ON KOENIGS' RATIOS FOR ITERATES OF REAL FUNCTIONS

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(Received 19 March 1968)

1. Introduction

In a recent note, M. Kuczma [5] has obtained a general result concerning real solutions $\phi(x)$ on the interval $0 \leqq x < a \leqq \infty$ of the Schröder functional equation

$$\phi[f(x)] = s\phi(x), \qquad 0 < s < 1 \tag{1.1}$$

providing the known real function satisfies the following (quite weak) conditions:

(1.2) $f(x)$ is continuous and strictly increasing in $[0, a)$;

(1.3) $f(0) = 0$ and $0 < f(x) < x$ for $x \in (0, a)$;

(1.4) $\lim_{x \to 0_+} \{f(x)/x\} = s$; and

(1.5) $f(x)/x$ is monotonic in $(0, a)$.

His theorem may be stated as follows:

THEOREM A. (Kuczma). *If conditions* (1.2)—(1.5) *hold, the limit*

$$\phi(x) = c \lim_{n\to\infty} \frac{f_n(x)}{f_n(d)}, \qquad x \in [0, a) \tag{1.6}$$

exists (*where c is an arbitrary constant, d is fixed in* $(0, a)$ *and* $f_n(x)$ *is the* n-th *functional iterate of* $f(x)$) *and defines a one-parameter family of solutions of* (1.1) *such that* $\phi(x)/x$ *is monotone on* $(0, a)$. *Moreover, all real solutions* ϕ *on* $(0, a)$ *of* (1.1), *such that* $\phi(x)/x$ *is monotone on this interval, belong to this family.*

We note before proceeding that (1.3) ensures that the functions $f_n(x)$ (with $f_0(x) = x$) are well defined for $n \geqq 0$ and positive on $(0, a)$. Moreover as $n \to \infty$ $f_n(x) \to 0$ by (1.2) and (1.3).

The ratios $\{f_n(x)/f_n(d)\}$ occurring in (1.6) are a modification introduced by Szekeres [8] of Koenigs' ratios $\{s^{-n}f_n(x)\}$ which occur in the classical theory (see e.g. [7], Chapter IV), of the Schröder equation, and whose asymptotic behaviour is of interest in its own right. The primary purpose of this paper is to completely specify the behaviour of Koenigs' ratios under the conditions (1.2)—(1.5) which ensure the validity of Kuczma's theorem.

A further section, § 3, briefly discusses our result in the context of the known theory of Koenigs' ratios and the Schröder equation. This leads in turn to a strengthening of a result due in part to each of Koenigs, Kneser and Szekeres.

2. The limit of Koenigs' ratios

THEOREM B. *Under conditions* (1.2)—(1.5), *the limit*

$$\phi(x) = \lim_{n\to\infty} \{s^{-n} f_n(x)\}, \qquad x \in [0, a) \tag{2.1}$$

exists in the sense that it is either positive, infinite, or zero throughout $(0, a)$. *A necessary and sufficient condition for* $\infty > \phi(x) > 0$ *in* $(0, a)$ *is*

$$\int_0^\delta \frac{|f(x)-sx|}{x^2}\, dx < \infty \tag{2.2}$$

where δ *is an arbitrary fixed number in* $(0, a)$. *Under condition* (2.2), $\phi(x)/x$ *is monotone in* $(0, a)$, *and* $\phi(x)$ *satisfies* (1.1), *with*

$$\lim_{x\to 0_+} \{\phi(x)/x\}(\equiv \phi'(0_+)) = 1.$$ [1]

PROOF. Let $g(x) = \{f(x)/sx\}$, $x \in (0, a)$. Then

$$s^{-n} f_n(x) = x \prod_{r=0}^{n-1} g[f_r(x)]. \tag{2.3}$$

Now, since either $0 < g(x) \leqq 1$ or $g(x) \geqq 1$ as $\{f(x)/x\}$ is decreasing or increasing, in view of (1.4) and (1.5), $\{s^{-n} f_n(x)\}$ approaches a positive finite limit if and only if

$$\sum_{n=0}^{\infty} |1-g[f_n(x)]| < \infty. \tag{2.4}$$

We now obtain geometric bounds for $f_n(x)$ as $n \to \infty$. For x sufficiently close to zero, and ε arbitrarily chosen in $(0, \min(s, 1-s))$, but fixed,

$$(s-\varepsilon)x \leqq f(x) \leqq (s+\varepsilon)x$$

from (1.4). Hence for x fixed in $(0, a)$, and $n \geqq n_0(x; \varepsilon)$

[1] Hence $\phi(x)$ belongs to the family defined by (1.6) if (2.2) holds, and is uniquely specified by $\phi'(0_+) = 1$.

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$$(s-\varepsilon)f_{n-1}(x) \leqq f_n(x) \leqq (s+\varepsilon)f_{n-1}(x)$$

so that

$$(2.5)\quad (s-\varepsilon)^n\{(s-\varepsilon)^{-n_0+1}f_{n_0-1}(x)\} \leqq f_n(x) \leqq (s+\varepsilon)^n\{(s+\varepsilon)^{-n_0+1}f_{n_0-1}(x)\}.$$

Now, since $g(x)$ is monotonic, from (1.4)

$$|1-g(A\alpha^j)| \geqq |1-g(A\alpha^{j+1})|$$

for arbitrary $A > 0$, $0 < \alpha < 1$, and $j \geqq j_0 \geqq 1$ where $A\alpha^{j_0} < a$. Hence by the Cauchy integral test

$$\sum_{j=j_0}^{\infty} |1-g(A\alpha^j)| \quad \text{and} \quad \int_{j_0}^{\infty} |1-g(A\alpha^y)|dy$$

converge or diverge together. Investigating the integral, put $A\alpha^y = w$ to obtain

$$-\frac{1}{\log\alpha}\int_0^{A\alpha^{j_0}} \frac{|1-g(w)|}{w}\,dw$$

which converges or diverges as

$$(2.6)\qquad \int_0^{\delta} \frac{|1-g(w)|}{w}\,dw \qquad (0 < \delta < a)$$

independently of the actual size of A and α.

Hence returning to (2.4) and using the geometric bounds for $f_n(x)$ in (2.5), in conjunction with the monotonicity of $g(x)$, it follows that (2.4) holds if and only if (2.6) holds, for any fixed $\delta \in (0, a)$. Hence the first assertion of the theorem.

Moreover, under (2.2), for $x \in [0, a)$

$$\begin{aligned}\phi[f(x)] &= \lim_{n\to\infty}\{s^{-n}f_{n+1}(x)\} = s\lim_{n\to\infty}\{s^{-(n+1)}f_{n+1}(x)\}\\ &= s\phi(x),\end{aligned}$$

and from the product representation

$$\frac{\phi(x)}{x} = \prod_{r=0}^{\infty} g[f_r(x)]$$

it follows that $\{\phi(x)/x\}$ is monotone in $(0, a)$, since g is, and from (1.2), $f_r(x)$ is.

Finally we may evaluate $\phi'(0_+) = \lim_{x\to 0_+}\{\phi(x)/x\}$ by noting that from (1.1) for $x \in [0, a)$

$$(2.7)\qquad \begin{aligned}\phi(x) &= \frac{\phi[f(x)]}{s} = \cdots = \frac{\phi[f_n(x)]}{s^n}\\ &= \frac{\phi[f_n(x)]}{f_n(x)}\cdot\frac{f_n(x)}{s^n}\end{aligned}$$

so that as $n \to \infty$, $\phi(x) = \phi'(0_+)\phi(x)$, whence $\phi'(0_+) = 1$ since $\infty > \phi(x) > 0$ for $x \in (0, a)$.

COROLLARY 1. *When the integral in* (2.2) *diverges,* $\phi(x) \equiv 0$ *or* $\equiv \infty$ *as* $\{f(x)/x\}$ *is decreasing or increasing.*

COROLLARY 2. *Under conditions* (1.2)—(1.4) *a real positive solution* $\phi(x)$ *on* $(0, a)$ *of* (1.1) *with* $\phi(0) = 0$ *may have* $0 < \phi'(0_+) < \infty$ *if and only if Koenigs' ratios* $\{s^{-n}f_n(x)\}$ *converge to a positive limit for all* x *in* $(0, a)$, *in which case* $\phi(x)$ *is a positive multiple of their limit.* (c.f. Kuczma [3], Theorem 8.3). (This is a direct consequence of (2.7).)

3. Complementary remarks and a general result

Since the existence of the inverse function, ϕ^{-1}, of a real solution of (1.1) is of importance in e.g. the theory of continuous Schröder iteration, we devote some discussion to this topic.

Even when condition (2.2) holds in addition to Kuczma's conditions (1.2)—(1.5), $\phi(x)$ the 'principal' Schröder function as defined by (1.6) (or (2.1)), although monotone and with $\phi'(0_+) = 1$, may still not be *continuous* and *strictly monotone* in $(0, a)$, so that ϕ^{-1} may not be defined.[2] We note however that the often cited example due to Szekeres [8] of pathological behaviour in this connection, viz.

$$(3.1) \qquad \begin{aligned} f(x) &= \frac{x}{2} + \frac{1}{3\pi} x^2 \sin\frac{\pi}{x}, \qquad & x > 0 \\ &= 0, & x = 0 \end{aligned}$$

while satisfying conditions (1.2)—(1.4) in a neighbourhood of zero, will not really serve here, since condition (1.5) is certainly not satisfied in any right neighbourhood of zero, although we note for future purposes that (2.2) is.

On the other hand, if Kuczma's condition (1.5) is replaced by the condition of convexity/concavity of $f(x)$ on $(0, a)$ (which in view of the other conditions implies monotonicity of $f(x)/x$), it is easy to see that, regardless of whether or not (2.2) holds, $\phi(x)$ as defined by (1.6) is invertible on $(0, a)$. (For related discussion, see Kuczma [3], § 12; [4], §§ 16—17.)

We need also to mention at this stage, that the work of Kneser ([2], § 2), and Szekeres ([8], § 5), implies that if conditions (1.2)—(1.3) hold, and in addition, as $x \to 0_+$

$$(3.2) \qquad f(x) = sx + O(x^{1+\gamma}), \qquad \gamma > 0$$

then the sequence $\{s^{-n}f_n(x)\}$ converges on $[0, a)$ to a solution $\phi(x)$ of the

[2] Lundberg [6], p. 200, has indicated this cannot happen if $f(x)/x$ is increasing in $(0, a)$.

Schröder equation, such that $\phi'(0_+) = 1$. It is interesting to note that these assumptions imply that (2.2) is satisfied if δ is small. This suggests that it may be possible to replace assumption (3.2) by (2.2) in general, providing (1.2)—(1.4) hold, to obtain the same conclusion. However, as the relevant portions of the proof of Theorem B then break down, since $g(x) = \{f(x)/sx\}$ is not necessarily monotone, we put this question aside, and pass onto a related one.

There is clearly a gap between our Theorem B and the Kneser-Szekeres result; to bridge it, it is necessary to find a condition which is equivalent to (2.2) when $\{f(x)/x\}$ is monotone, and implied by (3.2) — when the standard conditions (1.2)—(1.4) hold *a priori* in each case — but which is itself sufficient to yield convergence of Koenigs' ratios. The following result (Theorem C) is of the appropriate kind, as will be seen from its Corollary, in conjunction with Theorem D.

THEOREM C. *Suppose $f(x)$ satisfies conditions* (1.2)—(1.4) *and for some fixed ρ, $s < \rho < \min(1, 2s)$ also satisfies*

$$\sum_{n=0}^{\infty}\{\sup_{A_1\rho_1^n \leqq t \leqq A_2\rho_2^n} |1-g(t)|\} < \infty, \tag{3.3}$$

where $\rho_1 = 2s-\rho$, $\rho_2 = \rho$, for any two constants A_1, A_2 satisfying $0 < A_1 < A_2 < a$.

Then the sequence $\{s^{-n}f_n(x)\}$ converges on $[0, a)$ to a solution, $\phi(x)$, positive on $(0, a)$ with $\phi'(0_+) = 1$, of the Schröder equation.

PROOF. Suppose x is a fixed number in $[\beta_1, \beta_2] \subset (0, a)$. Then we may proceed as in the proof of Theorem B upto (2.5) but choose the fixed ε initially to be $\varepsilon = \rho - s$. Then we have for $n \geqq n_0(x; \varepsilon)$

$$\rho_1^n\{\rho_1^{-n_0+1}f_{n_0-1}(x)\} \leqq f_n(x) \leqq \rho_2^n\{\rho_2^{-n_0+1}f_{n_0-1}(x)\},$$

and since for $n \geqq 0$, $f_n(x)$ is strictly monotone increasing on $(0, a)$

$$\rho_1^n\{\rho_1^{-n_0+1}f_{n_0-1}(\beta_1)\} \leqq f_n(x) \leqq \rho_2^n\{\rho_2^{-n_0+1}f_{n_0-1}(\beta_2)\}.$$

Moreover, for $n \geqq n_1 = \max(n_0, r)$

$$\rho_1^{n-r}\{\rho_1^{r-n_0+1}f_{n_0-1}(\beta_1)\} \leqq f_n(x) \leqq \rho_2^{n-r}\{\rho_2^{r-n_0+1}f_{n_0-1}(\beta_2)\} \tag{3.4}$$

where $r = r(x; \varepsilon)$ is fixed and chosen so that

$$\rho_2^{r-n_0+1}f_{n_0-1}(\beta_2) < a.$$

Hence making the choice

$$A_i = \rho_i^{r-n_0+1}f_{n_0-1}(\beta_i), \qquad i = 1, 2$$

and invoking (3.4) it follows that for $n \geqq n_1$

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$$|1-g(f_n(x))| \leqq \{\sup_{A_1\rho_1^{n-r} \leqq t \leqq A_2\rho_2^{n-r}} |1-g(t)|\}$$

whence

$$\sum_n |1-g(f_n(x))| < \infty$$

by (3.3). Therefore the product

$$x\prod_{r=0}^{\infty}\{1-[1-g(f_r(x))]\} = x\prod_{r=0}^{\infty} g(f_r(x)) = \lim_{n\to\infty}\{s^{-n}f_n(x)\}$$

is absolutely convergent, and hence convergent [3] for arbitrary fixed x in arbitrary $[\beta_1, \beta_2] \subset (0, a)$ i.e. for arbitrary $x \in (0, a)$.

The remaining assertions of the theorem now follow easily, precisely as in the proof of Theorem B.

Corollary. *If $f(x)$ satisfies conditions* (1.1)—(1.3), *then* (3.2) *implies that* (3.3) *holds.*

Proof. From (3.2)

$$|1-g(t)| \leqq Ct^{\gamma}, \qquad \gamma > 0, \quad 0 < C = \text{const.},$$

for t sufficiently small and positive. Hence for *any* ρ_1, ρ_2, A_1, A_2 satisfying *only*

$$0 < \rho_1 < \rho_2 < 1, \qquad 0 < A_1 < A_2 < a \tag{3.5}$$

$$\sup_{A_1\rho_1^n \leqq t \leqq A_2\rho_2^n} |1-g(t)| \leqq CA_2^{\gamma}\rho_2^{n\gamma}$$

if n is sufficiently large, whence (3.3) follows easily.

Theorem D. *Under conditions* (1.1)—(1.4), *condition* (3.3) — *even with the weaker restrictions* (3.5) *on the constants* ρ_1, ρ_2, — *implies* (2.2). *If also $g(x)$ is monotone,* (3.3), *under the weaker restriction* (3.5) *on the ρ_i's, and* (2.2) *are equivalent.*

Proof. From (3.3), for any A_1, A_2, $0 < A_1 < A_2 < a$

$$\begin{aligned}
\infty &> \sum_{n=0}^{\infty}\{\sup_{A_1\rho_1^n \leqq t \leqq A_2\rho_2^n} |1-g(t)|\} \\
&\geqq \sum_{n=0}^{\infty}\{\sup_{A_1\rho_2^n \leqq t \leqq A_2\rho_2^n} |1-g(t)|\} \\
&= \frac{A_1}{A_2-A_1}\sum_{n=0}^{\infty}\left\{\frac{\sup_{A_1\rho_2^n \leqq t \leqq A_2\rho_2^n}|1-g(t)|}{A_1\rho_2^n}\right\}(A_2-A_1)\rho_2^n \\
&\geqq \frac{A_1}{A_2-A_1}\sum_{n=0}^{\infty}\left\{\sup_{A_1\rho_2^n \leqq t \leqq A_2\rho_2^n}\left(\frac{|1-g(t)|}{t}\right)\right\}(A_2-A_1)\rho_2^n,
\end{aligned}$$

[3] We note that assumption (1.3) precludes the possibility $g(x) = 0$ for any $x \in (0, a)$.

and since $A_2\rho_2^{n-1} \geqq A_1\rho_2^n$ on account of (3.5),

$$\geqq \frac{A_1}{A_2-A_1}\int_0^{A_2} \frac{|1-g(t)|}{t}\,dt$$

whence (2.2) follows immediately.

To prove the second assertion we need only prove that if $g(x)$ is monotone, (2.2) implies (3.3), in view of the above. The proof of Theorem B shows that (2.2) implies that

$$\sum_{j=0}^{\infty} |1-g(A\alpha^j)| < \infty$$

for every A, α satisfying only $0 < A < a$, $0 < \alpha < 1$. Hence for any A_i, ρ_i satisfying only (3.5)

$$\sum_{j=0}^{\infty} |1-g(A_i\rho_i^j)| < \infty, \qquad i = 1, 2. \tag{3.6}$$

But since $g(x)$ tends to unity monotonically

$$\sup_{A_1\rho_1^n \leqq t \leqq A_2\rho_2^n} |1-g(t)| = \max_{i=1,2} |1-g(A_i\rho_i^n)|$$

for $n \geqq 0$, whence our assertion follows from (3.6).

In conclusion we need to remark that the procedure in the proof of Theorem B (viz. the use of the integral 'test' after a geometric bounding of the iterates) is a generalisation of the technique in the proof of the main Lemma of [1]. That probabilistic result is basically due to D. Vere-Jones.

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