}_y\phi(y)\,dy \\
&\quad\overset{\text{by (4.7)}}{=} \int_{\partial Q_0} (\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y)\times n(y)|_+)\cdot\phi(y)\,dS(y) \\
&\qquad\quad + \int_{Q_0} \omega^2 H^0(x,y)\cdot\phi(y)\,dy \\
&\qquad\quad - \int_{Q_0} \epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)\cdot\operatorname{curl}_y\phi(y)\,dy.
\end{aligned}
\tag{4.16}
$$

Taking into account the representations (4.9) and (4.13), we find that

$$
\begin{aligned}
&\int_{Q_1} \{\operatorname{curl}_x(\epsilon_1^{-1}(y)\operatorname{curl}_x H^0(x,y))\cdot\phi(y) \\
&\qquad + \operatorname{curl}_x(\epsilon_1^{-1}(y)\operatorname{curl}_y H^1(x,y))\cdot\phi(y)\}\,dy \\
&\quad= \int_{Q_1} \operatorname{curl}_x\{\epsilon_1^{-1}(y)\,((I+\operatorname{curl} N(y))\operatorname{curl}_x u(x) \\
&\qquad + \operatorname{curl}_x\nabla_y v(x,y) + \operatorname{curl}_y\nabla_x v(x,y))\}\cdot\phi(y)\,dy \\
&\quad= \int_{Q_1} \operatorname{curl}_x\{\epsilon_1^{-1}(y)(I+\operatorname{curl} N(y))\operatorname{curl}_x u(x)\}\cdot\phi(y)\,dy,
\end{aligned}
\tag{4.17}
$$

where we again make use of the identity $\operatorname{curl}_x\nabla_y = -\operatorname{curl}_y\nabla_x$. Finally, equations (4.14)–(4.17) imply

$$
\begin{aligned}
&\int_{Q_1} \operatorname{curl}_x\{\epsilon_1^{-1}(y)(I+\operatorname{curl} N(y))\operatorname{curl}_x u(x)\}\cdot\phi(y)\,dy \\
&\qquad + \int_{Q_0} \epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)\cdot\operatorname{curl}_y\phi(y)\,dy \\
&\quad= \int_Q \omega^2 H^0(x,y)\cdot\phi(y)\,dy \quad \text{for all } \phi\in V.
\end{aligned}
\tag{4.18}
$$

In what follows we derive the system (2.4)–(2.6) by considering different choices of the test function ϕ in the identity (4.18).

Step 1. Choosing test functions $\phi \in [C_0^\infty(Q_0)]^3$ in (4.18), we find that

$$\operatorname{curl}_y(\epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y)) = \omega^2 H^0(x,y), \quad y \in Q_0.$$

Using the representation (4.9) and the identity $\operatorname{curl}_y \nabla_y = 0$, we arrive at (2.6).

Step 2. Choosing $\phi = \nabla_y \psi$ in (4.18), performing integration by parts, using the identity $\operatorname{div}_y \operatorname{curl}_x = -\operatorname{div}_x \operatorname{curl}_y$ and recalling (4.12) gives

$$\begin{aligned}\int_Q \omega^2 H^0(x,y)\cdot\nabla_y\psi\,dy \\ &= \int_{Q_1} \operatorname{curl}_x\{\epsilon_1^{-1}(y)(I+\operatorname{curl}N(y))\operatorname{curl}_x u(x)\}\cdot\nabla_y\psi(y)\,dy \\ &= -\int_{Q_1} \operatorname{div}_y\operatorname{curl}_x\{\epsilon_1^{-1}(y)(I+\operatorname{curl}N(y))\operatorname{curl}_x u(x)\}\cdot\psi(y)\,dy \\ &= \int_{Q_1} \operatorname{div}_x\operatorname{curl}_y\{\epsilon_1^{-1}(y)(I+\operatorname{curl}N(y))\operatorname{curl}_x u(x)\}\cdot\psi(y)\,dy = 0.\end{aligned}$$

Therefore, we deduce that

$$\operatorname{div}_y H^0(x,y) = 0, \quad y \in Q, \tag{4.19}$$

and, taking into account (4.9), we obtain the equation (2.5).

Step 3. Choosing $\phi(y) \equiv 1$ in the identity (4.18), we find, using the representation (4.9) once more, that (2.4) holds, where the matrix A^{hom} emerges as the result of integrating the expression $\epsilon_1^{-1}(y)(\operatorname{curl}N(y)+I)$ with respect to $y \in Q_1$.

In the next section we use the above formal construction of the series (2.2) to justify the two claims of Theorem 2.1.

§5. *Proof of Theorem 2.1.* For each $\eta > 0$, denote by $\mathcal{A}_\eta$ the operator in the space[5] $L^2_{\#\mathrm{sol}}(\mathbb{T})$ defined in a standard way by the bilinear form (cf. (2.1))

$$\int_{\mathbb{T}} \varepsilon_\eta^{-1}\left(\frac{\cdot}{\eta}\right)\operatorname{curl}u\cdot\operatorname{curl}v, \quad u,v \in [H^1_\#(\mathbb{T})]^3 \cap L^2_{\#\mathrm{sol}}(\mathbb{T}) =: \mathcal{H}.$$

For fixed ω in the spectrum of (2.4)–(2.6), let H^0 be a corresponding eigenfunction. Consider the (unique) solution $\widetilde{H}^\eta \in \mathcal{H}$ to the problem

$$(\mathcal{A}_\eta + I)\widetilde{H}^\eta = (\omega^2+1)H^0\left(\cdot,\frac{\cdot}{\eta}\right). \tag{5.1}$$

Denote also

$$\mathfrak{b}_\eta(u,v) := \int_{\mathbb{T}} \varepsilon_\eta^{-1}\left(\frac{\cdot}{\eta}\right)\operatorname{curl}u\cdot\operatorname{curl}v + \int_{\mathbb{T}} u\cdot v, \quad u,v \in [H^1_\#(\mathbb{T})]^3,$$

[5] We denote by $L^2_{\#\mathrm{sol}}(\mathbb{T})$ the closure of the set of smooth divergence-free vector fields on $\mathbb{T}$ with respect to the $L^2(\mathbb{T})$ norm.

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and (cf. (2.2))

$$H^{(2)}(\cdot,\eta) := H^0\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta H^1\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta^2 H^2\Big(\cdot,\frac{\cdot}{\eta}\Big), \tag{5.2}$$

where H^j, $j=1,2$, are solutions of the system of recurrence relations described in §4. The existence of solutions H^1, H^2 is guaranteed by a result established in [**1**, Lemma 3.4]. As these solutions are unique up to the addition of an element from V, we shall choose them as in Remark 5.1.

PROPOSITION 5.1. *There exists a constant $\widehat{C}>0$ such that the estimate*

$$|\mathfrak{b}_\eta(\widetilde{H}^\eta - H^{(2)}(\cdot,\eta),\varphi)| \leqslant \widehat{C}\eta\sqrt{\mathfrak{b}_\eta(\varphi,\varphi)} \tag{5.3}$$

holds for all $\varphi\in[H^1_\#(\mathbb{T})]^3$.

Proof. Using the definition of the function $\widetilde{H}^\eta$ and the recurrence relations (4.1)–(4.8) yields

$$\begin{aligned}
&\mathfrak{b}_\eta(\widetilde{H}^\eta - H^{(2)}(\cdot,\eta),\varphi)\\
&\quad= \int_{\mathbb{T}} \varepsilon_\eta^{-1}\Big(\frac{\cdot}{\eta}\Big)\operatorname{curl}\widetilde{H}^\eta\cdot\operatorname{curl}\varphi + \int_{\mathbb{T}}\widetilde{H}^\eta\cdot\varphi\\
&\qquad - \int_{\mathbb{T}} \varepsilon_\eta^{-1}\Big(\frac{\cdot}{\eta}\Big)\operatorname{curl}\Big(H^0\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta H^1\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta^2 H^2\Big(\cdot,\frac{\cdot}{\eta}\Big)\Big)\cdot\operatorname{curl}\varphi\\
&\qquad - \int_{\mathbb{T}}\Big(H^0\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta H^1\Big(\cdot,\frac{\cdot}{\eta}\Big) + \eta^2 H^2\Big(\cdot,\frac{\cdot}{\eta}\Big)\Big)\cdot\varphi\\
&\quad= \int_{\mathbb{T}} F^1(\cdot,\eta)\cdot\varphi + \int_{\mathbb{T}} F^2(\cdot,\eta)\cdot\eta\operatorname{curl}\varphi.
\end{aligned} \tag{5.4}$$

Here, F^1, F^2 are elements of $L^2(\mathbb{T})$ defined for a.e. $x\in\mathbb{T}$ by

$$\begin{aligned}
F^1(x,\eta) = -\eta\,(&\chi_0(y)\operatorname{curl}_x(\epsilon_0^{-1}(y)\operatorname{curl}_y H^0(x,y))\\
&+\chi_1(y)\,\{\operatorname{curl}_x(\epsilon_1^{-1}(y)\operatorname{curl}_x H^1(x,y))\\
&+\operatorname{curl}_x(\epsilon_1^{-1}(y)\operatorname{curl}_y H^2(x,y))\} + H^1(x,y) + \eta H^2(x,y))|_{y=x/\eta},\\
F^2(x,\eta) = -\eta\,(&\chi_0(y)\epsilon_0^{-1}(y)\,\{\operatorname{curl}_x H^0(x,y)\\
&+\operatorname{curl}_y H^1(x,y) + \eta\operatorname{curl}_x H^1(x,y)\\
&+\eta\operatorname{curl}_y H^2(x,y) + \eta^2\operatorname{curl}_x H^2(x,y)\}\\
&+\chi_1(y)\epsilon_1^{-1}(y)\operatorname{curl}_x H^2(x,y))|_{y=x/\eta}.
\end{aligned} \tag{5.5}$$

Notice that the functions $H^0=H^0(x,y)$, $H^1=H^1(x,y)$, $H^2=H^2(x,y)$ all belong to the space $C^\infty_\#(\mathbb{T},H^1_\#(Q))$. Indeed, this is seen to be true for H^0 by Proposition 3.1; in the case of $\omega=\alpha^k$ we choose $w\in C^\infty_\#(\mathbb{T})$. The assertions for H^1 and H^2 now follow from formula (4.13) for the corrector $H^1(x,y)$, and the boundary-value problem (4.5)–(4.6) for the function $H^2(x,y)$. It then follows from (5.5) (see, for example, [3, p. 1353]) that $\|F^1(\cdot,\eta)\|_{L^2(\mathbb{T})} \leqslant C\eta$, $\|F^2(\cdot,\eta)\|_{L^2(\mathbb{T})} \leqslant C\eta$, and by applying the Hölder inequality to (5.4) we deduce that

$$|\mathfrak{b}_\eta(\widetilde{H}^\eta - H^{(2)}(\cdot,\eta),\varphi)| \leqslant C\eta\biggl(\int_{\mathbb{T}}|\varphi|^2 + \int_{\mathbb{T}}|\eta\operatorname{curl}\varphi|^2\biggr)^{1/2},$$

as required. □

The above proposition implies the following statement.

THEOREM 5.1. *There exists a constant C such that the estimate*

$$\|\widetilde{H}^\eta - H^0(\cdot,\cdot/\eta)\|_{L^2(\mathbb{T})} \leqslant C\eta$$

holds for all η.

Proof. Setting $\varphi = \widetilde{H}^\eta - H^{(2)}(\cdot,\eta)$ in the estimate (5.3) yields

$$\widehat{C}^2\eta^2 \geqslant \mathfrak{b}_\eta(\widetilde{H}^\eta - H^{(2)}(\cdot,\eta), \widetilde{H}^\eta - H^{(2)}(\cdot,\eta)) \geqslant \|\widetilde{H}^\eta - H^{(2)}(\cdot,\eta)\|^2_{L^2(\mathbb{T})}.$$

The claim of the theorem now follows by noting that in view of (5.2), we have

$$\|H^{(2)}(\cdot,\eta) - H^0(\cdot,\cdot/\eta)\|_{L^2(\mathbb{T})} \leqslant \widetilde{C}\eta$$

for some $\widetilde{C} > 0$, and hence

$$\begin{aligned}\|\widetilde{H}^\eta - H^0(\cdot,\cdot/\eta)\|_{L^2(\mathbb{T})} &\leqslant \|\widetilde{H}^\eta - H^{(2)}(\cdot,\eta)\|_{L^2(\mathbb{T})}\\ &\quad + \|H^{(2)}(\cdot,\eta) - H^0(\cdot,\cdot/\eta)\|_{L^2(\mathbb{T})} \leqslant (\widehat{C}+\widetilde{C})\eta,\end{aligned}$$

as required. □

The claims of Theorem 2.1 now follow from the estimate

$$\begin{aligned}&\|((\omega^2+1)^{-1} - (\mathcal{A}_\eta + I)^{-1})H^0(\cdot,\cdot/\eta)\|_{L^2(\mathbb{T})}\\ &\quad\leqslant (\omega^2+1)^{-1}\|H^0(\cdot,\cdot/\eta) - \widetilde{H}^\eta\|_{L^2(\mathbb{T})} \leqslant C\eta,\end{aligned} \tag{5.6}$$

where we used the definition (5.1) of the function $\widetilde{H}^\eta$ and Theorem 5.1. Indeed, from [8, p. 109], we infer that the quantities $\operatorname{dist}((\omega^2+1)^{-1}, \operatorname{Sp}((\mathcal{A}_\eta+1)^{-1}))$ and $\operatorname{dist}((\omega^2+1)^{-1}H^0(\cdot,\cdot/\eta), X_\eta)$ are controlled above by the right-hand side of (5.6), which completes the proof of Theorem 2.1.

Remark 5.1. Note that $H^{(2)}$ is not solenoidal in general, but can be defined in such a way that it is "close" to a solenoidal field, thanks to equation (4.19) (equivalently, (2.5)) and the special choice of the function H^1 so that

$$\operatorname{div}_x H^0(x,y) + \operatorname{div}_y H^1(x,y) = 0 \quad \text{a.e. } (x,y) \in \mathbb{T}\times Q.$$

The function $H^{(2)}$ thus defined is η-close to the eigenspace X_η in the norm of $[H^1_\#(Q)]^3$.

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A. *Appendix. Symmetry of A^{hom} and $\Gamma(\omega)$ under rotations.* Suppose that $A \in [L^\infty(Q)]^{3\times 3}$ is symmetric such that $A \geqslant \nu I$ on Q_1, $\nu > 0$, and $A \equiv 0$ on Q_0. Consider the matrix

$$A^{\mathrm{hom}}_{pq} := \int_Q A(\operatorname{curl} N^q_p + \delta_{pq}), \qquad p, q \in \{1, 2, 3\},$$

where N^q is the unique solution to the problem (cf. [6], [1, Lemma 3.4] and (4.12) above for $A = \epsilon_1^{-1}\chi_1$, where χ_1 is the characteristic function of Q_1)

$$\operatorname{curl}(A[\operatorname{curl} N^q + e_q]) = 0, \qquad N^q \in \{u \in [H^1_\#(Q)]^3 : A \operatorname{curl} u = 0\}^\perp.$$

Here the superscript "⊥" denotes the orthogonal complement in $[H^1_\#(Q)]^3$. Notice that if, for fixed $\zeta \in \mathbb{R}^3$, we multiply each of the above equations by ζ_q, then we obtain

$$A^{\mathrm{hom}}\zeta = \int_Q A(\operatorname{curl} N_\zeta + \zeta), \tag{A.1}$$

where the vector N_ζ, whose components are $\sum_q N^q_p \zeta_q$, $p = 1, 2, 3$, is the unique solution to the problem

$$\operatorname{curl}(A[\operatorname{curl} N_\zeta + \zeta]) = 0, \qquad N_\zeta \in \{u \in [H^1_\#(Q)]^3 : A \operatorname{curl} u = 0\}^\perp. \tag{A.2}$$

It is clear that the matrix representation of the bounded linear mapping $\zeta \mapsto \int_Q A(\operatorname{curl} N_\zeta + \zeta)$ is equal to A^{hom}. The following property holds.

PROPOSITION A.1. *Suppose that σ is a rotation such that $\sigma Q = Q$ and assume that*

$$A(y) = \sigma^{-1} A(\sigma y)\sigma, \qquad y \in Q. \tag{A.3}$$

Then A^{hom} inherits the same symmetry, that is, one has

$$A^{\mathrm{hom}} = \sigma^{-1} A^{\mathrm{hom}} \sigma. \tag{A.4}$$

In particular, if (A.3) holds for all $\pi/2$-rotations, then one has $A^{\mathrm{hom}}_{kl} = A^{\mathrm{hom}}_{lk} = 0$ for all $l \neq k$.

Proof. For each $u \in [H^1_\#(Q)]^3$ let w be the solution of the vector equation

$$(\operatorname{curl}\operatorname{curl} w(y))_\alpha = \sum_{l,m,r,s=1}^{3} \sigma_{s\alpha}\epsilon_{slm}\sigma_{mr}\frac{\partial u_l(y)}{\partial y_r}, \quad \alpha = 1, 2, 3,$$

in the space

$$\left\{ w \in [H^1_\#(Q)]^3 : \operatorname{div} w = 0, \int_Q w = 0 \right\}.$$

It is clear that such a solution exists. We denote by $\hat{u}$ the vector field $\operatorname{curl} w$. A direct calculation, using the property $\sigma^{-1} = \sigma^\top$, yields

$$\operatorname{curl}_{y'} u(\sigma^{-1}y') = \sigma \operatorname{curl} \hat{u}(\sigma^{-1}y'). \tag{A.5}$$

Therefore, for all $\varphi \in [H^1_\#(Q)]^3$, the above equality and the assumption (A.3) imply

$$\begin{aligned}
&\int_Q A(y')\operatorname{curl}_{y'} u(\sigma^{-1}y')\cdot \operatorname{curl}_{y'} \varphi(\sigma^{-1}y')\,dy' \\
&\quad = \int_Q A(\sigma y)\sigma \operatorname{curl}\hat{u}(y)\cdot \sigma \operatorname{curl}\hat{\varphi}(y)\,dy \\
&\quad = \int_Q A(y)\operatorname{curl}\hat{u}(y)\cdot \operatorname{curl}\hat{\varphi}(y)\,dy.
\end{aligned}$$

Hence, a function $u \in [H^1_\#(Q)]^3$ solves

$$\begin{aligned}
&\int_Q A(y')\operatorname{curl}_{y'} u(\sigma^{-1}y')\cdot \operatorname{curl}_{y'} \varphi(\sigma^{-1}y')\,dy' \\
&\quad = \int_Q f(y')\cdot \operatorname{curl}_{y'} \varphi(\sigma^{-1}y')\,dy' \quad \text{for all } \varphi \in [H^1_\#(Q)]^3,
\end{aligned} \tag{A.6}$$

if and only if $\hat{u}$ solves

$$\begin{aligned}
&\int_Q A(y)\operatorname{curl}\hat{u}(y)\cdot \operatorname{curl}\hat{\varphi}(y)\,dy \\
&\quad = \int_Q \sigma^{-1} f(\sigma y)\cdot \operatorname{curl}\hat{\varphi}(y)\,dy \quad \text{for all } \hat{\varphi} \in [H^1_\#(Q)]^3.
\end{aligned} \tag{A.7}$$

Let us now prove (A.4). For fixed $\xi, \zeta \in \mathbb{R}^3$ let N_ξ be the unique solution to (A.2) and set $u(y) := N_\xi(\sigma y)$, $y \in Q$. By (A.1), assumption (A.3) and (A.5) we deduce that

$$\begin{aligned}
A^{\text{hom}}\xi\cdot\zeta &= \int_Q A(y')(\operatorname{curl}_{y'} N_\xi(y') + \xi)\cdot \zeta\,dy' \\
&= \int_Q A(y')(\operatorname{curl}_{y'} u(\sigma^{-1}y') + \xi)\cdot \zeta\,dy'
\end{aligned}$$

$$
\begin{aligned}
&\overset{(A.5)}{=} \int_Q A(y')(\sigma \operatorname{curl} \hat{u}(\sigma^{-1}y') + \xi) \cdot \zeta \, dy' \\
&\overset{y'=\sigma y}{=} \int_Q A(\sigma y)(\sigma \operatorname{curl} \hat{u}(y) + \xi) \cdot \zeta \, dy \\
&\overset{(A.3)}{=} \int_Q A(y)(\operatorname{curl} \hat{u}(y) + \sigma^{-1}\xi) \cdot \sigma^{-1}\zeta \, dy. \qquad (A.8)
\end{aligned}
$$

Since $N_\xi(y')$ solves (A.2), $u(\sigma^{-1}y')$ solves (A.6) for $f(y') = -A(y')\xi$ and therefore $\hat{u}$ solves (A.7) where, by (A.3), $\sigma^{-1}f(\sigma y) = -\sigma^{-1}A(\sigma y)\xi = -A(y)\sigma^{-1}\xi$. Hence, the solution $N_{\sigma^{-1}\xi}$ to (A.2), for $\zeta = \sigma^{-1}\xi$, is the projection of $\hat{u}$ onto the space $\{u \in [H^1_\#(Q)]^3 : A \operatorname{curl} u = 0\}^\perp$ and the expression in (A.8) equals $A^{\text{hom}}\sigma^{-1}\xi \cdot \sigma^{-1}\zeta$. The assertion (A.4) follows, in view of the arbitrary choice of ξ, ζ, and the equality $\sigma A^{\text{hom}}\sigma^{-1} = \sigma^{-1}A^{\text{hom}}\sigma$ which holds since σ is unitary and A^{hom} is symmetric. □

COROLLARY A.1. *If (A.3) holds for $\sigma = \sigma_k$, where σ_k is the rotation by π around the x_k-axis, then $A^{\text{hom}}_{kl} = 0$, for all $l \neq k$.*

Proof. Indeed, say for $k = 1$, (A.4) takes the form

$$
\begin{pmatrix}
A^{\text{hom}}_{11} & A^{\text{hom}}_{12} & A^{\text{hom}}_{13} \\
A^{\text{hom}}_{21} & A^{\text{hom}}_{22} & A^{\text{hom}}_{23} \\
A^{\text{hom}}_{31} & A^{\text{hom}}_{32} & A^{\text{hom}}_{33}
\end{pmatrix}
=
\begin{pmatrix}
A^{\text{hom}}_{11} & -A^{\text{hom}}_{12} & -A^{\text{hom}}_{13} \\
-A^{\text{hom}}_{21} & A^{\text{hom}}_{22} & A^{\text{hom}}_{23} \\
-A^{\text{hom}}_{31} & A^{\text{hom}}_{32} & A^{\text{hom}}_{33}
\end{pmatrix},
$$

and hence $A^{\text{hom}}_{12} = A^{\text{hom}}_{21} = A^{\text{hom}}_{13} = A^{\text{hom}}_{31} = 0$. □

Similarly, direct calculation proves the following statement.

COROLLARY A.2. *If (A.3) holds for $\sigma = \sigma_k$, where σ_k is the rotation by $\pi/2$ around the x_k-axis, then $A^{\text{hom}}_{kl} = 0$, for all $l \neq k$ and $A^{\text{hom}}_{ii} = A^{\text{hom}}_{jj}$, $i, j \neq k$.*

PROPOSITION A.2. *Let χ_0 be the characteristic function of the set Q_0. Suppose that the set Q and the coefficient $A = \epsilon_0^{-1}\chi_0 I$ are invariant under a rotation σ, that is, $\sigma Q = Q$ and $A = \epsilon_0^{-1}\chi_0 I$ satisfies (A.3), or equivalently,*

$$
\epsilon_0^{-1}(\sigma y)\chi_0(\sigma y) = \epsilon_0^{-1}(y)\chi_0(y) \quad \text{a.e. } y \in Q. \qquad (A.9)
$$

Then for all $\omega^2 \notin \{0\} \cup \{\alpha^k\}_{k=1}^\infty$ the matrix $\Gamma(\omega)$, defined by (3.9) and (3.3)–(3.6), satisfies the property

$$
\Gamma(\omega) = \sigma\Gamma(\omega)\sigma^{-1} = \sigma^{-1}\Gamma(\omega)\sigma.
$$

Proof. We make use of the representation (3.11) for $\Gamma(\omega)$ and of the equations (3.10) for the functions ϕ^k.

Multiplying (3.10) by $\psi \in [C_0^\infty(Q_0)]^3$ and integrating by parts yields

$$\begin{aligned}\int_{Q_0} \epsilon_0^{-1}(y)\,\mathrm{curl}\,\phi^k(y)\cdot\mathrm{curl}\,\psi(y)\,dy \\ = \alpha^k \int_{Q_0}\int_{Q_0} G(y-y')\,\mathrm{div}\,\phi^k(y')\,\mathrm{div}\,\psi(y)\,dy'\,dy \\ + \alpha^k \int_{Q_0} \phi^k(y)\cdot\psi(y)\,dy.\end{aligned} \tag{A.10}$$

We claim that the functions $\sigma\phi^k(\sigma^{-1}\cdot)$ satisfy the identity (A.10) with Q_0 replaced by $\sigma Q_0 := \{y \in Q : \sigma^{-1}y \in Q_0\}$. We show this by treating each term in (A.10) separately. It is clear that

$$\begin{aligned}\int_{\sigma Q_0} \sigma\phi^k(\sigma^{-1}\tilde{y})\cdot\sigma\psi(\sigma^{-1}\tilde{y})\,d\tilde{y} &= \int_{\sigma Q_0} \phi^k(\sigma^{-1}\tilde{y})\cdot\psi(\sigma^{-1}\tilde{y})\,d\tilde{y} \\ &\overset{\tilde{y}=\sigma y}{=} \int_{Q_0} \phi^k(y)\cdot\psi(y)\,dy.\end{aligned} \tag{A.11}$$

Furthermore, by utilizing the identity

$$\mathrm{curl}_{\tilde{y}}(\sigma\psi(\sigma^{-1}\tilde{y})) = (\mathrm{curl}\,\psi)(\sigma^{-1}\tilde{y}) \quad \text{a.e. } \tilde{y} \in \sigma Q_0 \text{ for all } \psi \in [H_0^1(Q_0)]^3,$$

which holds due to the fact that σ is a rotation, as well as property (A.9), we obtain

$$\begin{aligned}\int_{\sigma Q_0} \epsilon_0^{-1}(\tilde{y})\,\mathrm{curl}_{\tilde{y}}(\sigma\phi^k(\sigma^{-1}\tilde{y}))\cdot\mathrm{curl}_{\tilde{y}}(\sigma\psi(\sigma^{-1}\tilde{y}))\,d\tilde{y} \\ \overset{\tilde{y}=\sigma y}{=} \int_{Q_0} \epsilon_0^{-1}(y)\,\mathrm{curl}\,\phi^k(y)\cdot\mathrm{curl}\,\psi(y)\,dy.\end{aligned} \tag{A.12}$$

Finally, it is clear that $\mathrm{div}\,(\sigma F(\sigma^{-1}\cdot)) = (\mathrm{div}\,F)(\sigma^{-1}\cdot)$ for vector fields F and therefore

$$\begin{aligned}&\int_{\sigma Q_0}\int_{\sigma Q_0} G(\tilde{y}-\tilde{y}')\,\mathrm{div}_{\tilde{y}}(\sigma\phi^k(\sigma^{-1}\tilde{y}))\,\mathrm{div}_{\tilde{y}}(\sigma\psi(\sigma^{-1}\tilde{y}'))\,d\tilde{y}'\,d\tilde{y} \\ &\overset{\substack{\tilde{y}=\sigma y,\\ \tilde{y}'=\sigma y'}}{=} \int_{Q_0}\int_{Q_0} G(\sigma(y-y'))\,\mathrm{div}\,\phi^k(y)\,\mathrm{div}\,\psi(y')\,dy'\,dy \\ &= \int_{Q_0}\int_{Q_0} G(y-y')\,\mathrm{div}\,\phi^k(y)\,\mathrm{div}\,\psi(y')\,dy'\,dy,\end{aligned}$$

where the invariance of the Green function G under the rotation σ holds due to the assumption $\sigma Q = Q$.

The proof is concluded by combining the definition of $\Gamma(\omega)$ via (3.9), (3.3)–(3.6) and formula (3.11) applied twice, namely for the inclusion σQ_0, which coincides with Q_0 due to (A.9), and the inclusion Q_0 itself:

$$
\begin{aligned}
\Gamma(\omega) &= \omega^2 + \omega^4 \sum_{k=1}^{\infty} \frac{(\int_{\sigma Q_0} \sigma\phi^k(\sigma^{-1}\cdot)) \otimes (\int_{\sigma Q_0} \sigma\phi^k(\sigma^{-1}\cdot))}{\alpha^k - \omega^2} \\
&= \omega^2\sigma\sigma^{-1} + \omega^4 \sum_{k=1}^{\infty} \frac{(\int_{Q_0} \sigma\phi^k) \otimes (\int_{Q_0} \sigma\phi^k)}{\alpha^k - \omega^2} \\
&= \sigma\Gamma(\omega)\sigma^{-1}, \quad \omega^2 \notin \{0\} \cup \{\alpha^k\}_{k=1}^{\infty},
\end{aligned}
$$

as required. □

By analogy with Corollaries A.1 and A.2 we obtain the following statement.

COROLLARY A.3. *Under the conditions of Proposition A.2 with* $\sigma = \sigma_k$, *where* σ_k *is a rotation by* π *around the* x_k*-axis,* $\Gamma_{kl}(\omega) = 0$ *for all* $l \neq k$, $\omega^2 \notin \{0\} \cup \{\alpha^k\}_{k=1}^{\infty}$. *Moreover, if* σ_k *is a rotation by* $\pi/2$ *around the* x_k*-axis, then* $\Gamma_{ii}(\omega) = \Gamma_{jj}(\omega)$ *for* $i, j \neq k$.

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