*Forum of Mathematics, Sigma* (2021), **Vol. 9**:e25 1–45
doi:10.1017/fms.2021.19

RESEARCH ARTICLE

# The classification of symmetry protected topological phases of one-dimensional fermion systems

Chris Bourne [1,2] and Yoshiko Ogata [3]

[1]WPI-AIMR, Tohoku University, 2-1-1 Katahira, Aoba-ku, Sendai, 980-8577, Japan.
[2]RIKEN iTHEMS, Wako, Saitama 351-0198, Japan; E-mail: chris.bourne@tohoku.ac.jp.
[3]Graduate School of Mathematical Sciences, The University of Tokyo, Komaba, Tokyo, 153-8914, Japan; E-mail: yoshiko@ms.u-tokyo.ac.jp.

**Received:** 26 June 2020; **Revised:** 29 January 2021; **Accepted:** 16 February 2021

**2020 Mathematics Subject Classification:** *Primary* – 81R15; *Secondary* – 46L30

**Abstract**
We introduce an index for symmetry-protected topological (SPT) phases of infinite fermionic chains with an on-site symmetry given by a finite group $G$. This index takes values in $\mathbb{Z}_2 \times H^1(G,\mathbb{Z}_2) \times H^2(G,U(1)_{\mathfrak{p}})$ with a generalised Wall group law under stacking. We show that this index is an invariant of the classification of SPT phases. When the ground state is translation invariant and has reduced density matrices with uniformly bounded rank on finite intervals, we derive a fermionic matrix product representative of this state with on-site symmetry.

## 1. Introduction

The notion of symmetry-protected topological (SPT) phases was introduced by Gu and Wen [16]. We consider the set of all Hamiltonians with a prescribed symmetry and that have a unique gapped ground state in the bulk. Two Hamiltonians in this set are equivalent if there is a smooth path within the set connecting them. We may classify the Hamiltonians in this family by this equivalence relation. The equivalence class of a Hamiltonian with only on-site interactions is regarded as a trivial phase. If a phase is nontrivial, it is called an SPT phase (see also Remark 1.2).

A basic question is how to show that a given Hamiltonian belongs to an SPT phase. A mathematically natural approach for this problem is to define an invariant of the classification. This approach has been studied in the physics literature using matrix product states (MPS) [35, 36, 16, 12, 37]. MPS is a powerful framework introduced in [13], after the discovery of the famous Affleck-Kennedy-Lieb-Tasaki (AKLT) model [1]. Hastings showed that MPS approximates unique gapped ground states of quantum spin chains well [17]. However, we cannot comprehensively study invariants of the path-connected components of the space of unique gapped ground states via MPS only. Firstly, MPS are *translationally invariant* systems and we would like to define an invariant that does not require this assumption. Furthermore, an approximation of a gapped ground state by MPS may not be compatible with the path-connected components and so is insufficient to define an index in general. If the index is not defined for *all* unique gapped ground states, there is no way to discuss whether it is actually an invariant or not.

In [29, 30, 31], an index for SPT phases with on-site finite group symmetry and global reflection symmetry was defined for infinite quantum spin chains in a fully general setting. In these papers, it was proven that the index is actually an invariant of the classification of SPT phases. An important observation for stability of the index is the factorisation property of the automorphic equivalence. The key ingredient for the definition of the index is the split property of unique gapped ground states, proven by Matsui [23]. The index introduced in [29, 30] generalises the indices introduced for MPS in [35, 36, 16, 12, 37], where an MPS emerges naturally from a translation-invariant split state whose reduced density matrix has uniformly bounded rank on finite intervals [7, 23].

In this article, we are interested in the analogous problem for fermionic chains with on-site finite group symmetries. Fermionic SPT phases for finite systems in one dimension have already been extensively studied in the physics literature [14, 15, 11, 19, 20, 39]. In contrast to quantum spin chains, for parity-symmetric gapped ground states without additional symmetries, there are two distinct phases. A $\mathbb{Z}_2$-index to distinguish these phases in infinite systems was introduced in [4] and independently in [24]. It was outlined in [4] that this $\mathbb{Z}_2$-index is an invariant of the classification of unique parity-invariant gapped ground state phases using techniques from [29] and [28]. The aim of this article is to extend the analysis of fermionic gapped ground states to the case with an on-site symmetry; namely, a classification of one-dimensional fermionic SPT phases.

### *1.1. Setting and outline*

We assume that the reader has some familiarity with the basics of operator algebras and their application to quantum statistical mechanical systems; see [8, 9]. Throughout this article, we fix $d \in \mathbb{N}$. Let $\mathfrak{h} := l^2(\mathbb{Z}) \otimes \mathbb{C}^d$ and $\mathcal{A}$ be the CAR-algebra over $\mathfrak{h}$; that is, the universal $C^*$-algebra generated by the identity and $\{a(f)\}_{f\in\mathfrak{h}}$ such that $f \mapsto a(f)$ is anti-linear and

$$\{a(f_1), a(f_2)\} = 0, \qquad \{a(f_1), a(f_2)^*\} = \langle f_1, f_2\rangle. \tag{1.1}$$

For each subset $X$ of $\mathbb{Z}$, we set $\mathfrak{h}_X := l^2(X) \otimes \mathbb{C}^d$ and denote by $\mathcal{A}_X$ the CAR-algebra over $\mathfrak{h}_X$. We naturally regard $\mathcal{A}_X$ as a subalgebra of $\mathcal{A}$. We also use the notation $\mathcal{A}_R := \mathcal{A}_{\mathbb{Z}_{\geq 0}}$ and $\mathcal{A}_L := \mathcal{A}_{\mathbb{Z}_{<0}}$. We denote the set of all finite subsets in $\mathbb{Z}$ by $\mathfrak{S}_{\mathbb{Z}}$ and set $\mathcal{A}_{\rm loc} := \bigcup_{X\in\mathfrak{S}_{\mathbb{Z}}} \mathcal{A}_X$. Given a Hilbert space $\mathfrak{K}$, the fermionic Fock space of anti-symmetric tensors is denoted by $\mathcal{F}(\mathfrak{K})$. For a unitary/anti-unitary operator $U$ on $\mathbb{C}^d$, we denote the second quantisation of $U$ on the Fock space $\mathcal{F}(\mathbb{C}^d)$ by $\Gamma(U)$.

By the universality of the CAR-algebra, for any unitary/anti-unitary $w$ on $\mathfrak{h}$, we may define a linear/anti-linear automorphism $\beta_w$ on $\mathcal{A}$ such that $\beta_w(a(f)) = a(wf)$, $f \in \mathfrak{h}$. In particular, for $w = -\mathbb{I}$, we obtain the parity operator $\Theta := \beta_{-\mathbb{I}}$. For each $X \in \mathfrak{S}_{\mathbb{Z}}$, $\mathcal{A}_X$ is $\Theta$-invariant. We denote the restriction $\Theta|_{\mathcal{A}_X}$ by $\Theta_X$. For $\sigma = 0, 1$, the set of elements $A$ in $\mathcal{A}$ with $\Theta(A) = (-1)^\sigma A$ is denoted by $\mathcal{A}^{(\sigma)}$. Elements in $\mathcal{A}^{(0)}$ are said to be even and elements in $\mathcal{A}^{(1)}$ are said to be odd.

In this article, we consider an on-site symmetry given by a finite group $G$. We let $\mathrm{M}_d$ denote the algebra of $d \times d$ matrices with complex entries and consider a projective unitary/anti-unitary representation of $G$ on $\mathbb{C}^d$ relative to a group homomorphism $\mathfrak{p} : G \to \mathbb{Z}_2$.[1] That is, there is a projective representation $U$ of $G$ on $\mathbb{C}^d$ such that $U_g$ is unitary if $\mathfrak{p}(g) = 0$ and anti-unitary if $\mathfrak{p}(g) = 1$. Because $U$ is projective, there is a 2-cocycle $\upsilon : G \times G \to U(1)$ such that $U_g U_h = \upsilon(g, h) U_{gh}$ and for all $f, g, h \in G$

$$\upsilon(e, g) = 1 = \upsilon(g, e), \qquad \frac{\overline{\upsilon(g,h)}^{\mathfrak{p}(f)}\upsilon(f, gh)}{\upsilon(f, g)\upsilon(fg, h)} = 1, \tag{1.2}$$

where $\overline{z}^{\mathfrak{p}(f)} = z$ if $\mathfrak{p}(f) = 0$ and $\overline{z}^{\mathfrak{p}(f)} = \overline{z}$ if $\mathfrak{p}(f) = 1$. For a fixed homomorphism $\mathfrak{p}$, equivalence classes of such 2-cocycles give rise to the cohomology group $H^2(G, U(1)_{\mathfrak{p}})$.

For a fixed projective unitary/anti-unitary representation $U$ of $G$ on $\mathbb{C}^d$ relative to $\mathfrak{p} : G \to \mathbb{Z}_2$, we can extend this representation to an on-site representation $\bigoplus_{\mathbb{Z}} U$ on $l^2(\mathbb{Z}) \otimes \mathbb{C}^d$. We therefore can define the linear/anti-linear automorphism $\alpha$ on $\mathcal{A}$, where

$$\alpha_g := \beta_{(\bigoplus_{\mathbb{Z}} U_g)}, \quad g \in G. \tag{1.3}$$

[1]Throughout this article we use the presentation of $\mathbb{Z}_2$ as the additive group $\{0, 1\}$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

If $\mathfrak{p}(g) = 0$, then $\alpha_g$ is an automorphism on $\mathcal{A}$ and if $\mathfrak{p}(g) = 1$, then $\alpha_g$ is an anti-linear automorphism on $\mathcal{A}$. Note that $\alpha$ satisfies

$$\alpha_g \circ \Theta = \Theta \circ \alpha_g, \qquad \alpha_g(\mathcal{A}_X) = \mathcal{A}_X, \quad g \in G, \quad X \in \mathfrak{S}_{\mathbb{Z}}. \tag{1.4}$$

For each $g \in G$ and a state $\varphi$ on $\mathcal{A}$, we define a state $\varphi_g$ by $\varphi_g(A) = \varphi \circ \alpha_g(A)$, $A \in \mathcal{A}$ if $\mathfrak{p}(g) = 0$, and by $\varphi_g(A) = \varphi \circ \alpha_g(A^*)$, $A \in \mathcal{A}$ if $\mathfrak{p}(g) = 1$. We say that $\varphi$ is $\alpha$-invariant if $\varphi_g = \varphi$ for any $g \in G$.

In the latter half of the article we also consider space translations $\beta_{S_x}$, $x \in \mathbb{Z}$. Here the unitary $S_x$ is given by $S_x = s_x \otimes \mathbb{I}_{\mathbb{C}^d}$ with $s_x$ the shift by $x$ on $l^2(\mathbb{Z})$.

Throughout this article, for a state $\varphi$ on $\mathcal{A}_X$ (with $X$ a subset of $\mathbb{Z}$), $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ denotes a Gelfand-Naimark-Segal (GNS) triple of $\varphi$. When $\varphi$ is $\Theta_X$-invariant, then $\hat{\Gamma}_\varphi$ denotes the self-adjoint unitary on $\mathcal{H}_\varphi$ defined by $\hat{\Gamma}_\varphi \pi_\varphi(A)\Omega_\varphi = \pi_\varphi \circ \Theta_X(A)\Omega_\varphi$ for $A \in \mathcal{A}_X$. If $\varphi$ is $\alpha$-invariant, then we denote by $\hat{\alpha}_\varphi$ the extension of $\alpha|_{\mathcal{A}_X}$ to $\pi_\varphi(\mathcal{A}_X)''$.

The mathematical model of a one-dimensional fermionic system is fully specified by the interaction $\Phi$. An interaction is a map $\Phi$ from $\mathfrak{S}_{\mathbb{Z}}$ into $\mathcal{A}_{\mathrm{loc}}$ such that $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X) = \Phi(X)^*$ for $X \in \mathfrak{S}_{\mathbb{Z}}$. When we have $\Theta(\Phi(X)) = \Phi(X)$ for all $X \in \mathfrak{S}_{\mathbb{Z}}$, $\Phi$ is said to be even. We say that $\Phi$ is $\alpha$-invariant if we have $\alpha_g(\Phi(X)) = \Phi(X)$ for all $X \in \mathfrak{S}_{\mathbb{Z}}$ and $g \in G$. An interaction $\Phi$ is translation invariant if $\Phi(X + x) = \beta_{S_x}(\Phi(X))$, for all $x \in \mathbb{Z}$ and $X \in \mathfrak{S}_{\mathbb{Z}}$. Furthermore, an interaction $\Phi$ is finite range if there exists an $m \in \mathbb{N}$ such that $\Phi(X) = 0$ for any $X$ with diameter larger than $m$. We denote by $\mathcal{B}_f^e$ the set of all finite range even interactions $\Phi$ that satisfy

$$\sup_{X \in \mathfrak{S}_{\mathbb{Z}}} \|\Phi(X)\| < \infty. \tag{1.5}$$

For an interaction $\Phi$ and a finite set $\Lambda \in \mathfrak{S}_{\mathbb{Z}}$, we define the local Hamiltonian on $\Lambda$ by

$$H_{\Lambda,\Phi} := \sum_{X \subset \Lambda} \Phi(X). \tag{1.6}$$

The dynamics given by this local Hamiltonian is denoted by

$$\tau_t^{\Phi,\Lambda}(A) := e^{itH_{\Lambda,\Phi}} A e^{-itH_{\Lambda,\Phi}}, \quad t \in \mathbb{R}, \quad A \in \mathcal{A}. \tag{1.7}$$

If $\Phi$ belongs to $\mathcal{B}_f^e$, the limit

$$\tau_t^{\Phi}(A) = \lim_{\Lambda \to \mathbb{Z}} \tau_t^{\Phi,\Lambda}(A) \tag{1.8}$$

exists for each $A \in \mathcal{A}$ and $t \in \mathbb{R}$ and defines a strongly continuous one-parameter group of automorphisms $\tau^\Phi$ on $\mathcal{A}$ (see Appendix B). We denote the generator of $\tau^\Phi$ by $\delta_\Phi$.

For $\Phi \in \mathcal{B}_f^e$, a state $\varphi$ on $\mathcal{A}$ is called a $\tau^\Phi$-ground state if the inequality $-i\,\varphi(A^*\delta_\Phi(A)) \geq 0$ holds for any element $A$ in the domain $\mathcal{D}(\delta_\Phi)$ of $\delta_\Phi$. If $\varphi$ is a $\tau^\Phi$-ground state with GNS triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$, then there exists a unique positive operator $H_{\varphi,\Phi}$ on $\mathcal{H}_\varphi$ such that $e^{itH_{\varphi,\Phi}}\pi_\varphi(A)\Omega_\varphi = \pi_\varphi(\tau_t^\Phi(A))\Omega_\varphi$, for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. We call this $H_{\varphi,\Phi}$ the bulk Hamiltonian associated with $\varphi$. Note that $\Omega_\varphi$ is an eigenvector of $H_{\varphi,\Phi}$ with eigenvalue 0. The following definition clarifies what we mean by a model with a unique gapped ground state.

**Definition 1.1.** We say that a model with an interaction $\Phi \in \mathcal{B}_f^e$ has a unique gapped ground state if (i) the $\tau^\Phi$-ground state, which we denote as $\varphi$, is unique and (ii) there exists a $\gamma > 0$ such that $\sigma(H_{\varphi,\Phi}) \setminus \{0\} \subset [\gamma, \infty)$, where $\sigma(H_{\varphi,\Phi})$ is the spectrum of $H_{\varphi,\Phi}$.

Note that the uniqueness of $\varphi$ implies that 0 is a nondegenerate eigenvalue of $H_{\varphi,\Phi}$.

If $\varphi$ is a $\tau^\Phi$-ground state of an $\alpha$-invariant and $\Theta$-invariant interaction $\Phi \in \mathcal{B}_f^e$, then $\varphi \circ \Theta$ and $\varphi_g$ is also a $\tau^\Phi$-ground state for each $g \in G$. In particular, if $\varphi$ is a unique $\tau^\Phi$-ground state, it is pure,

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

$\Theta$-invariant and $\alpha$-invariant. We denote by $\mathcal{G}_f^{e,\alpha}$ the set of all $\alpha$-invariant interactions $\Phi \in \mathcal{B}_f^e$ with a unique gapped ground state.

Now the classification problem of SPT phases is the classification of $\mathcal{G}_f^{e,\alpha}$ with respect to the following equivalence relation: $\Phi_0, \Phi_1 \in \mathcal{G}_f^{e,\alpha}$ are equivalent if there is a smooth path in $\mathcal{G}_f^{e,\alpha}$ connecting them. (See Section 3 for a more precise definition.)

We now outline the main results of the article. In Section 2, we introduce an index for $\Theta$-invariant and $\alpha$-invariant gapped ground states in a fully general setting. This index takes value in $\mathbb{Z}_2 \times H^1(G,\mathbb{Z}_2) \times H^2(G,U(1)_{\mathfrak{p}})$, which is analogous to the indices introduced in [19] in the context of spin-topological quantum field theory (spin-TQFT) and [11, 20, 39] for the fermionic MPS setting. When $G$ is trivial, the index is $\mathbb{Z}_2$-valued and recovers the index studied in [4, 24]. The key ingredient for the definition is again the split property of unique gapped ground states for fermionic systems proven recently in [24]. In Section 3, we show that our defined index is an invariant of the classification; that is, it is stable along the smooth path in $\mathcal{G}_f^{e,\alpha}$.

Because our index takes values in a group, it suggests that one may compose fermionic SPT phases to obtain a new phase with index determined from the original systems. In the physics literature, this is achieved by stacking of systems; see [15, 39], for example. Mathematically this operation corresponds to a (graded) tensor product of ground states. In Section 4, we show that our index is indeed closed under this tensor product operation. However, despite the notation, the group operation on $\mathbb{Z}_2 \times H^1(G,\mathbb{Z}_2) \times H^2(G,U(1)_{\mathfrak{p}})$ is *not* the direct sum but rather a twisted product that follows a generalised Wall group law; cf. [40]. As an example, we consider the case of an anti-linear $\mathbb{Z}_2$-action (e.g., an on-site time-reversal symmetry) and show that our index takes values in $\mathbb{Z}_8$. This recovers the $\mathbb{Z}_8$-classification of time-reversal symmetric one-dimensional fermionic SPT phases noted in [14, 15] and extends this classification to infinite systems.

In Sections 5 and 6 we consider the unique ground state of a translation invariant $\Phi \in \mathcal{G}_f^{e,\alpha}$. For quantum spin systems, it is known that a representation of Cuntz algebra emerges from translation invariant pure split states [7, 21]. The generators of this Cuntz algebra representation give an operator product representation of the state and also implement the space translation. We find an analogous object for fermionic systems in Section 5. Because odd elements with disjoint support anti-commute in the CAR-algebra, the operator product representation and space translation is more complicated than the quantum spin chain setting. The results of Section 5 are then applied to the study of fermionic MPS in Section 6. When the rank of the reduced density matrices of the infinite volume ground state is uniformly bounded, we show that the ground state has a presentation as a fermionic MPS with on-site symmetry. We then show that our index agrees with and therefore extends the indices defined for fermionic MPS with an on-site symmetry in [7, 20, 39].

Basic properties of graded von Neumann algebras are reviewed in Appendix A. In Appendix B we adapt the Lieb-Robinson bound to the setting of lattice fermions (see also [10, 27]).

**Remark 1.2** (A note on terminology)**.** For the sake of clarity, let us more carefully specify the characterisation of an SPT phase used in this article. Given a $G$-symmetric unique gapped ground state, an SPT phase is often defined to be an equivalence class of ground states that can be connected to a ground state from an on-site interaction but that cannot be connected $G$-equivariantly. In this article, we define a $\mathbb{Z}_2 \times H^1(G,\mathbb{Z}_2) \times H^2(G,U(1)_{\mathfrak{p}})$-valued invariant for *any* unique gapped ground state of a one-dimensional fermionic interaction and do *not* assume that the ground state can be connected to a ground state from an on-site interaction without symmetry.

## 2. The index of fermionic SPT phases

### *2.1. Graded von Neumann algebras and dynamical systems*

In order to introduce the index, we first need to introduce type I central balanced graded $W^*$-$(G,\mathfrak{p})$-dynamical systems. Further details on graded von Neumann algebras can be found in Appendix A.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

**Definition 2.1.** A graded von Neumann algebra is a pair $(\mathcal{M}, \theta)$ with $\mathcal{M}$ a von Neumann algebra $\theta$ an involutive automorphism on $\mathcal{M}$, $\theta^2 = \mathrm{Id}$. If $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ and there is a self-adjoint unitary $\Gamma$ on $\mathcal{H}$ such that $\mathrm{Ad}_{\Gamma}|_{\mathcal{M}} = \theta$, then we call $(\mathcal{M}, \theta)$ a spatially graded von Neumann algebra acting with grading operator $\Gamma$. If $\theta$ is the identity automorphism, then we say that $(\mathcal{M}, \theta)$ is trivially graded.

We say that a graded von Neumann algebra $(\mathcal{M}, \theta)$ is of type $\lambda$, $\lambda \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, if $\mathcal{M}$ is type $\lambda$.

Given a graded von Neumann algebra $(\mathcal{M}, \theta)$, $\mathcal{M}$ is a direct sum of two self-adjoint $\sigma$-weakly closed linear subspaces as $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}$, where

$$\mathcal{M}^{(\sigma)} := \{x \in \mathcal{M} \mid \theta(x) = (-1)^{\sigma} x\}, \quad x \in \mathcal{M}, \ \sigma \in \{0, 1\}. \tag{2.1}$$

An element of $\mathcal{M}^{(\sigma)}$ is said to be homogeneous of degree $\sigma$ or even/odd for $\sigma = 0/\sigma = 1$, respectively. For a homogeneous $x \in \mathcal{M}$, its degree is denoted by $\partial x$. For graded von Neumann algebras $(\mathcal{M}_1, \theta_1)$, $(\mathcal{M}_2, \theta_2)$, a homomorphism $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a graded homomorphism if $\phi\big(\mathcal{M}_1^{(\sigma)}\big) \subset \mathcal{M}_2^{(\sigma)}$ for $\sigma = 0, 1$.

**Definition 2.2.** Let $(\mathcal{M}, \theta)$ be a graded von Neumann algebra. We say that $(\mathcal{M}, \theta)$ is balanced if $\mathcal{M}$ contains an odd self-adjoint unitary. If $Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}\mathbb{I}$ for the center $Z(\mathcal{M})$ of $\mathcal{M}$, we say that $(\mathcal{M}, \theta)$ is central.

We now consider dynamics on graded von Neumann algebras via a linear/anti-linear group action.

**Definition 2.3.** Let $G$ be a finite group and $\mathfrak{p} : G \to \mathbb{Z}_2$ be a group homomorphism. A graded $W^*$-$(G, \mathfrak{p})$-dynamical system $(\mathcal{M}, \theta, \hat{\alpha})$ is a graded von Neumann algebra $(\mathcal{M}, \theta)$ with an action $\hat{\alpha}$ of $G$ on $\mathcal{M}$ such that $\hat{\alpha}_g$ is a linear automorphism if $\mathfrak{p}(g) = 0$ and $\hat{\alpha}_g$ is an anti-linear automorphism if $\mathfrak{p}(g) = 1$, satisfying $\hat{\alpha}_g \circ \theta = \theta \circ \hat{\alpha}_g$.

We consider some key examples that will play an important role in defining our index. We fix a group homomorphism $\mathfrak{p} : G \to \mathbb{Z}_2$ and consider projective unitary/anti-unitary representations $V$ of $G$ relative to $\mathfrak{p}$ (see Subsection 1.1 for the definition).

**Example 2.4** $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$**.** Let $\mathcal{K}$ be a Hilbert space and set $\Gamma_{\mathcal{K}} := \mathbb{I}_{\mathcal{K}} \otimes \sigma_z$, a self-adjoint unitary on $\mathcal{K} \otimes \mathbb{C}^2$.[2] We set $\mathcal{R}_{0,\mathcal{K}} := \mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2$ and so $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}})$ is a spatially graded von Neumann algebra acting on $\mathcal{K} \otimes \mathbb{C}^2$ with grading operator $\Gamma_{\mathcal{K}}$. Let $V$ be a projective unitary/anti-unitary representation of $G$ on $\mathcal{K} \otimes \mathbb{C}^2$ relative to $\mathfrak{p}$. We also assume that there is a homomorphism $\mathfrak{q} : G \to \mathbb{Z}_2$ such that $\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}(g)} \Gamma_{\mathcal{K}}$. We then obtain a graded $W^*$-$(G, \mathfrak{p})$-dynamical system $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$.

We denote the set of all $W^*$-$(G, \mathfrak{p})$-dynamical systems of the form in Example 2.4 by $\mathcal{S}_0$.

**Example 2.5** $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$**.** Let $\mathcal{K}$ be a Hilbert space and set $\Gamma_{\mathcal{K}} := \mathbb{I}_{\mathcal{K}} \otimes \sigma_z$. Let $\mathfrak{C}$ be the subalgebra of $\mathrm{M}_2$ generated by $\sigma_x$ and set $\mathcal{R}_{1,\mathcal{K}} := \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$.[3] Then $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}})$ is a spatially graded von Neumann algebra acting on $\mathcal{K} \otimes \mathbb{C}^2$ with grading operator $\Gamma_{\mathcal{K}}$. Let $V$ be a projective unitary/anti-unitary representation of $G$ relative to $\mathfrak{p}$ such that $\mathrm{Ad}_{V_g}(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x) = (-1)^{\mathfrak{q}(g)} (\mathbb{I}_{\mathcal{K}} \otimes \sigma_x)$ and $\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}(g)} \Gamma_{\mathcal{K}}$ for $\mathfrak{q} : G \to \mathbb{Z}_2$ a group homomorphism. These assumptions imply that $\mathrm{Ad}_{V_g}\left(\mathcal{R}_{1,\mathcal{K}}\right) = \mathcal{R}_{1,\mathcal{K}}$ and so $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$ is a graded $W^*$-$(G, \mathfrak{p})$-dynamical system.

We denote the set of all $W^*$-$(G, \mathfrak{p})$-dynamical systems of the form of Example 2.5 by $\mathcal{S}_1$. Given a $W^*$-$(G, \mathfrak{p})$-dynamical systems in $\mathcal{S}_1$, we can construct a projective representation of $G$ on $\mathcal{K}$ from the projective representation on $\mathcal{K} \otimes \mathbb{C}^2$.

We first establish some notation. Let $C$ be the complex conjugation on $\mathbb{C}^2$ with respect to the standard basis. Given two group homomorphisms $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Hom}(G, \mathbb{Z}_2) \cong H^1(G, \mathbb{Z}_2)$, we can define a group

[2]In this article we use the following notation of Pauli matrices:

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[3]We may regard $\mathfrak{C}$ as Clifford algebra $\mathbb{C}l_1$ generated by $e_1 := \sigma_x$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

2-cocycle,

$$\epsilon(\mathfrak{q}_1, \mathfrak{q}_2)(g,h) = (-1)^{\mathfrak{q}_1(g)\mathfrak{q}_2(h)}, \quad g,h \in G. \tag{2.2}$$

**Remark 2.6.** Note that $[\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)] = [\epsilon(\mathfrak{q}_2,\mathfrak{q}_1)] \in H^2(G, U(1)_{\mathfrak{q}_1})$.

**Lemma 2.7.** *For* $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_1$, *there is a unique projective unitary/anti-unitary representation* $V^{(0)}$ *of* $G$ *on* $\mathcal{K}$ *relative to* $\mathfrak{p}$ *such that* $V_g = V_g^{(0)} \otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}(g)}$. *If* $[\tilde{v}]$ *and* $[v]$ *are the second cohomology classes associated to* $V$ *and* $V^{(0)}$ *respectively, then* $[\tilde{v}] = [v\,\epsilon(\mathfrak{q},\mathfrak{p})] \in H^2(G, U(1)_\mathfrak{p})$.

*Proof.* Because $\mathrm{Ad}_{V_g} \circ \mathrm{Ad}_{\Gamma_\mathcal{K}} = \mathrm{Ad}_{\Gamma_\mathcal{K}} \circ \mathrm{Ad}_{V_g}$, we have $\mathrm{Ad}_{V_g}(\mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\mathbb{I}_{\mathbb{C}^2}) = \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\mathbb{I}_{\mathbb{C}^2}$. Therefore, $\mathrm{Ad}_{V_g}$ induces a linear/anti-linear $*$-automorphism on $\mathcal{B}(\mathcal{K})$. Applying Wigner's theorem, there is a unitary/anti-unitary $\tilde{V}_g^{(0)}$ on $\mathcal{K}$ such that

$$\mathrm{Ad}_{V_g}(x \otimes \mathbb{I}_{\mathbb{C}^2}) = \mathrm{Ad}_{\tilde{V}_g^{(0)}}(x) \otimes \mathbb{I}_{\mathbb{C}^2}, \quad x \in \mathcal{B}(\mathcal{K}). \tag{2.3}$$

It is clear that $\tilde{V}^{(0)}$ gives a unitary/anti-unitary projective representation relative to $\mathfrak{p}$. Note that $V_g^*\big(\tilde{V}_g^{(0)} \otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}(g)}\big)$ is a unitary that commutes with $\mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\mathbb{I}_{\mathbb{C}^2}$, $\mathbb{I}_\mathcal{K} \otimes \sigma_x$, $\mathbb{I}_\mathcal{K} \otimes \sigma_z$ and therefore commutes with $\mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2$. Therefore, there is a $c(g) \in \mathbb{T}$ such that $V_g = c(g)\left(\tilde{V}_g^{(0)} \otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}(g)}\right)$. Setting $V_g^{(0)} := c(g)\tilde{V}_g^{(0)}$, we obtain $V_g = V_g^{(0)} \otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}(g)}$. Clearly, $V^{(0)}$ satisfies the required conditions. Because $\sigma_y^{\mathfrak{q}(g)}C^{\mathfrak{p}(h)} = (-1)^{\mathfrak{q}(g)\mathfrak{p}(h)}C^{\mathfrak{p}(h)}\sigma_y^{\mathfrak{q}(g)}$, we obtain the last statement. □

We introduce the following equivalence relation on graded $W^*$-$(G,\mathfrak{p})$-dynamical systems.

**Definition 2.8.** Let $G$ be a finite group and $\mathfrak{p} : G \to \mathbb{Z}_2$ be a group homomorphism. We say that two graded $W^*$-$(G,\mathfrak{p})$-dynamical systems $(\mathcal{M}_1, \theta_1, \hat{\alpha}^{(1)})$, $(\mathcal{M}_2, \theta_2, \hat{\alpha}^{(2)})$ are equivalent and write $(\mathcal{M}_1, \theta_1, \hat{\alpha}^{(1)}) \sim (\mathcal{M}_2, \theta_2, \hat{\alpha}^{(2)})$ if there is a $*$-isomorphism $\iota : \mathcal{M}_1 \to \mathcal{M}_2$ such that

$$\iota \circ \hat{\alpha}_g^{(1)} = \hat{\alpha}_g^{(2)} \circ \iota, \quad g \in G \tag{2.4}$$

$$\iota \circ \theta_1 = \theta_2 \circ \iota. \tag{2.5}$$

Clearly, this is an equivalence relation.

Using equivalence of $W^*$-$(G,\mathfrak{p})$-dynamical systems, we can reduce all type I balanced central graded $W^*$-$(G,\mathfrak{p})$-dynamical systems to the case of either Example 2.4 or 2.5.

**Proposition 2.9.** *Let* $(\mathcal{M}, \theta, \hat{\alpha})$ *be a graded* $W^*$-$(G,\mathfrak{p})$*-dynamical systems with* $(\mathcal{M},\theta)$ *balanced, central and type I. Then there is a* $\kappa \in \mathbb{Z}_2$ *and* $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ *such that* $(\mathcal{M},\theta,\hat{\alpha}) \sim (\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g})$.

*Proof.* Because $(\mathcal{M},\theta)$ is central, by Lemma A.2 either $\mathcal{M}$ is a factor or $Z(\mathcal{M})$ has an odd self-adjoint unitary $b \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$ such that

$$Z(\mathcal{M}) \cap \mathcal{M}^{(1)} = \mathbb{C}b. \tag{2.6}$$

We set $\kappa = 0$ for the former case, and $\kappa = 1$ for the latter case.
**(Case:** $\kappa = 0$**)** Suppose $\mathcal{M}$ is a type I factor. Because $(\mathcal{M},\theta)$ is balanced, there is an odd self-adjoint unitary $U \in \mathcal{M}^{(1)}$.

We claim that $\mathcal{M}^{(0)}$ is not a factor. If $\mathcal{M}^{(0)}$ is a factor, by Lemma A.1 it is of type I. Note then that $\mathrm{Ad}_U|_{\mathcal{M}^{(0)}}$ is an automorphism on the type I factor $\mathcal{M}^{(0)}$. By Wigner's theorem, there is a unitary $u \in \mathcal{M}^{(0)}$ such that $\mathrm{Ad}_U(x) = \mathrm{Ad}_u(x)$, $x \in \mathcal{M}^{(0)}$. Therefore, $u^*U \in \big(\mathcal{M}^{(0)}\big)'$. At the same time, $u^*U$ commutes with $U$ because $\mathrm{Ad}_U(u^*) = \mathrm{Ad}_u(u^*) = u^*$ for $u \in \mathcal{M}^{(0)}$. Hence, $u^*U \in \mathcal{M}' \cap \mathcal{M} = \mathbb{C}\mathbb{I}$. This is a contradiction because $u^*U$ is nonzero and odd. Hence, we conclude that $\mathcal{M}^{(0)}$ is not a factor.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Therefore, there is a projection $z$ in $Z(\mathcal{M}^{(0)})$ that is not 0 nor $\mathbb{I}$. For such a projection, we have $z + \mathrm{Ad}_U(z) \in \mathcal{M} \cap \left(\mathcal{M}^{(0)}\right)' \cap \{U\}' = Z(\mathcal{M}) = \mathbb{C}\mathbb{I}$, which then implies that $z + \mathrm{Ad}_U(z) = \mathbb{I}$. (We note that for orthogonal projections $p, q$ satisfying $p + q = t\mathbb{I}$ with $t \in \mathbb{R}$, either $p + q = \mathbb{I}$ or $p = 0$, $\mathbb{I}$ holds, by considering the spectrum of $p = t\mathbb{I} - q$.)

We claim $Z(\mathcal{M}^{(0)}) = \mathbb{C}z + \mathbb{C}\mathbb{I}$. Now, for any projection $s$ in $Z(\mathcal{M}^{(0)})$, $zs$ is a projection in $Z(\mathcal{M}^{(0)})$. Therefore, either $zs = 0$ or $zs + \mathrm{Ad}_U(zs) = \mathbb{I}$. The latter is possible only if $zs = z$ because $z + \mathrm{Ad}_U(z) = \mathbb{I}$. Similarly, we have $(\mathbb{I} - z)s = 0$ or $(\mathbb{I} - z)s = \mathbb{I} - z$. Hence, we have $Z(\mathcal{M}^{(0)}) = \mathbb{C}z + \mathbb{C}\mathbb{I}$, proving the claim.

Combining this with $\mathrm{Ad}_U(z) = \mathbb{I} - z$, $\mathcal{M}^{(0)}$ is a direct sum of two same-type factors $\mathcal{M}^{(0)}z$ and $\mathcal{M}^{(0)}(\mathbb{I} - z)$. Applying Lemma A.1, we see that $\mathcal{M}^{(0)}$ is of type I, and $\mathcal{M}^{(0)}z$ and $\mathcal{M}^{(0)}(\mathbb{I} - z)$ are type I factors.

Set $\Gamma := z - (\mathbb{I} - z)$. Note that $\mathrm{Ad}_\Gamma$ and $\theta$ are the identity on $\mathcal{M}^{(0)}$. We also have $\mathrm{Ad}_U(\Gamma) = (\mathbb{I} - z) - z = -\Gamma$; hence $\mathrm{Ad}_\Gamma(U) = -U = \theta(U)$. Therefore, we get

$$\theta(x) = \mathrm{Ad}_\Gamma(x), \quad x \in \mathcal{M}. \tag{2.7}$$

Next we claim that there is a Hilbert space $\mathcal{K}$ and a $*$-isomorphism $\iota : \mathcal{M} \to \mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2$ such that

$$\iota \circ \theta = \mathrm{Ad}_{\Gamma_\mathcal{K}} \circ \iota, \quad \text{and} \quad \iota(\Gamma) = \mathbb{I}_\mathcal{K} \otimes \sigma_z =: \Gamma_\mathcal{K}. \tag{2.8}$$

Because $\mathcal{M}$ is a type I factor, there is a Hilbert space $\hat{\mathcal{K}}$ and a $*$-isomorphism $\hat{\iota} : \mathcal{M} \to \mathcal{B}(\hat{\mathcal{K}})$. Let $\hat{\iota}(\Gamma) = Q_0 - Q_1$ be the spectral decomposition of a self-adjoint unitary $\hat{\iota}(\Gamma)$, with orthogonal projections $Q_0, Q_1$, corresponding to eigenvalues $1, -1$. Because we have $\mathrm{Ad}_{\hat{\iota}(\Gamma)} \circ \hat{\iota}(x) = \hat{\iota} \circ \mathrm{Ad}_\Gamma(x) = \hat{\iota} \circ \theta(x)$ for $x \in \mathcal{M}$ by (2.7), we have $\hat{\iota}(\mathcal{M}^{(0)}) = \mathcal{B}(Q_0\hat{\mathcal{K}}) \oplus \mathcal{B}(Q_1\hat{\mathcal{K}})$. Because $\mathrm{Ad}_\Gamma(U) = -U$, we have $\mathrm{Ad}_{\hat{\iota}(U)}(\hat{\iota}(\Gamma)) = -\hat{\iota}(\Gamma)$. From the spectral decomposition, we then have $\mathrm{Ad}_{\hat{\iota}(U)}(Q_0) = Q_1$ and $\mathrm{Ad}_{\hat{\iota}(U)}(Q_1) = Q_0$. We therefore see that $v := Q_0\hat{\iota}(U)Q_1$ is a unitary from $Q_1\hat{\mathcal{K}}$ onto $Q_0\hat{\mathcal{K}}$. We set $\mathcal{K} := Q_0\hat{\mathcal{K}}$ and define a unitary $W : \hat{\mathcal{K}} \to \mathcal{K} \otimes \mathbb{C}^2$ by

$$W \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \xi_0 \otimes e_0 + v\xi_1 \otimes e_1, \quad \xi_0 \in Q_0\hat{\mathcal{K}}, \quad \xi_1 \in Q_1\hat{\mathcal{K}}. \tag{2.9}$$

Here $\{e_0, e_1\}$ is the standard basis of $\mathbb{C}^2$. Note that $\mathrm{Ad}_W \circ \hat{\iota}(\Gamma) = \mathbb{I}_\mathcal{K} \otimes \sigma_z = \Gamma_\mathcal{K}$. Then $\iota := \mathrm{Ad}_W \circ \hat{\iota} : \mathcal{M} \to \mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2$ is a $*$-isomorphism satisfying (2.8), proving the claim.

Next we consider the action of $G$. Because $Z(\mathcal{M}^{(0)}) = \mathbb{C}z + \mathbb{C}(\mathbb{I} - z)$, $\Gamma = z - (\mathbb{I} - z)$ and $-\Gamma = -z + (\mathbb{I} - z)$ are the only self-adjoint unitaries in $Z(\mathcal{M}^{(0)}) \setminus \mathbb{C}\mathbb{I}$. Because $\hat{\alpha}_g$ preserves $\mathcal{M}^{(0)}$, $\hat{\alpha}_g(\Gamma)$ is a self-adjoint unitary in $Z(\mathcal{M}^{(0)}) \setminus \mathbb{C}\mathbb{I}$ and so $\hat{\alpha}_g(\Gamma) = (-1)^{\mathfrak{q}(g)}\Gamma$ for $\mathfrak{q}(g) = 0$ or $\mathfrak{q}(g) = 1$. Clearly, $\mathfrak{q} : G \to \mathbb{Z}_2$ is a group homomorphism.

Because $\iota \circ \hat{\alpha}_g \circ \iota^{-1}$ is a linear/anti-linear automorphism on $\mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2$, by Wigner's theorem there is a projective representation $V$ satisfying

$$\mathrm{Ad}_{V_g}(x) = \iota \circ \hat{\alpha}_g \circ \iota^{-1}(x), \quad x \in \mathcal{B}(\mathcal{K}) \otimes \mathrm{M}_2, \quad g \in G, \tag{2.10}$$

and where $V_g$ is unitary/anti-unitary depending on $\mathfrak{p}(g)$. Because $\hat{\alpha}_g(\Gamma) = (-1)^{\mathfrak{q}(g)}\Gamma$, we have

$$\mathrm{Ad}_{V_g}(\Gamma_\mathcal{K}) = (-1)^{\mathfrak{q}(g)}\Gamma_\mathcal{K}, \quad g \in G. \tag{2.11}$$

Hence, we obtain $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_0$. By (2.8) and (2.10), we also have $(\mathcal{M}, \theta, \hat{\alpha}) \sim (\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g})$.

**(Case:** $\kappa = 1$) Suppose that $\mathcal{M}$ has a self-adjoint unitary $b \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$ satisfying (2.6). Set $P_\pm := \frac{1 \pm b}{2}$, where $P_\pm$ are orthogonal projections in $Z(\mathcal{M})$ such that $P_+ + P_- = \mathbb{I}$. By (2.6), $Z(M) = \mathbb{C}b + \mathbb{C}\mathbb{I} = \mathbb{C}P_+ + \mathbb{C}P_-$. Because $\mathcal{M}$ is type I, $\mathcal{M}$ is a direct sum of the type I factors $\mathcal{M}P_+$ and $\mathcal{M}P_-$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

We claim that $\mathcal{M}^{(0)}$ is a type I factor. For any $x \in Z\left(\mathcal{M}^{(0)}\right)$, we have $x \in \mathcal{M}^{(0)} \cap \left(\mathcal{M}^{(0)}\right)' \cap \{b\}' = \mathcal{M}^{(0)} \cap \mathcal{M}' = Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}\mathbb{I}$, because $b$ is a self-adjoint unitary in $Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$. Hence, $Z(\mathcal{M}^{(0)}) = \mathbb{C}\mathbb{I}$ and by Lemma A.1, $\mathcal{M}^{(0)}$ is a type I factor.

Next we claim that there is a Hilbert space $\mathcal{K}$ and a $*$-isomorphism $\iota : \mathcal{M} \to \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$ such that

$$\iota \circ \theta = \mathrm{Ad}_{\Gamma_{\mathcal{K}}} \circ \iota, \quad \iota(b) = \mathbb{I}_{\mathcal{K}} \otimes \sigma_x, \tag{2.12}$$

for $\Gamma_{\mathcal{K}} = \mathbb{I}_{\mathcal{K}} \otimes \sigma_z$. (Recall Example 2.5 for $\mathfrak{C}$.) Because $\mathcal{M}^{(0)}$ is a type I factor, there is a Hilbert space $\mathcal{K}$ and a $*$-isomorphism $\iota_0 : \mathcal{M}^{(0)} \to \mathcal{B}(\mathcal{K})$. As $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(0)} b$, we may define a linear map $\iota : \mathcal{M} \to \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$ by

$$\iota(x + yb) := \iota_0(x) \otimes \mathbb{I} + \iota_0(y) \otimes \sigma_x, \quad x, y \in \mathcal{M}^{(0)}. \tag{2.13}$$

It can be easily checked that $\iota$ is a $*$-isomorphism satisfying (2.12).

Now we consider the group action. Because $Z(\mathcal{M}) \cap \mathcal{M}^{(1)} = \mathbb{C}b$, $b$ and $-b$ are the only self-adjoint unitaries in $Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$. Because $\hat{\alpha}_g$ commutes with the grading automorphism, $\hat{\alpha}_g(b)$ is a self-adjoint unitary in $Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$. Therefore, $\hat{\alpha}_g(b) = (-1)^{\mathfrak{q}(g)} b$ with $\mathfrak{q} : G \to \mathbb{Z}_2$ a group homomorphism.

Because $\hat{\alpha}_g(\mathcal{M}^{(0)}) = \mathcal{M}^{(0)}$ and $\iota(\mathcal{M}^{(0)}) = \mathcal{B}(\mathcal{K}) \otimes \mathbb{C}\mathbb{I}$ by (2.12), $\iota \circ \hat{\alpha}_g \circ \iota^{-1}$ induces a linear/anti-linear automorphism on $\mathcal{B}(\mathcal{K})$ that is implemented by a unitary/anti-unitary $V_g^{(0)}$ on $\mathcal{K}$ by Wigner's theorem. That is,

$$\iota \circ \hat{\alpha}_g \circ \iota^{-1}(a \otimes \mathbb{I}_{\mathbb{C}^2}) = \mathrm{Ad}_{V_g^{(0)}}(a) \otimes \mathbb{I}_{\mathbb{C}^2}, \quad a \in \mathcal{B}(\mathcal{K}), \quad g \in G, \tag{2.14}$$

with $V^{(0)}$ a projective unitary/anti-unitary representation of $G$ on $\mathcal{K}$ relative to $\mathfrak{p}$. Set $V_g := V_g^{(0)} \otimes C^{\mathfrak{p}(g)} \sigma_y^{\mathfrak{q}(g)}$, with the complex conjugation $C$ on $\mathbb{C}^2$ with respect to the standard basis. Clearly, $V$ is also a projective unitary/anti-unitary representation of $G$ on $\mathcal{K} \otimes \mathbb{C}^2$ relative to $\mathfrak{p}$. We then have

$$\mathrm{Ad}_{V_g}(a \otimes \mathbb{I}_{\mathbb{C}^2}) = \iota \circ \hat{\alpha}_g \circ \iota^{-1}(a \otimes \mathbb{I}_{\mathbb{C}^2}), \quad a \in \mathcal{B}(\mathcal{K}), \tag{2.15}$$

$$\mathrm{Ad}_{V_g}(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x) = (-1)^{\mathfrak{q}(g)}(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x) = \iota \circ \hat{\alpha}_g(b) = \iota \circ \hat{\alpha}_g \circ \iota^{-1}(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x). \tag{2.16}$$

Combining these identities, we obtain

$$\mathrm{Ad}_{V_g} \circ \iota(x) = \iota \circ \hat{\alpha}_g(x), \quad x \in \mathcal{M}. \tag{2.17}$$

We also have

$$\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}(g)} \Gamma_{\mathcal{K}}. \tag{2.18}$$

Hence, we obtain $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_1$ such that $(\mathcal{M}, \theta, \hat{\alpha}) \sim (\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$. □

**Definition 2.10.** Let $(\mathcal{M}, \theta, \hat{\alpha})$ be a graded $W^*$-$(G, \mathfrak{p})$-dynamical system with $(\mathcal{M}, \theta)$ balanced, central and type I. By Proposition 2.9, there is a $\kappa \in \mathbb{Z}_2$ and $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ such that $(\mathcal{M}, \theta, \hat{\alpha}) \sim (\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$. Let $\mathfrak{q} : G \to \mathbb{Z}_2$ be a group homomorphism such that $\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}(g)} \Gamma_{\mathcal{K}}$ and $[v]$ the second cohomology class associated to the projective representation $V_g$ if $\kappa = 0$ and $V_g^{(0)}$ (from Lemma 2.7) if $\kappa = 1$. We define an index of $(\mathcal{M}, \theta, \hat{\alpha})$ by

$$\mathrm{Ind}(\mathcal{M}, \theta, \hat{\alpha}) := (\kappa, \mathfrak{q}, [v]) \in \mathbb{Z}_2 \times H^1(G, \mathbb{Z}_2) \times H^2(G, U(1)_{\mathfrak{p}}). \tag{2.19}$$

**Lemma 2.11.** *The quantity* $\mathrm{Ind}(\mathcal{M}, \theta, \hat{\alpha})$ *is well defined and independent of the choice of* $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ *such that* $(\mathcal{M}, \theta, \hat{\alpha}) \sim (\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

*Proof.* Suppose that both $(\mathcal{R}_{\kappa_1,\mathcal{K}_1}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_1}}, \mathrm{Ad}_{V_g^{(1)}}) \in \mathcal{S}_{\kappa_1}$ and $(\mathcal{R}_{\kappa_2,\mathcal{K}_2}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_2}}, \mathrm{Ad}_{V_g^{(2)}}) \in \mathcal{S}_{\kappa_2}$ are equivalent to $(\mathcal{M}, \theta, \hat{\alpha})$, via $*$-isomorphisms $\iota_i : \mathcal{M} \to \mathcal{R}_{\kappa_i,\mathcal{K}_i}$, $i = 1, 2$, respectively. Then $\iota_2 \circ \iota_1^{-1} : \mathcal{R}_{\kappa_1,\mathcal{K}_1} \to \mathcal{R}_{\kappa_2,\mathcal{K}_2}$ is a $*$-isomorphism such that for all $g \in G$,

$$\iota_2 \circ \iota_1^{-1} \circ \mathrm{Ad}_{V_g^{(1)}} = \mathrm{Ad}_{V_g^{(2)}} \circ \iota_2 \circ \iota_1^{-1}, \qquad \iota_2 \circ \iota_1^{-1} \circ \mathrm{Ad}_{\Gamma_{\mathcal{K}_1}} = \mathrm{Ad}_{\Gamma_{\mathcal{K}_2}} \circ \iota_2 \circ \iota_1^{-1}. \tag{2.20}$$

Let $(\kappa_i, \mathfrak{q}_i, [v_i])$ be indices obtained from $(\mathcal{R}_{\kappa_i,\mathcal{K}_i}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_i}}, \mathrm{Ad}_{V_g^{(i)}})$, for $i = 1, 2$. Because of the $*$-isomorphism $\iota_2 \circ \iota_1^{-1}$, we clearly have $\kappa_1 = \kappa_2$. If $\kappa_1 = \kappa_2 = 0$, then both of $\iota_i^{-1}(\mathbb{I}_{\mathcal{K}_i} \otimes \sigma_z)$, $i = 1, 2$, are self-adjoint unitaries in $Z(\mathcal{M}^{(0)}) \setminus \mathbb{C}\mathbb{I}$. From the proof of Proposition 2.9, this means that $\iota_2 \circ \iota_1^{-1}(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) = \pm(\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_z)$. Hence, we get

$$\begin{aligned}
(-1)^{\mathfrak{q}_1(g)} \iota_2 \circ \iota_1^{-1}(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) &= \iota_2 \circ \iota_1^{-1} \circ \mathrm{Ad}_{V_g^{(1)}}(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) \\
&= \mathrm{Ad}_{V_g^{(2)}} \circ \iota_2 \circ \iota_1^{-1}(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) \\
&= \pm \mathrm{Ad}_{V_g^{(2)}}(\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_z) \\
&= \pm(-1)^{\mathfrak{q}_2(g)}(\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_z) \\
&= (-1)^{\mathfrak{q}_2(g)} \iota_2 \circ \iota_1^{-1}(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z).
\end{aligned} \tag{2.21}$$

We therefore obtain that $\mathfrak{q}_1(g) = \mathfrak{q}_2(g)$. When $\kappa_1 = \kappa_2 = 1$, an analogous argument for $\iota_i^{-1}(\mathbb{I}_{\mathcal{K}_i} \otimes \sigma_x) \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$, $i = 1, 2$ implies $\mathfrak{q}_1(g) = \mathfrak{q}_2(g)$.

If $\kappa_1 = \kappa_2 = 0$, the $*$-isomorphism $\iota_2 \circ \iota_1^{-1} : \mathcal{B}(\mathcal{K}_1) \otimes \mathrm{M}_2 \to \mathcal{B}(\mathcal{K}_2) \otimes \mathrm{M}_2$ is implemented by a unitary $W : \mathcal{K}_1 \otimes \mathbb{C}^2 \to \mathcal{K}_2 \otimes \mathbb{C}^2$. Hence, we see from (2.20) that $\mathrm{Ad}_{W V_g^{(1)} W^*}(x) = \mathrm{Ad}_{V_g^{(2)}}(x)$ for all $x \in \mathcal{B}(\mathcal{K}_1) \otimes \mathrm{M}_2$. This means that $[v_1] = [v_2]$. If $\kappa_1 = \kappa_2 = 1$, the restriction of the $*$-isomorphism $\iota_2 \circ \iota_1^{-1}$ onto $\mathcal{B}(\mathcal{K}_1)$ induces a $*$-isomorphism from $\mathcal{B}(\mathcal{K}_1)$ to $\mathcal{B}(\mathcal{K}_2)$. Therefore, there is a unitary $W : \mathcal{K}_1 \to \mathcal{K}_2$ such that $\iota_2 \circ \iota_1^{-1}(x \otimes \mathbb{I}) = \mathrm{Ad}_W(x) \otimes \mathbb{I}$, for all $x \in \mathcal{B}(\mathcal{K}_1)$. Therefore, from (2.20) we have $\mathrm{Ad}_{W (V_g^{(1)})^{(0)} W^*}(x) = \mathrm{Ad}_{(V_g^{(2)})^{(0)}}(x)$ for all $x \in \mathcal{B}(\mathcal{K}_1)$. This means that $[v_1] = [v_2]$. □

Proposition 2.9, Lemma 2.11 and the fact that equivalence of $W^*$-$(G, \mathfrak{p})$-dynamical systems is an equivalence relation gives us the following.

**Proposition 2.12.** *Let* $(\mathcal{M}_1, \theta_1, \hat{\alpha}_1)$, $(\mathcal{M}_2, \theta_2, \hat{\alpha}_2)$ *be graded* $W^*$-$(G, \mathfrak{p})$*-dynamical systems of balanced, central and type I graded von Neumann algebras. If* $(\mathcal{M}_1, \theta_1, \hat{\alpha}_1) \sim (\mathcal{M}_2, \theta_2, \hat{\alpha}_2)$, *then* $\mathrm{Ind}(\mathcal{M}_1, \theta_1, \hat{\alpha}_1) = \mathrm{Ind}(\mathcal{M}_2, \theta_2, \hat{\alpha}_2)$.

### 2.2. *The index for pure split states*

We now define an index to fermionic SPT phases. For each $\Theta$-invariant and $\alpha$-invariant state $\mathcal{A}$, $(\pi_\varphi(\mathcal{A}_R)'', \mathrm{Ad}_{\hat{\Gamma}_\varphi}, \hat{\alpha}_\varphi)$ is a graded $W^*$-$(G, \mathfrak{p})$-dynamical system.

We first review the split property and recent results of Matsui [24] that relate the split property to unique gapped ground states of the CAR-algebra. Given a state $\varphi$ on $\mathcal{A}$, $\varphi|_{\mathcal{A}_R}$ denotes the restriction of $\varphi$ to $\mathcal{A}_R$ and $\pi_{\varphi|_{\mathcal{A}_R}}$ is the GNS representation of $\mathcal{A}_R$ from this restricted state.

**Definition 2.13.** Let $\varphi$ be a pure $\Theta$-invariant state on $\mathcal{A}$. We say that $\varphi$ satisfies the split property if $\pi_{\varphi|_{\mathcal{A}_R}}(\mathcal{A}_R)''$ is a type I von Neumann algebra.

If $\varphi$ is a pure $\Theta$-invariant state satisfying the split property, then there is an approximate statistical independence between the half-infinite restrictions $\varphi|_{\mathcal{A}_R}$ and $\varphi|_{\mathcal{A}_L}$. It is shown in [23] that pure states whose entanglement entropy is *uniformly* bounded on finite regions satisfy the split property. Hence, the split property of pure states is closely related to the area law of entanglement entropy in one-dimensional

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

systems. See [34, 33] for further applications of the split property to Lieb-Schulz-Mattis-type theorems in the setting of quantum spin chains.

Recall the notation $\mathcal{B}^e_f$, which denotes the set of all finite-range even interactions that satisfy the bound (1.5). Similarly, $\mathcal{G}^{e,\alpha}_f$ denotes the set of all $\alpha$-invariant interactions $\Phi \in \mathcal{B}^e_f$, with a unique gapped ground state.

**Theorem 2.14** ([24])**.** *Let $\varphi$ be a unique gapped $\tau^\Phi$-ground state of an interaction $\Phi \in \mathcal{B}^e_f$. Then $\varphi$ satisfies the split property.*

To apply Matsui's result to graded $W^*$-$(G, \mathfrak{p})$-dynamical systems, we must first relate the split ground state of an interaction $\Phi \in \mathcal{G}^{e,\alpha}_f$ to balanced and central graded type I von Neumann algebras. To show this, we first note the following.

**Lemma 2.15.** *Let $\varphi$ be a $\Theta$-invariant pure state on $\mathcal{A}$. Then*

(i) $Z(\pi_\varphi(\mathcal{A}_R)'') \cap \left(\pi_\varphi(\mathcal{A}_R)''\right)^{(0)} = \mathbb{C}\mathbb{I}$.
(ii) *The representations $\pi_{\varphi|_{\mathcal{A}_R}}$ and $(\pi_\varphi)|_{\mathcal{A}_R}$, the restriction of $\pi_\varphi$ to $\mathcal{A}_R$, are quasi-equivalent.*

*Proof.* (i) We have that

$$Z(\pi_\varphi(\mathcal{A}_R)'') \cap \left(\pi_\varphi(\mathcal{A}_R)''\right)^{(0)} \subset \pi_\varphi(\mathcal{A}_L)' \cap \pi_\varphi(\mathcal{A}_R)' = \pi_\varphi(\mathcal{A})' = \mathbb{C}\mathbb{I}, \tag{2.22}$$

where the last equality is because $\varphi$ is pure.

(ii) Let $\hat{\Gamma}_\varphi$ be a self-adjoint unitary on $\mathcal{H}_\varphi$ given by $\hat{\Gamma}_\varphi \pi_\varphi(A)\Omega_\varphi = \pi_\varphi \circ \Theta(A)\Omega_\varphi$, $A \in \mathcal{A}$. Let $p$ denote the the orthogonal projection onto $\overline{\pi_\varphi(\mathcal{A}_R)\Omega_\varphi}$. Then $(p\mathcal{H}_\varphi, \pi_\varphi(\cdot)|_{\mathcal{A}_R} p, \Omega_\varphi)$ is a GNS triple of $\varphi|_{\mathcal{A}_R}$. To show (ii), it suffices to show that $\tau : \pi_\varphi(\mathcal{A}_R)'' \to (\pi_\varphi(\mathcal{A}_R)p)''$ defined by $\tau(x) = xp$ is a $*$-isomorphism. It is standard to see that $\tau$ is a surjective $*$-homomorphism. To see that $\tau$ is injective, note that from (i) and Lemma A.2, either $\pi_\varphi(\mathcal{A}_R)''$ is factor or $Z(\pi_\varphi(\mathcal{A}_R)'') = \mathbb{C}\mathbb{I} + \mathbb{C}b$ with some self-adjoint unitary $b \in Z(\pi_\varphi(\mathcal{A}_R)'') \cap (\pi_\varphi(\mathcal{A}_R)'')^{(1)}$. For the former case, $\tau$ is clearly injective. For the latter case, let $b = P_+ - P_-$ be the spectral decomposition. Because $b$ is odd, we have $\mathrm{Ad}_{\hat{\Gamma}_\varphi}(P_\pm) = P_\mp$. If $\tau$ is not injective, the kernel of $\tau$ is either $\pi_\varphi(\mathcal{A}_R)'' P_+$ or $\pi_\varphi(\mathcal{A}_R)'' P_-$. If $\tau(P_+) = 0$, then we have $P_+\Omega_\varphi$=0. We then have

$$P_-\Omega_\varphi = \hat{\Gamma}_\varphi P_+ \hat{\Gamma}_\varphi \Omega_\varphi = \hat{\Gamma}_\varphi P_+ \Omega_\varphi = 0. \tag{2.23}$$

Hence, we obtain $\Omega_\varphi = (P_+ + P_-)\Omega_\varphi = 0$, which is a contradiction. Similarly, we have $\tau(P_-) \neq 0$. Therefore, $\tau$ is injective. □

**Lemma 2.16.** *Let $\varphi$ be a split pure $\Theta$-invariant and $\alpha$-invariant state on $\mathcal{A}$. Then $\pi_\varphi(\mathcal{A}_R)''$ is balanced and central with respect to the grading given by $\hat{\Gamma}_\varphi$ and type I. The triple $(\pi_\varphi(\mathcal{A}_R)'', \mathrm{Ad}_{\hat{\Gamma}_\varphi}, \hat{\alpha}_\varphi)$ is a graded $W^*$-$(G, \mathfrak{p})$-dynamical system.*

*Proof.* Because $\varphi$ is pure and $\Theta$-invariant, $\pi_\varphi(\mathcal{A}_R)''$ is central by part (i) of Lemma 2.15. Because $\varphi$ is split, $\pi_{\varphi|_{\mathcal{A}_R}}(\mathcal{A}_R)''$ is type I by definition. Because $(\pi_\varphi)|_{\mathcal{A}_R}$ is quasi-equivalent to $\pi_{\varphi|_{\mathcal{A}_R}}$ by part (ii) of Lemma 2.15, $\pi_\varphi(\mathcal{A}_R)''$ is also type I. It is also balanced because $\mathcal{A}_R$ has an odd self-adjoint unitary. Because $\alpha_g \circ \Theta = \Theta \circ \alpha_g$ for all $g \in G$, we have $\left(\hat{\alpha}_\varphi\right)_g \circ \mathrm{Ad}_{\hat{\Gamma}_\varphi} = \mathrm{Ad}_{\hat{\Gamma}_\varphi} \circ \left(\hat{\alpha}_\varphi\right)_g$. □

**Remark 2.17.** Consider the setting of Lemma 2.16. Let $\varphi_R := \varphi|_{\mathcal{A}_R}$. Then $(\pi_{\varphi_R}(\mathcal{A}_R)'', \mathrm{Ad}_{\hat{\Gamma}_{\varphi_R}}, \hat{\alpha}_{\varphi_R})$ is also a graded $W^*$-$(G, \mathfrak{p})$-dynamical system of a balanced, central and type I graded von Neumann algebra with

$$(\pi_\varphi(\mathcal{A}_R)'', \mathrm{Ad}_{\hat{\Gamma}_\varphi}, \hat{\alpha}_\varphi) \sim (\pi_{\varphi_R}(\mathcal{A}_R)'', \mathrm{Ad}_{\hat{\Gamma}_{\varphi_R}}, \hat{\alpha}_{\varphi_R}). \tag{2.24}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

From Lemma 2.16, we see that our index of $W^*$-$(G, \mathfrak{p})$-dynamical systems can be applied to split, pure, $\Theta$-invariant and $\alpha$-invariant states on $\mathcal{A}$. In particular, we may define an index for $\Phi \in \mathcal{G}_f^{e,\alpha}$.

**Definition 2.18.** Let $\varphi$ be a $\Theta$-invariant, $\alpha$-invariant, split and pure state on $\mathcal{A}$ with $\varphi_R := \varphi|_{\mathcal{A}_R}$. We set

$$\operatorname{ind}\varphi := \operatorname{Ind}(\pi_\varphi(\mathcal{A}_R)'', \operatorname{Ad}_{\hat{\Gamma}_\varphi}, \hat{\alpha}_\varphi) = \operatorname{Ind}(\pi_{\varphi_R}(\mathcal{A}_R)'', \operatorname{Ad}_{\hat{\Gamma}_{\varphi_R}}, \hat{\alpha}_{\varphi_R}). \tag{2.25}$$

For interactions $\Phi \in \mathcal{G}_f^{e,\alpha}$, we define the index of $\Phi$ by $\operatorname{ind}(\Phi) := \operatorname{ind}(\varphi_\Phi)$, with $\varphi_\Phi$ the unique ground state of $\Phi$.

## 3. The stability of the index

In this section we prove that $\operatorname{ind}(\Phi)$ is an invariant of the classification of SPT phases. That is, for a path of interactions $\{\Phi(s)\}_{s\in[0,1]}$ satisfying Assumption 3.2, we show that $\operatorname{ind}(\Phi(0)) = \operatorname{ind}(\Phi(1))$.

For each $N \in \mathbb{N}$, we denote $[-N, N] \cap \mathbb{Z}$ by $\Lambda_N$. Let $\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N}$ be the conditional expectation with respect to the trace state; see [2]. We consider the following subset of $\mathcal{A}$.

**Definition 3.1.** Let $f : (0,\infty) \to (0,\infty)$ be a continuous decreasing function with $\lim_{t\to\infty} f(t) = 0$. For each $A \in \mathcal{A}$, let

$$\|A\|_f := \|A\| + \sup_{N\in\mathbb{N}} \left( \frac{\|A - \mathbb{E}_N(A)\|}{f(N)} \right). \tag{3.1}$$

We denote by $\mathcal{D}_f$ the set of all $A \in \mathcal{A}$ such that $\|A\|_f < \infty$.

We consider a path in $\mathcal{G}_f^{e,\alpha}$ satisfying the following conditions.

**Assumption 3.2.** Let $[0,1] \ni s \mapsto \Phi(s) \in \mathcal{B}_f^e$ be a path of interactions on $\mathcal{A}$. We assume the following:

(i) For each $X \in \mathfrak{S}_\mathbb{Z}$, the map $[0,1] \ni s \mapsto \Phi(X; s) \in \mathcal{A}_X$ is continuous and piecewise $C^1$. We denote by $\dot{\Phi}(X; s)$ the corresponding derivatives. The interaction obtained by differentiation is denoted by $\dot{\Phi}(s)$ for each $s \in [0,1]$.

(ii) There is a number $R \in \mathbb{N}$ such that $X \in \mathfrak{S}_\mathbb{Z}$ and $\operatorname{diam}(X) \geq R$ implies $\Phi(X; s) = 0$ for all $s \in [0,1]$.

(iii) For each $s \in [0,1]$, $\Phi(s) \in \mathcal{G}_f^{e,\alpha}$. We denote the unique $\tau^{\Phi(s)}$-ground state by $\varphi_s$.

(iv) Interactions are bounded as follows:

$$\sup_{s\in[0,1]} \sup_{X\in\mathfrak{S}_\mathbb{Z}} \left( \|\Phi(X; s)\| + |X| \left\|\dot{\Phi}(X; s)\right\| \right) < \infty. \tag{3.2}$$

(v) Setting

$$b(\varepsilon) := \sup_{Z\in\mathfrak{S}_\mathbb{Z}} \sup_{\substack{s,s_0\in[0,1],\\ 0<|s-s_0|<\varepsilon}} \left\| \frac{\Phi(Z; s) - \Phi(Z; s_0)}{s - s_0} - \dot{\Phi}(Z; s_0) \right\| \tag{3.3}$$

for each $\varepsilon > 0$, we have $\lim_{\varepsilon\to 0} b(\varepsilon) = 0$.

(vi) There exists a $\gamma > 0$ such that $\sigma(H_{\varphi_s,\Phi(s)}) \setminus \{0\} \subset [\gamma, \infty)$ for all $s \in [0,1]$, where $\sigma(H_{\varphi_s,\Phi(s)})$ is the spectrum of $H_{\varphi_s,\Phi(s)}$.

(vii) There exists $0 < \beta < 1$ satisfying the following: Set $\zeta(t) := e^{-t^\beta}$. Then for each $A \in D_\zeta$, $\varphi_s(A)$ is differentiable with respect to $s$, and there is a constant $C_\zeta$ such that

$$|\dot{\varphi}_s(A)| \leq C_\zeta \|A\|_\zeta, \tag{3.4}$$

for any $A \in D_\zeta$.

The main result of this section is the following.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

**Theorem 3.3.** *Let* $[0,1] \ni s \mapsto \Phi(s) \in \mathcal{B}_f^e$ *be a path of interactions on* $\mathcal{A}$ *satisfying Assumption 3.2. Then* $\mathrm{ind}(\Phi(0)) = \mathrm{ind}(\Phi(1))$.

The proof relies on the idea introduced in [29]; that is, using the factorisation property of automorphic equivalence. Namely, we note the following.

**Proposition 3.4.** *Let* $[0,1] \ni s \mapsto \Phi(s) \in \mathcal{B}_f^e$ *be a path of interactions on* $\mathcal{A}$ *satisfying Assumption 3.2. Let* $\varphi_s$ *be the unique* $\tau^{\Phi(s)}$*-ground state for each* $s \in [0,1]$. *Then there is an automorphism* $\Xi$ *on* $\mathcal{A}$ *and a unitary* $u \in \mathcal{A}$ *such that for all* $g \in G$,

$$\begin{aligned} &\Xi(\mathcal{A}_L) = \mathcal{A}_L, && \Xi(\mathcal{A}_R) = \mathcal{A}_R, && \Xi \circ \Theta = \Theta \circ \Xi, && \Xi \circ \alpha_g = \alpha_g \circ \Xi, \\ &\Theta(u) = u, && \alpha_g(u) = u, && \varphi_1 = \varphi_0 \circ \mathrm{Ad}_u \circ \Xi. && \end{aligned}$$

In Appendix B, we prove the Lieb-Robinson bound and a locality estimate for lattice fermion systems. Having them, the proof of Proposition 3.4 is the same as that of [25, Theorem 1.3] and [29, Proposition 3.5].

To prove Theorem 3.3, we first prove a preparatory lemma.

**Lemma 3.5.** *Let* $\varphi_1, \varphi_2$ *be pure* $\Theta$*-invariant states on* $\mathcal{A}$. *If* $\varphi_1$ *and* $\varphi_2$ *are quasi-equivalent, then* $\varphi_1|_{\mathcal{A}_R}$ *and* $\varphi_2|_{\mathcal{A}_R}$ *are quasi-equivalent.*

*Proof.* Let $\pi_i, \pi_{i,R}$ be GNS representations of $\varphi_i$ and $\varphi_i|_{\mathcal{A}_R}$ respectively for $i = 1, 2$. By Lemma 2.15, there are $*$-isomorphisms $\tau_i : \pi_i(\mathcal{A}_R)'' \to \pi_{i,R}(\mathcal{A}_R)''$ for $i = 1, 2$ such that $\tau_i \circ \pi_i(A) = \pi_{i,R}(A)$ $A \in \mathcal{A}_R$. Because $\varphi_1$ and $\varphi_2$ are quasi-equivalent, there is a $*$-isomorphism $\tau : \pi_1(\mathcal{A})'' \to \pi_2(\mathcal{A})''$ such that $\tau \circ \pi_1(A) = \pi_2(A)$, for $A \in \mathcal{A}$. The restriction of $\tau$ to $\pi_1(\mathcal{A}_R)''$ gives a $*$-isomorphism $\tau_R$ : $\pi_1(\mathcal{A}_R)'' \to \pi_2(\mathcal{A}_R)''$. Hence, we obtain a $*$-isomorphism $\hat{\tau} := \tau_2 \circ \tau_R \circ \tau_1^{-1} : \pi_{1,R}(\mathcal{A}_R)'' \to \pi_{2,R}(\mathcal{A}_R)''$ such that $\hat{\tau} \circ \pi_{1,R}(A) = \pi_{2,R}(A)$, $A \in \mathcal{A}_R$. Therefore, $\varphi_1|_{\mathcal{A}_R}$ and $\varphi_2|_{\mathcal{A}_R}$ are quasi-equivalent. □

Now we are ready to prove the theorem.

*Proof of Theorem 3.3.* Let $(\mathcal{H}_i, \pi_i, \Omega_i)$ be the GNS triple of the states $\varphi_i|_{\mathcal{A}_R}$ for $i = 0, 1$. Let $\Gamma_i$ be a self-adjoint unitary given by $\Gamma_i \pi_i(A)\Omega_i = \pi_i \circ \Theta(A)\Omega_i$, $A \in \mathcal{A}_R$. Let $\hat{\alpha}_i$ be the extension of $\alpha|_{\mathcal{A}_R}$ to $\pi_i(\mathcal{A}_R)''$. From Proposition 2.12 and Remark 2.17, it suffices to show that $(\pi_0(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_0}, \hat{\alpha}_0) \sim (\pi_1(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_1}, \hat{\alpha}_1)$. Recalling the $*$-automorphism $\Xi$ from Proposition 3.4, $\Xi(\mathcal{A}_R) = \mathcal{A}_R$ and so $\Xi_R := \Xi|_{\mathcal{A}_R}$ defines a $*$-automorphism on $\mathcal{A}_R$. Note that $(\mathcal{H}_0, \pi_0 \circ \Xi_R, \Omega_0)$ is a GNS triple of $\varphi_0|_{\mathcal{A}_R} \circ \Xi_R$. The state $\varphi_1 = \varphi_0 \circ \mathrm{Ad}_u \circ \Xi$ is quasi-equivalent to $\varphi_0 \circ \Xi$. Because $\Xi \circ \Theta = \Theta \circ \Xi$, both $\varphi_0 \circ \Xi$ and $\varphi_1$ are $\Theta$-invariant pure states. Applying Lemma 3.5, $\varphi_1|_{\mathcal{A}_R}$ and $\varphi_0 \circ \Xi|_{\mathcal{A}_R} = \varphi_0|_{\mathcal{A}_R} \circ \Xi_R$ are quasi-equivalent. Hence, there is a $*$-isomorphism

$$\tau : \pi_0 \circ \Xi_R(\mathcal{A}_R)'' = \pi_0(\mathcal{A}_R)'' \to \pi_1(\mathcal{A}_R)'', \quad \tau \circ \pi_0 \circ \Xi_R(A) = \pi_1(A), \quad A \in \mathcal{A}_R. \tag{3.5}$$

Using properties of the quasi-equivalence $\tau$ and automorphism $\Xi_R$, we see that

$$\begin{aligned} \tau \circ \hat{\alpha}_{0,g} \circ \pi_0 \circ \Xi_R(A) &= \tau \circ \pi_0 \circ \alpha_g \circ \Xi_R(A) = \tau \circ \pi_0 \circ \Xi_R \circ \alpha_g(A) \\ &= \pi_1 \circ \alpha_g(A) = \hat{\alpha}_{1,g} \circ \pi_1(A) = \hat{\alpha}_{1,g} \circ \tau \circ \pi_0 \circ \Xi_R(A), \end{aligned} \tag{3.6}$$

$$\begin{aligned} \tau \circ \mathrm{Ad}_{\Gamma_0} \circ \pi_0 \circ \Xi_R(A) &= \tau \circ \pi_0 \circ \Theta \circ \Xi_R(A) = \tau \circ \pi_0 \circ \Xi_R \circ \Theta(A) \\ &= \pi_1 \circ \Theta(A) = \mathrm{Ad}_{\Gamma_1} \circ \pi_1(A) = \mathrm{Ad}_{\Gamma_1} \circ \tau \circ \pi_0 \circ \Xi_R(A) \end{aligned} \tag{3.7}$$

for all $A \in \mathcal{A}_R$. Hence, we obtain

$$\tau \circ \hat{\alpha}_{0,g}(x) = \hat{\alpha}_{1,g} \circ \tau(x), \quad \tau \circ \mathrm{Ad}_{\Gamma_0}(x) = \mathrm{Ad}_{\Gamma_1} \circ \tau(x), \quad x \in \pi_0(\mathcal{A}_R)''. \tag{3.8}$$

This completes the proof. □

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

## 4. Stacking and group law of fermionic SPT phases

### *4.1. The graded tensor product*

Let $(\mathcal{M}_1, \mathrm{Ad}_{\Gamma_1})$ and $(\mathcal{M}_2, \mathrm{Ad}_{\Gamma_2})$ be spatially graded von Neumann algebras acting on on $\mathcal{H}_1$, $\mathcal{H}_2$ with grading operators $\Gamma_1, \Gamma_2$. We define a product and involution on the algebraic tensor product $\mathcal{M}_1 \odot \mathcal{M}_2$ by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\partial b_1 \partial a_2}(a_1 a_2 \hat{\otimes} b_1 b_2),$$
$$(a \hat{\otimes} b)^* = (-1)^{\partial a \partial b} a^* \hat{\otimes} b^* \tag{4.1}$$

for homogeneous elementary tensors. The algebraic tensor product with this multiplication and involution is a $*$-algebra, denoted $\mathcal{M}_1 \hat{\odot} \mathcal{M}_2$. On the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\pi(a \hat{\otimes} b) := a\Gamma_1^{\partial b} \otimes b \tag{4.2}$$

for homogeneous $a \in \mathcal{M}_1$, $b \in \mathcal{M}_2$ defines a faithful $*$-representation of $\mathcal{M}_1 \hat{\odot} \mathcal{M}_2$. We call the von Neumann algebra generated by $\pi(\mathcal{M}_1 \hat{\odot} \mathcal{M}_2)$ the graded tensor product of $(\mathcal{M}_1, \mathcal{H}_1, \Gamma_1)$ and $(\mathcal{M}_2, \mathcal{H}_2, \Gamma_2)$ and denote it by $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$. It is simple to check that $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ is a spatially graded von Neumann algebra with a grading operator $\Gamma_1 \otimes \Gamma_2$.

For $a \in \mathcal{M}_1$ and homogeneous $b \in \mathcal{M}_2$, we denote $\pi(a \hat{\otimes} b)$ by $a \hat{\otimes} b$, embedding $\mathcal{M}_1 \hat{\odot} \mathcal{M}_2$ in $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$. Note that $\partial(a \hat{\otimes} b) = \partial(a) + \partial(b)$ for homogeneous $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$.

Fix a finite group $G$ and a homomorphism $\mathfrak{p} : G \to \mathbb{Z}_2$. Let $(\mathcal{M}_1, \mathrm{Ad}_{\Gamma_1}, \alpha_1)$ and $(\mathcal{M}_2, \mathrm{Ad}_{\Gamma_2}, \alpha_2)$ be graded $W^*$-$(G, \mathfrak{p})$-dynamical systems, where $(\mathcal{M}_1, \mathrm{Ad}_{\Gamma_1})$ and $(\mathcal{M}_2, \mathrm{Ad}_{\Gamma_2})$ are spatially graded, balanced, central and type I. We may define an action $\alpha_1 \hat{\otimes} \alpha_2$ of $G$ on $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ by

$$\left(\alpha_1 \hat{\otimes} \alpha_2\right)_g (a \hat{\otimes} b) = \alpha_{1,g}(a) \hat{\otimes} \alpha_{2,g}(b), \quad g \in G \tag{4.3}$$

for all homogeneous $a \in \mathcal{M}_1$ and $b \in \mathcal{M}_2$; see Lemma A.8.

### *4.2. Stacking and the group law*

In this section, we show that $W^*$-$(G, \mathfrak{p})$-dynamical systems of balanced, central, type I and spatially graded von Neumann algebras are closed under graded tensor products. Furthermore, our index from Definition 2.10 obeys a twisted group law (a generalised Wall group law) under this operation.

**Theorem 4.1.** *Let $(\mathcal{M}_1, \mathrm{Ad}_{\Gamma_1}, \alpha_1)$, $(\mathcal{M}_2, \mathrm{Ad}_{\Gamma_2}, \alpha_2)$ be graded $W^*$-$(G, \mathfrak{p})$-dynamical systems with balanced, central and spatially graded type I von Neumann algebras. Then the triple $(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2, \mathrm{Ad}_{\Gamma_1 \otimes \Gamma_2}, \alpha_1 \hat{\otimes} \alpha_2)$ is a graded $W^*$-$(G, \mathfrak{p})$-dynamical system with a balanced, central and spatially graded type I von Neumann algebra. If $\mathrm{Ind}(\mathcal{M}_i, \mathrm{Ad}_{\Gamma_i}, \alpha_i) = (\kappa_i, \mathfrak{q}_i, [v_i])$, $i = 1, 2$, then*

$$\begin{aligned}\mathrm{Ind}(&\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2, \mathrm{Ad}_{\Gamma_1 \otimes \Gamma_2}, \alpha_1 \hat{\otimes} \alpha_2) \\ &= \left(\kappa_1 + \kappa_2,\ \mathfrak{q}_1 + \mathfrak{q}_2 + \kappa_1\kappa_2\mathfrak{p},\ [v_1\, v_2\, \epsilon_{\mathfrak{p}}(\kappa_1, \mathfrak{q}_1, \kappa_2, \mathfrak{q}_2)]\right),\end{aligned} \tag{4.4}$$

*where $\epsilon_{\mathfrak{p}}(\kappa_1, \mathfrak{q}_1, \kappa_2, \mathfrak{q}_2)$ is a group 2-cocycle defined by*

$$\epsilon_{\mathfrak{p}}(\kappa_1, \mathfrak{q}_1, \kappa_2, \mathfrak{q}_2)(g, h) = (-1)^{\mathfrak{q}_1(g)\mathfrak{q}_2(h) + (\kappa_1 - \kappa_2)(\kappa_1\mathfrak{q}_2(g) + \kappa_2\mathfrak{q}_1(g))\cdot\mathfrak{p}(h)}, \quad g, h \in G. \tag{4.5}$$

**Remarks 4.2.**

(i) One can check that (4.4) gives an abelian group law, which is not surprising because of the corresponding properties of the graded tensor product.

(ii) The group law (4.4) is a little cumbersome in full generality but simplifies in many examples of interest. For example, if $\alpha_1$ and $\alpha_2$ are linear group actions, $\mathfrak{p}(g) = 0$ for all $g \in G$, we recover the

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

more familiar twisted sum formula

$$(\kappa_1,\ \mathfrak{q}_1,\ [v_1])\cdot(\kappa_2,\ \mathfrak{q}_2,\ [v_2]) = (\kappa_1+\kappa_2,\ \mathfrak{q}_1+\mathfrak{q}_2,\ [v_1\, v_2\, \epsilon(\mathfrak{q}_1,\mathfrak{q}_2)]). \tag{4.6}$$

*Proof.* By Lemma A.5 and Lemma 2.12, we may assume that

$$(\mathcal{M}_i, \mathrm{Ad}_{\Gamma_i}, \alpha_i) = (\mathcal{R}_{\kappa_i,\mathcal{K}_i}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_i}}, \mathrm{Ad}_{V_i}) \in \mathcal{S}_{\kappa_i}. \tag{4.7}$$

Let

$$\mathrm{Ind}\left(\mathcal{R}_{\kappa_i,\mathcal{K}_i}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_i}}, \mathrm{Ad}_{V_i}\right) = (\kappa_i, \mathfrak{q}_i, [v_i]), \quad i=1,2. \tag{4.8}$$

We would like to show that

$$\left(\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_1}}\ \hat{\otimes}\ \mathrm{Ad}_{\Gamma_{\mathcal{K}_1}}, \mathrm{Ad}_{V_1}\ \hat{\otimes}\ \mathrm{Ad}_{V_2}\right) \sim (\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_V) \in \mathcal{S}_\kappa \tag{4.9}$$

for suitably chosen $\kappa = 0, 1$, Hilbert space $\mathcal{K}$ and projective representation $V$ on $\mathcal{K}\otimes\mathbb{C}^2$, satisfying

$$\mathrm{Ind}(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) = \left(\kappa_1+\kappa_2,\ \mathfrak{q}_1+\mathfrak{q}_2+\kappa_1\kappa_2\mathfrak{p},\ [v_1\, v_2\, \epsilon_{\mathfrak{p}}(\kappa_1,\mathfrak{q}_1,\kappa_2,\mathfrak{q}_2)]\right). \tag{4.10}$$

**(Case:** $\kappa_1 = 0$ **or** $\kappa_2 = 0$**)**
We set the following notation:

$$\mathcal{K} := \mathcal{K}_1\otimes\mathcal{K}_2\otimes\mathbb{C}^2, \qquad \lambda = \begin{cases} 1, & \text{if } \kappa_1=\kappa_2=0,\\ 2, & \text{if } \kappa_1=1,\ \kappa_2=0,\\ 3, & \text{if } \kappa_1=0,\ \kappa_2=1,\end{cases} \tag{4.11}$$

and define the unitary $v : \mathbb{C}^2\otimes\mathcal{K}_2 \to \mathcal{K}_2\otimes\mathbb{C}^2$,

$$v(\xi\otimes\eta) = \eta\otimes\xi, \quad \xi\in\mathbb{C}^2, \quad \eta\in\mathcal{K}_2. \tag{4.12}$$

Using the standard basis $\{e_0, e_1\}$ of $\mathbb{C}^2$, we define the unitaries $w_1, w_2, w_3$ on $\mathbb{C}^2\otimes\mathbb{C}^2$ by

$$\begin{aligned}
&w_1(e_0\otimes e_0) = e_0\otimes e_0, && w_1(e_1\otimes e_1) = e_1\otimes e_0, && w_1(e_1\otimes e_0) = e_0\otimes e_1,\\
&w_1(e_0\otimes e_1) = e_1\otimes e_1, && w_2(e_0\otimes e_0) = e_0\otimes e_0, && w_2(e_1\otimes e_1) = e_1\otimes e_0,\\
&w_2(e_1\otimes e_0) = e_0\otimes e_1, && w_2(e_0\otimes e_1) = -e_1\otimes e_1, && w_3(e_0\otimes e_0) = e_0\otimes e_0,\\
&w_3(e_1\otimes e_1) = e_1\otimes e_0, && w_3(e_1\otimes e_0) = e_1\otimes e_1, && w_3(e_0\otimes e_1) = e_0\otimes e_1.
\end{aligned}$$

By direct calculation, we may check

$$\mathrm{Ad}_{w_\lambda}(\sigma_z\otimes\sigma_z) = \mathbb{I}_{\mathbb{C}^2}\otimes\sigma_z, \quad \lambda = 1,2,3, \tag{4.13}$$

$$\mathrm{Ad}_{w_2}(\sigma_x\otimes\sigma_z) = \mathbb{I}_{\mathbb{C}^2}\otimes\sigma_x, \quad \mathrm{Ad}_{w_3}(\mathbb{I}_{\mathbb{C}^2}\otimes\sigma_x) = \mathbb{I}_{\mathbb{C}^2}\otimes\sigma_x. \tag{4.14}$$

We now define unitary $U_\lambda : \mathcal{K}_1\otimes\mathbb{C}^2\otimes\mathcal{K}_2\otimes\mathbb{C}^2 \to \mathcal{K}\otimes\mathbb{C}^2$ such that

$$U_\lambda := (\mathbb{I}_{\mathcal{K}_1}\otimes\mathbb{I}_{\mathcal{K}_2}\otimes w_\lambda)(\mathbb{I}_{\mathcal{K}_1}\otimes v\otimes\mathbb{I}_{\mathbb{C}^2}), \quad \lambda = 1,2,3. \tag{4.15}$$

By (4.13), we have

$$\mathrm{Ad}_{U_\lambda}\left(\Gamma_{\mathcal{K}_1}\otimes\Gamma_{\mathcal{K}_2}\right) = \Gamma_{\mathcal{K}}, \quad \lambda = 1,2,3; \tag{4.16}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

hence, for $x \in \mathcal{R}_{\kappa_1,\mathcal{K}_1} \hat{\otimes} \mathcal{R}_{\kappa_2,\mathcal{K}_2}$,

$$\mathrm{Ad}_{U_\lambda} \circ \left(\mathrm{Ad}_{\Gamma_{\mathcal{K}_1}} \hat{\otimes} \mathrm{Ad}_{\Gamma_{\mathcal{K}_2}}\right)(x) = \mathrm{Ad}_{\Gamma_{\mathcal{K}}} \circ \mathrm{Ad}_{U_\lambda}(x), \quad \lambda = 1, 2, 3. \tag{4.17}$$

By (4.14), when $\lambda = 2$, for $(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_x) \hat{\otimes} (\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_z) \in \mathcal{R}_{\kappa_1,\mathcal{K}_1} \hat{\otimes} \mathcal{R}_{\kappa_2,\mathcal{K}_2}$ we have

$$\mathrm{Ad}_{U_2}\left((\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_x) \hat{\otimes} (\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_z)\right) = \mathbb{I}_{\mathcal{K}} \otimes \sigma_x. \tag{4.18}$$

Similarly, when $\lambda = 3$, for $(\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) \hat{\otimes} (\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_x) \in \mathcal{R}_{\kappa_1,\mathcal{K}_1} \hat{\otimes} \mathcal{R}_{\kappa_2,\mathcal{K}_2}$,

$$\mathrm{Ad}_{U_3}\left((\mathbb{I}_{\mathcal{K}_1} \otimes \sigma_z) \hat{\otimes} (\mathbb{I}_{\mathcal{K}_2} \otimes \sigma_x)\right) = \mathbb{I}_{\mathcal{K}} \otimes \sigma_x. \tag{4.19}$$

Let $[\tilde{v}_i]$ be the second cohomology class associated to the projective representation $V_i$, $i = 1, 2$. We set

$$V_g := \mathrm{Ad}_{U_\lambda}\left(V_{1,g} \otimes V_{2,g} \Gamma_{\mathcal{K}_2}^{\mathfrak{q}_1(g)}\right), \quad g \in G, \quad \lambda = 1, 2, 3. \tag{4.20}$$

This gives a projective unitary/anti-unitary representation $V$ of $G$ on $\mathcal{K} \otimes \mathbb{C}^2$ relative to $\mathfrak{p}$. Using that $\mathrm{Ad}_{V_{2,g}}(\Gamma_{\mathcal{K}_2}) = (-1)^{\mathfrak{q}_2(g)} \Gamma_{\mathcal{K}_2}$ for $g \in G$, the second cohomology class associated to $V$ is equal to $[\tilde{v}_1 \, \tilde{v}_2 \, \epsilon(\mathfrak{q}_1, \mathfrak{q}_2)] \in H^2(G, U(1)_{\mathfrak{p}})$, where $\epsilon(\mathfrak{q}_1, \mathfrak{q}_2)$ is given in (2.2). By Lemmas A.6 and A.7, we have that for $x \in \mathcal{R}_{\kappa_1,\mathcal{K}_1} \hat{\otimes} \mathcal{R}_{\kappa_2,\mathcal{K}_2}$, $g \in G$ and any $\lambda = 1, 2, 3$,

$$\mathrm{Ad}_{V_g} \circ \mathrm{Ad}_{U_\lambda}(x) = \mathrm{Ad}_{U_\lambda} \circ \mathrm{Ad}_{V_{1,g} \otimes V_{2,g} \Gamma_{\mathcal{K}_2}^{\mathfrak{q}_1(g)}}(x) = \mathrm{Ad}_{U_\lambda} \circ \left(\alpha_{1,g} \hat{\otimes} \alpha_{2,g}\right)(x). \tag{4.21}$$

In particular, for $\lambda = 2, 3$, we also have

$$\mathrm{Ad}_{V_g}(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x) = (-1)^{\mathfrak{q}_1(g) + \mathfrak{q}_2(g)} (\mathbb{I}_{\mathcal{K}} \otimes \sigma_x), \quad g \in G, \tag{4.22}$$

from (4.18) and (4.19).

By (4.16), we have

$$\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = \mathrm{Ad}_{U_\lambda} \circ \mathrm{Ad}_{V_{1,g} \otimes V_{2,g} \Gamma_{\mathcal{K}_2}^{\mathfrak{q}_1(g)}}\left(\Gamma_{\mathcal{K}_1} \otimes \Gamma_{\mathcal{K}_2}\right) = (-1)^{\mathfrak{q}_1(g) + \mathfrak{q}_2(g)} \Gamma_{\mathcal{K}}, \quad g \in G. \tag{4.23}$$

Having set up the required preliminaries, we now consider the $W^*$-$(G, \mathfrak{p})$-dynamical system $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ and show equivalence with the graded tensor product in the three cases where $\kappa_1$ or $\kappa_2 = 0$.

(i)-1 For $\lambda = 1$ (i.e., $\kappa_1 = \kappa_2 = 0$), we set $\kappa = 0$ and note from (4.23) that $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$. In this case, $[\tilde{v}_i] = [v_i]$ and $\epsilon_{\mathfrak{p}}(0, \mathfrak{q}_1, 0, \mathfrak{q}_2) = \epsilon(\mathfrak{q}_1, \mathfrak{q}_2)$. Hence, the second cohomology class of $V$ is $[v_1 v_2 \epsilon_{\mathfrak{p}}(0, \mathfrak{q}_1, 0, \mathfrak{q}_2)]$. With this and (4.23), the index of $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$ is given by (4.10). So we just need to show equivalence of the $W^*$-$(G, \mathfrak{p})$-dynamical system with the graded tensor product. The equivalence is given by a $*$-isomorphism

$$\iota := \mathrm{Ad}_{U_1} : \mathcal{B}(\mathcal{K}_1 \otimes \mathbb{C}^2 \otimes \mathcal{K}_2 \otimes \mathbb{C}^2) = \mathcal{R}_{\kappa_1,\mathcal{K}_1} \hat{\otimes} \mathcal{R}_{\kappa_2,\mathcal{K}_2} \to \mathcal{B}(\mathcal{K} \otimes \mathbb{C}^2) = \mathcal{R}_{0,\mathcal{K}}. \tag{4.24}$$

By (4.17) and (4.21), $\iota$ satisfies the required conditions (2.4) and (2.5) for equivalence of $W^*$-$(G, \mathfrak{p})$-dynamical systems.

(i)-2 For $\lambda = 2$ (i.e., $\kappa_1 = 1$, $\kappa_2 = 0$), set $\kappa = 1$. By (4.22) and (4.23), we see that $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$. Note that $[\tilde{v}_1] = [v_1 \, \epsilon(\mathfrak{q}_1, \mathfrak{p})] \in H^2(G, U(1)_{\mathfrak{p}})$ (see Lemma 2.7 and Definition 2.10), with $\tilde{v}_2 = v_2$. Hence, the second cohomology associated to our projective representation $V$ is

$$\left[\tilde{v}_1 \, \tilde{v}_2 \, \epsilon(\mathfrak{q}_1, \mathfrak{q}_2)\right] = \left[v_1 \, v_2 \, \epsilon(\mathfrak{q}_1, \mathfrak{p}) \, \epsilon(\mathfrak{q}_1, \mathfrak{q}_2)\right]. \tag{4.25}$$

Combining this and (4.23), the second cohomology associated to the projective representation $V^{(0)}$ (cf. Lemma 2.7 and Definition 2.10) is

$$\begin{aligned}\left[\tilde{v}_1\,\tilde{v}_2\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\,\epsilon(\mathfrak{q}_1+\mathfrak{q}_2,\mathfrak{p})\right] &= \left[v_1\,v_2\,\epsilon(\mathfrak{q}_1,\mathfrak{p})\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\,\epsilon(\mathfrak{q}_1+\mathfrak{q}_2,\mathfrak{p})\right]\\ &= \left[v_1\,v_2\,\epsilon_{\mathfrak{p}}(1,\mathfrak{q}_1,0,\mathfrak{q}_2)\right].\end{aligned}$$

From this and (4.23), we see that the index of $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ is given by (4.10).

Now we show that $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$ is equivalent to the graded tensor product (4.9). From Lemma A.4, the commutant of $\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}$ is $\mathbb{C}\mathbb{I}_{\mathcal{K}_1\otimes\mathbb{C}^2\otimes\mathcal{K}_2\otimes\mathbb{C}^2} + \mathbb{C}\mathbb{I}_{\mathcal{K}_1}\otimes\sigma_x\otimes\mathbb{I}_{\mathcal{K}_2}\otimes\sigma_z$. Note that by (4.18), $\mathrm{Ad}_{U_2}$ maps the commutant to $\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathbb{I}_{\mathbb{C}^2} + \mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\sigma_x = (\mathcal{R}_{\kappa,\mathcal{K}})'$. Therefore, we have $\mathrm{Ad}_{U_2}(\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}) = \mathcal{R}_{\kappa,\mathcal{K}}$. Hence, $\iota := \mathrm{Ad}_{U_2}|_{\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}}$ defines a $*$-isomorphism $\iota : \mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2} \to \mathcal{R}_{\kappa,\mathcal{K}}$. By (4.17) and (4.21), $\iota$ satisfies the required conditions of an equivalence of $W^*$-$(G,\mathfrak{p})$-dynamical systems.

(i)-3 For $\lambda = 3$ (i.e., $\kappa_1 = 0$, $\kappa_2 = 1$), we set $\kappa = 1$. By (4.23) and (4.22), we see that $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$. We also have that $[\tilde{v}_1] = [v_1]$ and $[\tilde{v}_2] = [v_2\,\epsilon(\mathfrak{q}_2,\mathfrak{p})]$. Hence, the second cohomology class associated to $V$ is

$$\left[\tilde{v}_1\,\tilde{v}_2\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\right] = \left[v_1\,v_2\,\epsilon(\mathfrak{q}_2,\mathfrak{p})\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\right]. \tag{4.26}$$

Hence, from (4.23) the cohomology class associated to $V^{(0)}$ is

$$\left[\tilde{v}_1\,\tilde{v}_2\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\,\epsilon(\mathfrak{q}_1+\mathfrak{q}_2,\mathfrak{p})\right] = \left[v_1\,v_2\,\epsilon_{\mathfrak{p}}(0,\mathfrak{q}_1,1,\mathfrak{q}_2)\right] \tag{4.27}$$

and the index of $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ is given by (4.10).

We now show that $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$ is equivalent to the graded tensor product. From Lemma A.4, the commutant of $\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}$ is $\mathbb{C}\mathbb{I}_{\mathcal{K}_1\otimes\mathbb{C}^2\otimes\mathcal{K}_2\otimes\mathbb{C}^2} + \mathbb{C}\mathbb{I}_{\mathcal{K}_1\otimes\mathbb{C}^2\otimes\mathcal{K}_2}\otimes\sigma_x$, which by (4.19) is mapped to $\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathbb{I}_{\mathbb{C}^2} + \mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\sigma_x = (\mathcal{R}_{\kappa,\mathcal{K}})'$ by $\mathrm{Ad}_{U_3}$. Therefore, $\mathrm{Ad}_{U_3}(\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}) = \mathcal{R}_{\kappa,\mathcal{K}}$ and $\iota := \mathrm{Ad}_{U_3}|_{\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}}$ define a $*$-isomorphism $\iota : \mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2} \to \mathcal{R}_{\kappa,\mathcal{K}}$ and implement an equivalence of $W^*$-$(G,\mathfrak{p})$-dynamical systems.

**(Case:** $\kappa_1 = \kappa_2 = 1$)

Set $\kappa := 0$ and $\mathcal{K} := \mathcal{K}_1\otimes\mathcal{K}_2$. We define a projective representation $V$ of $G$ on $\mathcal{K}\otimes\mathbb{C}^2$ relative to $\mathfrak{p}$ by

$$V_g := V_{1,g}^{(0)}\otimes V_{2,g}^{(0)}\otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}_1(g)}\sigma_x^{\mathfrak{q}_2(g)+\mathfrak{p}(g)}, \quad g\in G. \tag{4.28}$$

Here $V_i^{(0)}$ is the projective representation on $\mathcal{K}_1$ such that $V_{i,g} = V_{i,g}^{(0)}\otimes C^{\mathfrak{p}(g)}\sigma_y^{\mathfrak{q}_i(g)}$ for $i = 1,2$ (see Lemma 2.7). Then we have

$$\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}_1(g)+\mathfrak{q}_2(g)+\mathfrak{p}(g)}\Gamma_{\mathcal{K}}, \quad g\in G. \tag{4.29}$$

Hence, $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$. Because $\sigma_y$ anti-commutes with $\sigma_x$ and $C$, and $\sigma_x$ commutes with $C$, the second cohomology class associated to the projective representation $V$ is

$$\left[v_1\,v_2\,\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)\right] = \left[v_1\,v_2\,\epsilon_{\mathfrak{p}}(1,\mathfrak{q}_1,1,\mathfrak{q}_2)\right], \tag{4.30}$$

where we recall that $[\epsilon(\mathfrak{q}_1,\mathfrak{q}_2)] = [\epsilon(\mathfrak{q}_2,\mathfrak{q}_1)]$. Hence, the triple $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$ has index given by (4.10).

Now we show (4.9) for the constructed $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_{\mathcal{K}}}, \mathrm{Ad}_{V_g})$. Regarding $\mathfrak{C}$ as a graded von Neumann algebra $(\mathfrak{C}, \mathrm{Ad}_{\sigma_z}) \subset \mathrm{M}_2$, there is a $*$-isomorphism $\iota_0 : \mathfrak{C}\hat{\otimes}\mathfrak{C} \to \mathrm{M}_2$ such that

$$\iota_0(\mathbb{I}\hat{\otimes}\mathbb{I}) = \mathbb{I}, \quad \iota_0(\sigma_x\hat{\otimes}\mathbb{I}) := \sigma_x, \quad \iota_0(\mathbb{I}\hat{\otimes}\sigma_x) := \sigma_y, \quad \iota_0(\sigma_x\hat{\otimes}\sigma_x) := i\sigma_z. \tag{4.31}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Noting $\mathrm{Ad}_{\mathbb{I}_{\mathcal{K}_1}\otimes v\otimes \mathbb{I}_{\mathbb{C}^2}}\left(\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}\right) = \mathcal{B}(\mathcal{K}_1)\otimes\mathcal{B}(\mathcal{K}_2)\otimes\left(\mathfrak{C}\hat{\otimes}\mathfrak{C}\right)$ with $v$ in (4.12), we obtain a $*$-isomorphism $\iota : \mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}\to\mathcal{B}(\mathcal{K})\otimes\mathrm{M}_2 = \mathcal{R}_{\kappa,\mathcal{K}}$ given by

$$\iota(x) := (\mathrm{id}_{\mathcal{K}}\otimes\iota_0)\circ\mathrm{Ad}_{\mathbb{I}_{\mathcal{K}_1}\otimes v\otimes\mathbb{I}_{\mathbb{C}^2}}(x),\quad x\in\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}. \tag{4.32}$$

We then have

$$\begin{aligned}\mathrm{Ad}_{V_g}\circ\iota\left((a\otimes\sigma_x)\ \hat{\otimes}\ (b\otimes\mathbb{I}_{\mathbb{C}^2})\right) &= \mathrm{Ad}_{V_g}\left(a\otimes b\otimes\sigma_x\right)\\ &= \mathrm{Ad}_{V_{1,g}^{(0)}}(a)\otimes\mathrm{Ad}_{V_{2,g}^{(0)}}(b)\otimes(-1)^{\mathfrak{q}_1(g)}\sigma_x\\ &= \iota\left(\mathrm{Ad}_{V_{1,g}}(a\otimes\sigma_x)\hat{\otimes}\left(\mathrm{Ad}_{V_{2,g}}(b\otimes\mathbb{I}_{\mathbb{C}^2})\right)\right)\\ &= \iota\circ\left(\alpha_{1,g}\hat{\otimes}\alpha_{2,g}\right)\left((a\otimes\sigma_x)\ \hat{\otimes}\ (b\otimes\mathbb{I}_{\mathbb{C}^2})\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Ad}_{V_g}\circ\iota\left((a\otimes\mathbb{I}_{\mathbb{C}^2})\ \hat{\otimes}\ (b\otimes\sigma_x)\right) &= \mathrm{Ad}_{V_g}\left(a\otimes b\otimes\sigma_y\right)\\ &= \mathrm{Ad}_{V_{1,g}^{(0)}}(a)\otimes\mathrm{Ad}_{V_{2,g}^{(0)}}(b)\otimes(-1)^{\mathfrak{q}_2(g)}\sigma_y\\ &= \iota\left(\mathrm{Ad}_{V_{1,g}}(a\otimes\mathbb{I}_{\mathbb{C}^2})\hat{\otimes}\left(\mathrm{Ad}_{V_{2,g}}(b\otimes\sigma_x)\right)\right)\\ &= \iota\circ\left(\alpha_{1,g}\hat{\otimes}\alpha_{2,g}\right)\left((a\otimes\mathbb{I}_{\mathbb{C}^2})\ \hat{\otimes}\ (b\otimes\sigma_x)\right)\end{aligned}$$

for all $a\in\mathcal{B}(\mathcal{K}_1)$, $b\in\mathcal{B}(\mathcal{K}_2)$. Because the elements $(a\otimes\sigma_x)\hat{\otimes}(b\otimes\mathbb{I}_{\mathbb{C}^2})$ and $(a\otimes\mathbb{I}_{\mathbb{C}^2})\hat{\otimes}(b\otimes\sigma_x)$ generate $\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}$, we see that $\mathrm{Ad}_{V_g}\circ\iota(x) = \iota\circ\left(\alpha_{1,g}\hat{\otimes}\alpha_{2,g}\right)(x)$ for $x\in\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}$. We also see from (4.31) that $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}\circ\iota(x) = \iota\circ\left(\mathrm{Ad}_{\Gamma_1}\ \hat{\otimes}\ \mathrm{Ad}_{\Gamma_2}\right)(x)$ for $x\in\mathcal{R}_{\kappa_1,\mathcal{K}_1}\hat{\otimes}\mathcal{R}_{\kappa_2,\mathcal{K}_2}$. Hence, we obtain (4.9). □

**Example 4.3** (Time-reversal symmetry and the $\mathbb{Z}_8$-classification)**.** As a simple example, let us consider fermionic SPT phases with time-reversal symmetry. That is, we take $G = \mathbb{Z}_2 = \{0,1\}$ with $\mathfrak{p}(1) = 1$. We let $\alpha = \alpha_1$ be the anti-linear $*$-automorphism of order 2 from the nontrivial element. Therefore, if $G$ acts on a balanced, central and type I von Neumann algebra, then $\alpha$ is implemented on a graded Hilbert space $\mathcal{K}$ by $\mathrm{Ad}_R$ with $R$ anti-unitary. Following [29], we can ensure that $R^2 = \pm\mathbb{I}_{\mathcal{K}}$ and so the group 2-cocycle is determined by the sign of $R^2$.

The data $\mathbb{Z}_2\times H^1(\mathbb{Z}_2,\mathbb{Z}_2)\times H^2(\mathbb{Z}_2,U(1)_{\mathfrak{p}})$ from Theorem 4.1 is wholly determined by the triple $[\kappa;\varepsilon,\pm]$, where $\varepsilon = \mathfrak{q}(1)\in\mathbb{Z}_2$ and $\pm$ is the sign of $R^2$. Our choice of notation is so that our results can easily be compared with [26, Appendix A] and [40]. Following (4.4), the triple has the (abelian) composition law under stacking

$$\begin{aligned}{}[0;\varepsilon_1,\xi_1][0,\varepsilon_2,\xi_2] &= [0;\varepsilon_1+\varepsilon_2,(-)^{\varepsilon_1\varepsilon_2}\xi_1\xi_2]\\ [0;\varepsilon_1,\xi_1][1,\varepsilon_2,\xi_2] &= [1;\varepsilon_1+\varepsilon_2,(-)^{\varepsilon_1+\varepsilon_1\varepsilon_2}\xi_1\xi_2]\\ [1;\varepsilon_1,\xi_1][1,\varepsilon_2,\xi_2] &= [0;\varepsilon_1+\varepsilon_2+1,(-)^{\varepsilon_1\varepsilon_2}\xi_1\xi_2].\end{aligned}$$

One therefore sees that $\mathbb{Z}_2\times H^1(\mathbb{Z}_2,\mathbb{Z}_2)\times H^2(\mathbb{Z}_2,U(1)_{\mathfrak{p}})\cong\mathbb{Z}_8$ with generator $[1;0,+]$. Hence, we recover and extend the $\mathbb{Z}_8$-classification of time-reversal symmetric fermionic SPT phases in one dimension considered for finite systems in [14, 15, 11].

## 5. Translation-invariant states

In this section, we derive a representation of pure, split, translation-invariant and $\alpha$-invariant states in terms of a finite set of operators on Hilbert spaces. The idea of the proof is the same as quantum spin case (cf. [6, 21]), although anti-commutativity results in richer structures.

Recall the integer shift $S_x$ on $l^2(\mathbb{Z}) \otimes \mathbb{C}^d$, $x \in \mathbb{Z}$, which defines the $*$-automorphism $\beta_{S_x} \in \mathrm{Aut}(\mathcal{A})$. Let $\omega$ be a pure, split, $\alpha$-invariant and translation-invariant state on $\mathcal{A}$. In particular, such states are $\Theta$-invariant (see [9, Example 5.2.21]). By Proposition 2.9 and Lemma 2.16, the graded $W^*$-$(G,\mathfrak{p})$-dynamical system $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ associated to $\omega$ is equivalent to some $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$. We denote this $\kappa$ by $\kappa_\omega$. The space translation lifts to an endomorphism on $\pi_\omega(\mathcal{A}_R)''$.

**Lemma 5.1.** *Let $\omega$ be a pure, split, $\alpha$-invariant and translation-invariant state on $\mathcal{A}$. Suppose that the graded $W^*$-$(G,\mathfrak{p})$-dynamical system $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ associated to $\omega$ is equivalent to $(\mathcal{R}_{\kappa,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_\kappa$, via a $*$-isomorphism $\iota : \pi_\omega(\mathcal{A}_R)'' \to \mathcal{R}_{\kappa,\mathcal{K}}$. Then there is an injective $*$-endomorphism $\rho$ on $\mathcal{R}_{\kappa,\mathcal{K}}$ such that*

$$\iota \circ \pi_\omega \circ \beta_{S_1}(A) = \rho \circ \iota \circ \pi_\omega(A), \quad A \in \mathcal{A}_R. \tag{5.1}$$

*Furthermore, we have*

$$a\rho(b) - (-1)^{\partial a \partial b}\rho(b)a = 0, \tag{5.2}$$

*for homogeneous $a \in \iota \circ \pi_\omega(\mathcal{A}_{\{0\}})$ and $b \in \mathcal{R}_{\kappa,\mathcal{K}}$.*

*Proof.* By the translation invariance of $\omega$, the space translation $\beta_{S_1}$ is lifted to an automorphism $\hat{\beta}_{S_1}$ on $\pi_\omega(\mathcal{A})''$. Restricting $\hat{\beta}_{S_1}$ to $\pi_\omega(\mathcal{A}_R)''$, we obtain an injective $*$-endomorphism $\tilde{\beta}$ on $\pi_\omega(\mathcal{A}_R)''$. We then see that $\rho := \iota \circ \tilde{\beta} \circ \iota^{-1} : \mathcal{R}_{\kappa,\mathcal{K}} \to \mathcal{R}_{\kappa,\mathcal{K}}$ is an injective endomorphism on $\mathcal{R}_{\kappa,\mathcal{K}}$ satisfying (5.1). Because $\beta_{S_1}(\mathcal{A}_R) \subset \mathcal{A}_{\mathbb{Z}\geq 1}$, we see that $a_0\beta_{S_1}(a_1) - (-1)^{\partial a_0 \partial a_1}\beta_{S_1}(a_1)a_0 = 0$ for homogeneous $a_0 \in \mathcal{A}_{\{0\}}$ and $a_1 \in \mathcal{A}_R$. Then, because $\rho\left(\mathcal{R}_{\kappa,\mathcal{K}}\right) = \left(\iota \circ \pi_\omega \circ \beta_{S_1}\left(\mathcal{A}_R\right)\right)''$, Equation (5.2) follows. □

Let $\mathcal{P}$ be the power set $\mathcal{P} = \mathcal{P}(\{1,\dots,d\}) = 2^{\{1,\dots,d\}}$ of $\{1,\dots,d\}$. We denote the parity of the number of the elements in $\mu \in \mathcal{P}$ by $|\mu| = \#\mu \bmod 2$. We denote by $\{\psi_\mu\}_{\mu\in\mathcal{P}}$ the standard basis of $\mathcal{F}(\mathbb{C}^d)$. Namely, with the Fock vacuum $\Omega_d$ of $\mathcal{F}(\mathbb{C}^d)$ and the standard basis $\{e_i\}_{i=1}^d$ of $\mathbb{C}^d$, $\psi_\mu$ for $\mu \neq \emptyset$ is given by $\psi_\mu = C_\mu a^*(e_{\mu_1})a^*(e_{\mu_2})\cdots a^*(e_{\mu_l})\Omega_d$ with $l = \#\mu$, $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ with $\mu_1 < \mu_2 \cdots < \mu_l$ and a suitable normalisation factor $C_\mu \in \mathbb{C} \setminus \{0\}$. For the empty set $\mu = \emptyset$, we set $\psi_\emptyset := \Omega_d$.

We denote the matrix units of $\mathcal{A}_{\{0\}} \simeq \mathcal{B}(\mathcal{F}(\mathbb{C}^d)) \simeq \mathrm{M}_{2^d}$ associated to the standard basis $\{\psi_\mu\}_{\mu\in\mathcal{P}}$ by $\{E^{(0)}_{\mu,\nu}\}$, $\mu,\nu \in \mathcal{P}$. Because $\Theta$ is implemented by the second quantisation of $-\mathbb{I}_{\mathbb{C}^d}$,

$$\Gamma(-\mathbb{I}) = \sum_{\mu\in\mathcal{P}} (-1)^{|\mu|} E^{(0)}_{\mu\mu} \in \mathcal{A}_{\{0\}}, \tag{5.3}$$

we see that

$$\Theta(E^{(0)}_{\mu,\nu}) = (-1)^{|\mu|+|\nu|} E^{(0)}_{\mu,\nu}, \qquad \mu,\nu \in \mathcal{P}. \tag{5.4}$$

We set $E^{(x)}_{\mu,\nu} := \beta_{S_x}\left(E^{(0)}_{\mu,\nu}\right)$ for general $x \in \mathbb{Z}$. Clearly, $\{E^{(x)}_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}}$ are matrix units of $\mathcal{A}_{\{x\}}$.

**Lemma 5.2.** *Let $\omega$ be a pure, split and translation-invariant state on $\mathcal{A}$ and $\hat{\beta}_{S_n}$ be the extension of $\beta_{S_n}$ to $\pi_\omega(\mathcal{A})''$, i.e. $\hat{\beta}_{S_n} \circ \pi_\omega(A) = \pi_\omega \circ \beta_{S_n}(A)$, $A \in \mathcal{A}$.*

(i) *If $x \in (\pi_\omega(\mathcal{A}_R)'')^{(0)}$, then* $\sigma$-weak $\lim_{n\to\infty} \hat{\beta}_{S_n}(x) = \langle\Omega_\omega, x\Omega_\omega\rangle \mathbb{I}_{\mathcal{H}_\omega}$.
(ii) *If $x \in (\pi_\omega(\mathcal{A}_R)'')^{(1)}$ and $\pi_\omega(\mathcal{A}_R)''$ is a factor, then*

$$\sigma\text{-weak} \lim_{n\to\infty} \pi_\omega\left(\Gamma(-\mathbb{I})\beta_{S_1}\left(\Gamma(-\mathbb{I})\right)\cdots\beta_{S_{n-1}}\left(\Gamma(-\mathbb{I})\right)\right)\hat{\beta}_{S_n}(x) = 0 = \langle\Omega_\omega, x\Omega_\omega\rangle \mathbb{I}_{\mathcal{H}_\omega}. \tag{5.5}$$

(iii) *If $x \in (\pi_\omega(\mathcal{A}_R)'')^{(1)}$ and $Z\left(\pi_\omega(\mathcal{A}_R)''\right) \cap \left(\pi_\omega(\mathcal{A}_R)''\right)^{(1)} \neq \{0\}$, then* $\sigma$-weak $\lim_{n\to\infty} \hat{\beta}_{S_n}(x) = 0 = \langle\Omega_\omega, x\Omega_\omega\rangle$.

*Proof.* First we note from the $\sigma$-weak continuity of $\hat{\beta}_{S_n}$ that

$$\hat{\beta}_{S_n}\left((\pi_\omega(\mathcal{A}_R)'')^{(\sigma)}\right) \subset \left((\pi_\omega \circ \beta_{S_n}(\mathcal{A}_R))''\right)^{(\sigma)}, \quad n \in \mathbb{N}, \quad \sigma = 0, 1. \tag{5.6}$$

(i) By (5.6), we have $\hat{\beta}_{S_n}\left((\pi_\omega(\mathcal{A}_R)'')^{(0)}\right) \subset \pi_\omega(\mathcal{A}_{[0,n-1]})'$. Therefore, for any $x \in (\pi_\omega(\mathcal{A}_R)'')^{(0)}$, any $\sigma$-weak accumulation point $z$ of $\{\hat{\beta}_{S_n}(x)\}$ belongs to $\pi_\omega(\mathcal{A}_R)' \cap (\pi_\omega(\mathcal{A}_R)'')^{(0)}$. But $\pi_\omega(\mathcal{A}_R)' \cap (\pi_\omega(\mathcal{A}_R)'')^{(0)} = \mathbb{C}\mathbb{I}_{\mathcal{H}_\omega}$ by Lemma 2.15. Hence, we have $z \in \mathbb{C}\mathbb{I}_{\mathcal{H}_\omega}$. Because $\langle \Omega_\omega, \hat{\beta}_{S_n}(x)\Omega_\omega\rangle = \langle \Omega_\omega, x\Omega_\omega\rangle$, this means $z = \langle \Omega_\omega, x\Omega_\omega\rangle \mathbb{I}_{\mathcal{H}_\omega}$. Because this holds for any accumulation point, we obtain $\sigma$-weak $\lim_{n\to\infty} \hat{\beta}_{S_n}(x) = \langle \Omega_\omega, x\Omega_\omega\rangle \mathbb{I}_{\mathcal{H}_\omega}$.

(ii) Suppose that $\pi_\omega(\mathcal{A}_R)''$ is a factor and set $Y_n := \Gamma(-\mathbb{I})\beta_{S_1}(\Gamma(-\mathbb{I}))\cdots\beta_{S_{n-1}}(\Gamma(-\mathbb{I}))$. Note that $\mathrm{Ad}_{Y_n}(B) = \Theta(B)$ for any $B \in \mathcal{A}_{[0,n-1]}$. Therefore, by (5.6), we have $\pi_\omega(Y_n)\hat{\beta}_{S_n}\left((\pi_\omega(\mathcal{A}_R)'')^{(1)}\right) \subset \pi_\omega(\mathcal{A}_{[0,n-1]})'$. Hence, for any $x \in (\pi_\omega(\mathcal{A}_R)'')^{(1)}$, any $\sigma$-weak accumulation point $z$ of the set $\{\pi_\omega(Y_n)\hat{\beta}_{S_n}(x)\}$ belongs to $\pi_\omega(\mathcal{A}_R)' \cap (\pi_\omega(\mathcal{A}_R)'')^{(1)} = \{0\}$. As such, $z = 0$. Because this holds for any accumulation point, we obtain (ii).

(iii) Suppose that $Z(\pi_\omega(\mathcal{A}_R)'') \cap (\pi_\omega(\mathcal{A}_R)'')^{(1)} \neq \{0\}$. By (5.6), we have that

$$\hat{\beta}_{S_n}\left((\pi_\omega(\mathcal{A}_R)'')^{(1)}\right) \subset \pi_\omega(\mathcal{A}_R)'' \cap \pi_\omega(\mathcal{A}_{[0,n-1]})'\Gamma_\omega. \tag{5.7}$$

Therefore, for any $x \in (\pi_\omega(\mathcal{A}_R)'')^{(1)}$, any $\sigma$-weak accumulation point $z$ of $\{\hat{\beta}_{S_n}(x)\}$ belongs to $(\pi_\omega(\mathcal{A}_R)'\Gamma_\omega) \cap (\pi_\omega(\mathcal{A}_R)'')^{(1)}$. Because $Z(\pi_\omega(\mathcal{A}_R)'')$ has an odd element, $\pi_\omega(\mathcal{A}_R)''$ is not a factor. Lemma A.3 then implies that $\pi_\omega(\mathcal{A}_R)'\Gamma_\omega \cap \pi_\omega(\mathcal{A}_R)'' = \{0\}$. Hence, we have $z = 0$. Because this holds for any accumulation point, we obtain (iii). □

Before stating the result, we fix some notation. Given the operators $\{W_\mu\}_{\mu\in\mathcal{P}}$ we define the completely positive (CP) map $T_{\mathbf{W}}$ by

$$T_{\mathbf{W}}(x) = \sum_{\mu\in\mathcal{P}} W_\mu x W_\mu^*.$$

Because the algebraic structure of the von Neumann algebra of interest changes depending on whether $\kappa_\omega = 0, 1$, we treat each case separately, though the general strategy of proof is the same.

### 5.1. *Case:* $\kappa_\omega = 0$

Recall that $\Gamma(U_g)$ denotes the second quantisation of $U_g$ on $\mathcal{F}(\mathbb{C}^d)$. In this subsection we prove the following.

**Theorem 5.3.** *Let $\omega$ be a pure $\alpha$-invariant and translation-invariant split state on $\mathcal{A}$. Suppose that the graded $W^*$-$(G, \mathfrak{p})$-dynamical system $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ associated to $\omega$ is equivalent to $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_0$, via a $*$-isomorphism $\iota : \pi_\omega(\mathcal{A}_R)'' \to \mathcal{B}(\mathcal{K} \otimes \mathbb{C}^2)$. Let $\rho$ be the $*$-endomorphism on $\mathcal{R}_{0,\mathcal{K}}$ given in Lemma 5.1. Then there is a set of isometries $\{B_\mu\}_{\mu\in\mathcal{P}}$ on $\mathcal{K} \otimes \mathbb{C}^2$ such that $B_\nu^* B_\mu = \delta_{\mu,\nu}\mathbb{I}$,*

$$\rho \circ \iota \circ \pi_\omega(A) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{B_\mu \Gamma_\mathcal{K}^{|\mu|}} \circ \iota \circ \pi_\omega(A) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{\Gamma_\mathcal{K}^{|\mu|} B_\mu} \circ \iota \circ \pi_\omega(A), \quad A \in \mathcal{A}_R, \tag{5.8}$$

*and*

$$\iota \circ \pi_\omega\left(E^{(0)}_{\mu_0,\nu_0} E^{(1)}_{\mu_1,\nu_1} \cdots E^{(N)}_{\mu_N,\mu_N}\right) = (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|} B_{\mu_0} \cdots B_{\mu_N} B_{\nu_N}^* \cdots B_{\nu_0}^* \tag{5.9}$$

*for all $N \in \mathbb{N} \cup \{0\}$ and $\mu_0, \dots \mu_N, \nu_0, \dots, \nu_N \in \mathcal{P}$. The operators $B_\mu$ have homogeneous parity and are such that $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}(B_\mu) = (-1)^{|\mu|+\sigma_0} B_\mu$, with some uniform $\sigma_0 \in \{0,1\}$. Furthermore,*

$$\sigma\text{-weak}\lim_{N\to\infty} T_{\mathbf{B}}^N \circ \iota(x) = \langle \Omega_\omega, x\Omega_\omega \rangle \mathbb{I}_{\mathcal{K}\otimes\mathbb{C}^2}, \quad x \in \pi_\omega(\mathcal{A}_R)'' \tag{5.10}$$

*and for each $g \in G$, there is some $c_g \in \mathbb{T}$ such that*

$$\sum_{\mu\in\mathcal{P}} \langle \psi_\mu, \Gamma(U_g)\psi_\nu \rangle B_\mu = c_g V_g B_\nu V_g^*. \tag{5.11}$$

We will prove this result in several steps. First we note some properties of endomorphisms of operators on graded Hilbert spaces and the Cuntz algebra.

**Proposition 5.4.** *Let $\mathcal{H}$ be a Hilbert space with a self-adjoint unitary $\Gamma$ that gives a grading for $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M}$ be a finite type I von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ with matrix units $\{E_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}} \subset \mathcal{M}$ spanning $\mathcal{M}$. Assume that*

$$\mathrm{Ad}_\Gamma(E_{\mu,\nu}) = (-1)^{|\mu|+|\nu|} E_{\mu,\nu}, \qquad \mu,\nu \in \mathcal{P} \tag{5.12}$$

*and set $\Gamma_0 := \sum_{\mu\in\mathcal{P}} (-1)^{|\mu|} E_{\mu\mu}$. Let $\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a graded, unital $*$-endomorphism such that $\rho(a)b - (-1)^{\partial a \partial b} b\rho(a) = 0$ for $a \in \mathcal{B}(\mathcal{H})$ $b \in \mathcal{M}$ with homogeneous grading. Suppose further that $\mathcal{B}(\mathcal{H}) = \rho(\mathcal{B}(\mathcal{H})) \vee \mathcal{M}$. Then there exist isometries $\{S_\mu\}_{\mu\in\mathcal{P}}$ on $\mathcal{H}$ with the property that*

$$S_\nu^* S_\mu = \delta_{\mu,\nu}\mathbb{I}, \qquad \rho(x) = \sum_\mu S_\mu x S_\mu^* \tag{5.13}$$

*for all $\mu,\nu \in \mathcal{P}$ and $x \in \mathcal{B}(\mathcal{H})$. The operators $S_\mu$ have homogeneous parity and are such that $\mathrm{Ad}_\Gamma(S_\mu) = (-1)^{|\mu|+\sigma_0} S_\mu$ with some uniform $\sigma_0 \in \{0,1\}$. Furthermore, setting $B_\mu := (\Gamma_0\Gamma)^{|\mu|} S_\mu$, for $\mu \in \mathcal{P}$, we have $B_\nu^* B_\mu = \delta_{\mu,\nu}\mathbb{I}$,*

$$\rho(x) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{B_\mu} \circ \mathrm{Ad}_{\Gamma^{|\mu|}}(x), \quad x \in \mathcal{B}(\mathcal{H}), \tag{5.14}$$

*and*

$$E_{\mu_0,\nu_0}\rho(E_{\mu_1,\nu_1}) \cdots \rho^N(E_{\mu_N,\mu_N}) = (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|} B_{\mu_0} \cdots B_{\mu_N} B_{\nu_N}^* \cdots B_{\nu_0}^* \tag{5.15}$$

*for all $N \in \mathbb{N} \cup \{0\}$ and $\mu_0, \dots, \mu_N, \nu_0, \dots, \nu_N \in \mathcal{P}$. The operators $B_\mu$ have homogeneous parity such that $\mathrm{Ad}_\Gamma(B_\mu) = (-1)^{|\mu|+\sigma_0} B_\mu$, with the same $\sigma_0$ as above. If there are isometries $\{T_\mu\}_{\mu\in\mathcal{P}}$ such that*

$$T_\nu^* T_\mu = \delta_{\mu,\nu}\mathbb{I}, \quad T_\mu T_\nu^* = E_{\mu,\nu}, \quad \rho(x) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{T_\mu} \circ \mathrm{Ad}_{\Gamma^{|\mu|}}(x), \quad x \in \mathcal{B}(\mathcal{H}), \tag{5.16}$$

*then there is some $c \in \mathbb{T}$ such that $T_\mu = cB_\mu$, for all $\mu \in \mathcal{P}$.*

To study the situation, we note the following general property.

**Lemma 5.5.** *Let $\mathcal{H}$ be a Hilbert space with a self-adjoint unitary $\Gamma$ that gives a grading for $\mathcal{B}(\mathcal{H})$. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be $\mathrm{Ad}_\Gamma$-invariant von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ with $\mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{B}(\mathcal{H})$. Suppose that $\mathcal{M}_1$ is a type I factor with a self-adjoint unitary $\Gamma_1 \in \mathcal{M}_1$ such that $\mathrm{Ad}_{\Gamma_1}(x) = \mathrm{Ad}_\Gamma(x)$ for all $x \in \mathcal{M}_1$. Suppose further that*

$$ab - (-1)^{\partial a \partial b} ba = 0, \quad \text{for homogeneous} \quad a \in \mathcal{M}_1, b \in \mathcal{M}_2. \tag{5.17}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

*Then there are Hilbert spaces* $\mathcal{H}_1, \mathcal{H}_2$ *and a unitary* $V : \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2$ *such that*

$$\mathrm{Ad}_V(\mathcal{M}_1) = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_2}. \tag{5.18}$$

*Furthermore, there are self-adjoint unitaries* $\tilde{\Gamma}_i$ *on* $\mathcal{H}_i$ *with* $i = 1, 2$ *such that*

$$\mathrm{Ad}_V(\Gamma) = \tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2, \quad \mathrm{Ad}_V(\Gamma_1) = \tilde{\Gamma}_1 \otimes \mathbb{I}_{\mathcal{H}_2}. \tag{5.19}$$

*The commutant of* $\mathcal{M}_2$ *is given by*

$$\mathcal{M}_2' = \mathcal{M}_1^{(0)} + \mathcal{M}_1^{(1)}\Gamma_1\Gamma. \tag{5.20}$$

*If* $p$ *is an even minimal projection in* $\mathcal{M}_1$*, then* $\mathcal{M}_2 \cdot p = \mathcal{B}(p\mathcal{H})$*.*

We note that if $\mathcal{M}_1$ is a type I factor, Wigner's theorem guarantees the existence of a self-adjoint unitary $\Gamma_1 \in \mathcal{M}_1$ such that $\mathrm{Ad}_{\Gamma_1}(x) = \mathrm{Ad}_{\Gamma}(x)$ for all $x \in \mathcal{M}_1$.

*Proof.* Because $\mathcal{M}_1$ is a type I factor, by [38, Chapter V, Theorem 1.31] there are Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and a unitary $V : \mathcal{H} \to \mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying (5.18). Because $\Gamma_1 \in \mathcal{M}_1$ and $\Gamma\Gamma_1 \in \mathcal{M}_1'$, there are self-adjoint unitaries $\tilde{\Gamma}_i$ on $\mathcal{H}_i$ with $i = 1, 2$ satisfying (5.19). Clearly, $\mathrm{Ad}_{\Gamma_1}(\Gamma_1) = \Gamma_1$ and so $\Gamma_1$ is an even element of $\mathcal{M}_1$.

Note that $\mathcal{N} := \mathcal{M}_2^{(0)} + \mathcal{M}_2^{(1)}\Gamma_1$ is a von Neumann subalgebra of $\mathcal{M}_1'$ by (5.17). Therefore, $\mathrm{Ad}_V(\mathcal{N})$ is a von Neumann subalgebra of $\mathbb{I}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)$. Because

$$\mathcal{M}_2 = \mathcal{M}_2^{(0)} + \mathcal{M}_2^{(1)}\Gamma_1\Gamma_1 \subset \mathcal{M}_1 \vee \mathcal{N}, \qquad \mathcal{M}_1 \subset \mathcal{M}_1 \vee \mathcal{N}, \qquad \mathcal{M}_1 \vee \mathcal{M}_2 = \mathcal{B}(\mathcal{H}),$$

we have $\mathcal{M}_1 \vee \mathcal{N} = \mathcal{B}(\mathcal{H})$. Combining with (5.18), this means

$$\mathrm{Ad}_V(\mathcal{M}_2^{(0)} + \mathcal{M}_2^{(1)}\Gamma_1) = \mathrm{Ad}_V(\mathcal{N}) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2). \tag{5.21}$$

Now we associate the grading given by $\tilde{\Gamma}_i$ to $\mathcal{B}(\mathcal{H}_i)$ for $i = 1, 2$, and regard $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ as $\mathcal{B}(\mathcal{H}_1) \hat{\otimes} \mathcal{B}(\mathcal{H}_2)$, the graded tensor product of $(\mathcal{B}(\mathcal{H}_1), \mathcal{H}_1, \tilde{\Gamma}_1)$ and $(\mathcal{B}(\mathcal{H}_2), \mathcal{H}_2, \tilde{\Gamma}_2)$. Because $\mathrm{Ad}_V(\Gamma) = \tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2$, $\mathrm{Ad}_V : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_1) \hat{\otimes} \mathcal{B}(\mathcal{H}_2)$ is a graded $*$-isomorphism. Considering the even and odd subspaces of (5.21), we obtain

$$\mathrm{Ad}_V(\mathcal{M}_2^{(0)}) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)^{(0)}, \quad \mathrm{Ad}_V(\mathcal{M}_2^{(1)})\,\mathrm{Ad}_V(\Gamma_1) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)^{(1)} \tag{5.22}$$

and so

$$\begin{aligned}\mathrm{Ad}_V(\mathcal{M}_2) = \mathrm{Ad}_V(\mathcal{M}_2^{(0)} + \mathcal{M}_2^{(1)}) &= \mathbb{C}\mathbb{I}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{H}_2)^{(0)} + \mathbb{C}\tilde{\Gamma}_1 \otimes \mathcal{B}(\mathcal{H}_2)^{(1)} \\ &= \mathbb{C}\mathbb{I}_{\mathcal{H}_1} \hat{\otimes} \mathcal{B}(\mathcal{H}_2),\end{aligned} \tag{5.23}$$

where $\mathbb{C}\mathbb{I}_{\mathcal{H}_1} \hat{\otimes} \mathcal{B}(\mathcal{H}_2)$ is a graded tensor product of $(\mathbb{C}\mathbb{I}_{\mathcal{H}_1}, \mathcal{H}_1, \tilde{\Gamma}_1)$ and $(\mathcal{B}(\mathcal{H}_2), \mathcal{H}_2, \tilde{\Gamma}_2)$.

We now consider the commutant of $\mathcal{M}_2$. Applying Lemma A.4, we see that

$$\mathrm{Ad}_V(\mathcal{M}_2') = \mathcal{B}(\mathcal{H}_1)^{(0)} \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_2} + \mathcal{B}(\mathcal{H}_1)^{(1)} \otimes \mathbb{C}\tilde{\Gamma}_2 = \mathrm{Ad}_V\left(\mathcal{M}_1^{(0)} + \mathcal{M}_1^{(1)}\Gamma_1\Gamma\right). \tag{5.24}$$

Hence, we obtain (5.20).

Let $p$ be a minimal projection in $\mathcal{M}_1$ and suppose that it is even. Then $\mathrm{Ad}_V(p)$ is a minimal projection in $\mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C}\mathbb{I}_{\mathcal{H}_2}$. Therefore, there is a rank 1 projection $r$ on $\mathcal{H}_1$ such that $\mathrm{Ad}_V(p) = r \otimes \mathbb{I}_{\mathcal{H}_2}$. Because $p$ is even and $\mathrm{Ad}_V$ is a graded $*$-isomorphism, we have $\mathrm{Ad}_{\tilde{\Gamma}_1}(r) = r$. Because $r$ is rank 1, this means

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

that $\tilde{\Gamma}_1 r = \pm r$. Therefore, using (5.23), we have

$$\begin{aligned}\mathrm{Ad}_V(\mathcal{M}_2 p) &= \mathbb{C}r \otimes \mathcal{B}(\mathcal{H}_2)^{(0)} + \mathbb{C}\tilde{\Gamma}_1 r \otimes \mathcal{B}(\mathcal{H}_2)^{(1)} \\ &= \mathbb{C}r \otimes \left(\mathcal{B}(\mathcal{H}_2)^{(0)} \pm \mathcal{B}(\mathcal{H}_2)^{(1)}\right) = \mathbb{C}r \otimes \mathcal{B}(\mathcal{H}_2) = \mathrm{Ad}_V\left(p\mathcal{B}(\mathcal{H})p\right). \qquad (5.25)\end{aligned}$$

Hence, we obtain $\mathcal{M}_2 p = p\mathcal{B}(\mathcal{H})p = \mathcal{B}(p\mathcal{H})$. □

**Lemma 5.6.** *Consider the setting of Proposition 5.4. Then the following hold:*

(i) $\rho(\mathcal{B}(\mathcal{H}))' = \mathcal{M}^{(0)} + \mathcal{M}^{(1)}\Gamma_0\Gamma$.
(ii) *Let* $\hat{E}_{\mu,\nu} = E_{\mu,\nu}(\Gamma_0\Gamma)^{|\mu|+|\nu|}$. *Then* $\{\hat{E}_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}}$ *are matrix units in* $\rho(\mathcal{B}(\mathcal{H}))'$ *spanning* $\rho(\mathcal{B}(\mathcal{H}))'$,
(iii) *For all* $\mu \in \mathcal{P}$, *the map*

$$\rho_\mu : \mathcal{B}(\mathcal{H}) \ni x \mapsto \rho(x)E_{\mu,\mu} \in \mathcal{B}(E_{\mu,\mu}\mathcal{H}) \qquad (5.26)$$

*is a* $*$*-isomorphism.*

*Proof.* Note that $\mathrm{Ad}_\Gamma(x) = \mathrm{Ad}_{\Gamma_0}(x)$ for $x \in \mathcal{M}$. Applying Lemma 5.5 with $\mathcal{M}_1 = \mathcal{M}$, $\mathcal{M}_2 = \rho(\mathcal{B}(\mathcal{H}))$ and $\Gamma_1 = \Gamma_0$, we immediately obtain (i). Because $\{E_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}}$ are matrix units spanning $\mathcal{M}$ and satisfying (5.12), we see from (i) that

$$\rho(\mathcal{B}(\mathcal{H}))' = \mathcal{M}^{(0)} + \mathcal{M}^{(1)}\Gamma_0\Gamma = \mathrm{span}_{\mu,\nu\in\mathcal{P}}\left\{E_{\mu,\nu}\left(\Gamma_0\Gamma\right)^{|\mu|+|\nu|}\right\} = \mathrm{span}_{\mu,\nu\in\mathcal{P}}\left\{\hat{E}_{\mu,\nu}\right\}. \qquad (5.27)$$

Because $\Gamma_0\Gamma$ commutes with $E_{\mu,\nu}$, it is straightforward to check that $\{\hat{E}_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}}$ are matrix units. Hence, we obtain (ii).

For part (iii), we first note that because $E_{\mu,\mu}$ is even, $[\rho(x), E_{\mu,\mu}] = 0$ for all $x \in \mathcal{B}(\mathcal{H})$. Therefore, there is a well-defined $*$-homomorphism

$$\rho_\mu : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(E_{\mu,\mu}\mathcal{H}), \qquad \rho_\mu(x) = \rho(x)E_{\mu,\mu}, \ \ x \in \mathcal{B}(\mathcal{H}).$$

Because $\mathcal{B}(\mathcal{H})$ is a factor, $\rho_\mu$ is injective. To see that $\rho_\mu$ is surjective, we note that $E_{\mu\mu}$ is a minimal projection of $\mathcal{M}$ and it is even. Then applying Lemma 5.5 with $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = \rho(\mathcal{B}(\mathcal{H}))$, we obtain $\rho(\mathcal{B}(\mathcal{H})) \cdot E_{\mu\mu} = \mathcal{B}(E_{\mu\mu}\mathcal{H})$ and so $\rho_\mu$ is surjective. □

We now prove Proposition 5.4, which we split into two lemmas. We recall the matrix units $\{E_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}} \subset \mathcal{M}$ and $\hat{E}_{\mu,\nu} = E_{\mu,\nu}(\Gamma_0\Gamma)^{|\mu|+|\nu|}$ from Lemma 5.6.

**Lemma 5.7** (First part of Proposition 5.4)**.** *Consider the setting of Proposition 5.4. Then there exist isometries* $\{S_\mu\}_{\mu\in\mathcal{P}}$ *on* $\mathcal{H}$ *with the property that for all* $\mu, \nu \in \mathcal{P}$ *and* $x \in \mathcal{B}(\mathcal{H})$,

$$S_\nu^* S_\mu = \delta_{\mu,\nu}\mathbb{I}, \quad S_\mu S_\nu^* = \hat{E}_{\mu,\nu}, \quad \rho(x)E_{\mu,\mu} = S_\mu x S_\mu^*, \quad \rho(x) = \sum_{\mu\in\mathcal{P}} S_\mu x S_\mu^*. \qquad (5.28)$$

*The operators* $S_\mu$ *have homogeneous parity and are such that* $\mathrm{Ad}_\Gamma\left(S_\mu\right) = (-1)^{|\mu|+\sigma_0}S_\mu$, *with some uniform* $\sigma_0 \in \{0, 1\}$. *They also satisfy* $\Gamma_0 S_\mu = (-1)^{|\mu|}S_\mu$.

*Proof.* By part (iii) of Lemma 5.6, $\rho_\mu$ in (5.26) is a $*$-isomorphism $\mathcal{B}(\mathcal{H}) \xrightarrow{\rho_\mu} \mathcal{B}(E_{\mu,\mu}\mathcal{H})$. Therefore, we can apply Wigner's theorem to obtain a unitary $w_\mu : \mathcal{H} \to E_{\mu,\mu}\mathcal{H}$ such that $\rho_\mu = \mathrm{Ad}_{w_\mu}$. Note that

$$w_\mu^* w_\nu = w_\mu^* E_{\mu,\mu}E_{\nu,\nu}w_\nu = \delta_{\mu,\nu}\,\mathbb{I}_{\mathcal{H}}, \quad \mu, \nu \in \mathcal{P}.$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

We also see that, because $\sum_\mu E_{\mu,\mu} = \mathbb{I}$,

$$\rho(x) = \sum_\mu \rho(x) E_{\mu,\mu} = \sum_\mu \rho_\mu(x) = \sum_\mu w_\mu x w_\mu^*, \quad x \in \mathcal{B}(\mathcal{H}).$$

We use the above property to compute that for any $x \in \mathcal{B}(\mathcal{H})$,

$$w_\mu w_\nu^* \rho(x) = w_\mu w_\nu^* \Big(\sum_\lambda w_\lambda x w_\lambda^*\Big) = w_\mu x w_\nu^* = \Big(\sum_\lambda w_\lambda x w_\lambda^*\Big) w_\mu w_\nu^* = \rho(x) w_\mu w_\nu^*.$$

Therefore, $w_\mu w_\nu^* \in \rho(\mathcal{B}(\mathcal{H}))'$ for any $\mu, \nu \in \mathcal{P}$.

Summarising our results so far, we have obtained a collection of operators $\{w_\mu w_\nu^*\}_{\mu,\nu\in\mathcal{P}}$ in $\rho(\mathcal{B}(\mathcal{H}))'$ such that

$$\hat{E}_{\mu,\mu} w_\mu w_\nu^* \hat{E}_{\nu,\nu} = w_\mu w_\nu^*. \tag{5.29}$$

From (5.29) and (ii) of Lemma 5.6, there is some $c_{\mu\nu} \in \mathbb{C}$ such that

$$w_\mu w_\nu^* = c_{\mu\nu} \hat{E}_{\mu,\nu}.$$

Note that $c_{\mu\nu} = \overline{c_{\nu\mu}}$. Because of the definition, we have $w_\mu w_\mu^* = \hat{E}_{\mu\mu}$ and we see that $c_{\mu\mu} = 1$. On the other hand, because of $w_\nu^* w_\nu = \mathbb{I}_\mathcal{H}$, we have

$$c_{\mu\lambda} \hat{E}_{\mu,\lambda} = w_\mu w_\lambda^* = w_\mu w_\nu^* w_\nu w_\lambda^* = c_{\mu\nu} c_{\nu\lambda} \hat{E}_{\mu,\lambda}$$

and so $c_{\mu\lambda} = c_{\mu\nu} c_{\nu\lambda}$. In particular, $1 = c_{\mu\mu} = c_{\mu\nu} c_{\nu\mu} = |c_{\mu\nu}|^2$ and so $c_{\mu\nu} \in \mathbb{T}$. Now set $\mu_0 := \emptyset \in \mathcal{P}$ and define $S_\mu = c_{\mu_0\mu} w_\mu$ for every $\mu \in \mathcal{P}$. Then because of the above properties of $c_{\mu\nu}$, the collection $\{S_\mu\}_{\mu\in\mathcal{P}}$ has the same algebraic properties as $\{w_\mu\}$ as well as that $S_\mu S_\nu^* = \hat{E}_{\mu,\nu}$ as required. Hence, we obtain (5.28).

Next, we recall the grading operator $\Gamma_0 = \sum_\mu (-1)^{|\mu|} E_{\mu,\mu}$ of $\mathcal{M}$. Because $S_\mu$ is an isometry onto $E_{\mu,\mu}\mathcal{H}$,

$$\Gamma_0 S_\mu = \Gamma_0 E_{\mu,\mu} S_\mu = (-1)^{|\mu|} E_{\mu,\mu} S_\mu = (-1)^{|\mu|} S_\mu.$$

We now consider the grading of $S_\mu$, $\mathrm{Ad}_\Gamma(S_\mu)$. We compute that for any $x \in \mathcal{B}(\mathcal{H})$,

$$\Gamma S_\mu x S_\mu^* \Gamma = \Gamma \rho(x) E_{\mu,\mu} \Gamma = \Gamma \rho(x) \Gamma E_{\mu,\mu} = \rho(\Gamma x \Gamma) E_{\mu,\mu} = S_\mu \Gamma x \Gamma S_\mu^* \tag{5.30}$$

because $E_{\mu,\mu}$ is even and $\rho$ commutes with the grading. Multiplying (5.30) $\Gamma S_\mu^*$ from the left and $\Gamma S_\mu$ from the right, we see that $\Gamma S_\mu^* \Gamma S_\mu \in \mathcal{B}(\mathcal{H})' = \mathbb{C}\mathbb{I}_\mathcal{H}$. Note that $\Gamma S_\mu^* \Gamma S_\mu$ is unitary because $\mathrm{Ad}_\Gamma(E_{\mu\mu}) = E_{\mu\mu}$. So $S_\mu^* \Gamma S_\mu = e^{i\varphi}\Gamma$ with some $e^{i\varphi} \in \mathbb{T}$. Multiplying this identity by $S_\mu$ from the left and by $\Gamma$ from the right, we obtain $\Gamma S_\mu \Gamma = E_{\mu\mu} \Gamma S_\mu \Gamma = S_\mu S_\mu^* \Gamma S_\mu \Gamma = e^{i\varphi} S_\mu$. But because $(\mathrm{Ad}_\Gamma)^2 = \mathrm{id}$, $(e^{i\varphi})^2 = 1$ and $\mathrm{Ad}_\Gamma(S_\mu) = (-1)^{b_\mu} S_\mu$ with some $b_\mu = 0, 1$.

Let us further examine the grading of the operator $S_\mu$. We compute that

$$\begin{aligned}\Gamma \hat{E}_{\mu,\nu} \Gamma &= \Gamma E_{\mu,\nu} (\Gamma_0 \Gamma)^{|\mu|+|\nu|} \Gamma = \Gamma E_{\mu,\nu} \Gamma (\Gamma_0 \Gamma)^{|\mu|+|\nu|} \\ &= (-1)^{|\mu|+|\nu|} E_{\mu,\nu} (\Gamma_0 \Gamma)^{|\mu|+|\nu|} = (-1)^{|\mu|+|\nu|} \hat{E}_{\mu,\nu}\end{aligned}$$

and we also find

$$\Gamma \hat{E}_{\mu,\nu} \Gamma = \Gamma S_\mu \Gamma \Gamma S_\nu^* \Gamma = (-1)^{b_\mu} (-1)^{b_\nu} S_\mu S_\nu^* = (-1)^{b_\mu + b_\nu} \hat{E}_{\mu,\nu}.$$

Therefore, $|\mu| + |\nu| = b_\mu + b_\nu \in \mathbb{Z}_2$. By setting $\mu_0 := \emptyset \in \mathcal{P}$ and $\sigma_0 := b_{\mu_0}$, we have that $\Gamma S_\mu \Gamma = (-1)^{|\mu|+\sigma_0} S_\mu$ for all $\mu \in \mathcal{P}$. □

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

**Lemma 5.8** (Second half of Proposition 5.4)**.** *Consider the setting of Proposition 5.4. For $S_\mu$ of Lemma 5.7, set $B_\mu := (\Gamma_0\Gamma)^{|\mu|}S_\mu$, for $\mu \in \mathcal{P}$. Then $B_\nu^* B_\mu = \delta_{\mu,\nu}\mathbb{I}$, $B_\mu B_\nu^* = E_{\mu,\nu}$,*

$$\rho(x) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{B_\mu} \circ \mathrm{Ad}_{\Gamma^{|\mu|}}(x) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{\Gamma^{|\mu|}} \circ \mathrm{Ad}_{B_\mu}(x), \quad x \in \mathcal{B}(\mathcal{H}),$$

$$E_{\mu_0,\nu_0}\rho(E_{\mu_1,\nu_1})\cdots\rho^N(E_{\mu_N,\mu_N}) = (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|} B_{\mu_0}\cdots B_{\mu_N}B_{\nu_N}^*\cdots B_{\nu_0}^*, \tag{5.31}$$

*for all $N \in \mathbb{N}\cup\{0\}$ and $\mu_0, \ldots \mu_N, \nu_0, \ldots, \nu_N \in \mathcal{P}$. The operators $B_\mu$ have homogeneous parity and are such that $\mathrm{Ad}_\Gamma\left(B_\mu\right) = (-1)^{|\mu|+\sigma_0}B_\mu$, with the same $\sigma_0 \in \{0,1\}$ as in Lemma 5.7. If there are isometries $\{T_\mu\}_{\mu\in\mathcal{P}}$ satisfying (5.16), then there is some $c \in \mathbb{T}$ such that $T_\mu = cB_\mu$, for all $\mu \in \mathcal{P}$.*

*Proof.* From Lemma 5.7, we check that

$$B_\mu^* B_\nu = S_\mu^*(\Gamma_0\Gamma)^{|\mu|+|\nu|}S_\nu = S_\mu^* S_\nu \Gamma^{|\mu|+|\nu|}(-1)^{(|\nu|+\sigma_0)(|\mu|+|\nu|)}(-1)^{|\nu|(|\mu|+|\nu|)} = \delta_{\mu,\nu}\mathbb{I}. \tag{5.32}$$

We also have from Lemma 5.7 that

$$B_\mu B_\nu^* = (\Gamma_0\Gamma)^{|\mu|}S_\mu S_\nu^*(\Gamma_0\Gamma)^{|\nu|} = (\Gamma_0\Gamma)^{|\mu|}E_{\mu,\nu}(\Gamma_0\Gamma)^{|\mu|+|\nu|}(\Gamma_0\Gamma)^{|\nu|} = E_{\mu,\nu} \tag{5.33}$$

because $\Gamma_0\Gamma$ commutes with $\mathcal{M}$. Because $S_\mu$ has homogenous parity and $\Gamma_0\Gamma$ is even, $B_\mu$ has the same homogeneous parity as $S_\mu$. In particular, $\mathrm{Ad}_\Gamma\left(B_\mu\right) = (-1)^{|\mu|+\sigma_0}B_\mu$, with the same $\sigma_0 \in \{0,1\}$ as in Lemma 5.7. This implies that the endomorphism $\mathrm{Ad}_{B_\mu}$ respects the grading on $\mathcal{B}(\mathcal{H})$; that is, $\mathrm{Ad}_\Gamma \circ \mathrm{Ad}_{B_\mu} = \mathrm{Ad}_{B_\mu} \circ \mathrm{Ad}_\Gamma$. Furthermore, using that $\Gamma_0 S_\mu = (-1)^{|\mu|}S_\mu$, $\mathrm{Ad}_{S_\mu} = \mathrm{Ad}_{\Gamma^{|\mu|}B_\mu} = \mathrm{Ad}_{B_\mu\Gamma^{|\mu|}}$. We therefore see that for $x \in \mathcal{B}(\mathcal{H})$,

$$\rho(x) = \sum_{\mu\in\mathcal{P}} S_\mu x S_\mu^* = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{B_\mu} \circ \mathrm{Ad}_{\Gamma^{|\mu|}}(x).$$

A simple induction argument using that $\mathrm{Ad}_{B_\mu}$ commutes with $\mathrm{Ad}_\Gamma$ gives that

$$\rho^N(x) = \sum_{\lambda_0,\ldots,\lambda_{N-1}\in\mathcal{P}} \mathrm{Ad}_{B_{\lambda_0}\cdots B_{\lambda_{N-1}}} \circ \mathrm{Ad}_{\Gamma^{|\lambda_0|+\cdots|\lambda_{N-1}|}}(x). \tag{5.34}$$

We now consider $\rho(E_{\mu,\nu})$. Recalling (5.33) and that $\mathrm{Ad}_\Gamma(E_{\mu,\nu}) = (-1)^{|\mu|+|\nu|}E_{\mu,\nu}$, we see that

$$\rho(E_{\mu,\nu}) = \sum_\lambda B_\lambda \Gamma^{|\lambda|}E_{\mu,\nu}\Gamma^{|\lambda|}B_\lambda^* = \sum_\lambda (-1)^{|\lambda|(|\mu|+|\nu|)}B_\lambda B_\mu B_\nu^* B_\lambda^*.$$

From this, (5.33) and (5.32), we have

$$E_{\mu_0,\nu_0}\rho(E_{\mu_1,\nu_1}) = B_{\mu_0}B_{\nu_0}^*\sum_\lambda(-1)^{|\lambda|(|\mu_1|+|\nu_1|)}B_\lambda B_{\mu_1}B_{\nu_1}^*B_\lambda^* = (-1)^{|\nu_0|(|\mu_1|+|\nu_1|)}B_{\mu_0}B_{\mu_1}B_{\nu_1}^*B_{\nu_0}^*.$$

This proves Equation (5.31) in the case of $N = 1$. We now assume that the equality is true for $N$ and consider $N+1$. Using Equations (5.32), (5.33), (5.34), we compute that

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

$$
\begin{aligned}
&E_{\mu_0,\nu_0}\rho(E_{\mu_1,\nu_1})\cdots\rho^N(E_{\mu_N,\nu_N})\rho^{N+1}(E_{\mu_{N+1},\nu_{N+1}})\\
&=(-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}B_{\mu_0}\cdots B_{\mu_N}B^*_{\nu_N}\cdots B^*_{\nu_0}\,\rho^{N+1}(E_{\mu_{N+1},\nu_{N+1}})\\
&=(-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}B_{\mu_0}\cdots B_{\mu_N}B^*_{\nu_N}\cdots B^*_{\nu_0}\Big(\sum_{\lambda_0,\dots,\lambda_N}\mathrm{Ad}_{B_{\lambda_0}\cdots B_{\lambda_N}}\circ\mathrm{Ad}_{\Gamma^{\sum_{j=0}^{N}|\lambda_j|}}(E_{\mu_{N+1},\nu_{N+1}})\Big)\\
&=(-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}B_{\mu_0}\cdots B_{\mu_N}\Big((-1)^{(|\mu_{N+1}|+|\nu_{N+1}|)\sum_{j=0}^{N}|\nu_j|}B_{\mu_{N+1}}B^*_{\nu_{N+1}}\Big)B^*_{\nu_N}\cdots B^*_{\nu_0}\\
&=(-1)^{\sum_{k=1}^{N+1}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}B_{\mu_0}\cdots B_{\mu_N}B_{\mu_{N+1}}B^*_{\nu_{N+1}}B^*_{\nu_N}\cdots B^*_{\nu_0}
\end{aligned}
$$

as required.

To show the last statement, suppose that $\{T_\mu\}_{\mu\in\mathcal{P}}\subset\mathcal{B}(\mathcal{H})$ satisfy (5.16). Because

$$\sum_{\lambda\in\mathcal{P}}\mathrm{Ad}_{T_\lambda}\circ\mathrm{Ad}_{\Gamma^{|\lambda|}}(x)=\rho(x)=\sum_{\lambda\in\mathcal{P}}\mathrm{Ad}_{B_\lambda}\circ\mathrm{Ad}_{\Gamma^{|\lambda|}}(x),\quad x\in\mathcal{B}(\mathcal{H}),\tag{5.35}$$

multiplying (5.35) by $T^*_\nu$ from the left and by $B_\nu$ from the right, we obtain

$$\mathrm{Ad}_{\Gamma^{|\nu|}}(x)\cdot T^*_\nu B_\nu=T^*_\nu B_\nu\cdot\mathrm{Ad}_{\Gamma^{|\nu|}}(x),\quad x\in\mathcal{B}(\mathcal{H}).\tag{5.36}$$

Hence, we obtain $T^*_\nu B_\nu\in\mathbb{C}\mathbb{I}_{\mathcal{H}}$; that is, we have $T^*_\nu B_\nu=c_\mu\mathbb{I}_{\mathcal{H}}$ for some $c_\mu\in\mathbb{C}$. We then have

$$B_\nu=E_{\nu\nu}B_\nu=T_\nu T^*_\nu B_\nu=c_\nu T_\nu.\tag{5.37}$$

By $B^*_\nu B_\nu=T^*_\nu T_\nu=\mathbb{I}_{\mathcal{H}}$, we see that $c_\nu\in\mathbb{T}$. Furthermore, from $B_\mu B^*_\nu=T_\mu T^*_\nu=E_{\mu\nu}$, we see that $c_\mu=c_\nu=:c\in\mathbb{T}$. □

Lemmas 5.7 and 5.8 complete the proof of Proposition 5.4. We are ready to show Theorem 5.3.

*Proof of Theorem 5.3.* We fix a $W^*$-$(G,\mathfrak{p})$-dynamical system $(\mathcal{R}_{0,\mathcal{K}},\mathrm{Ad}_{\Gamma_\mathcal{K}},\mathrm{Ad}_{V_g})\in\mathcal{S}_0$ that is equivalent to $(\pi_\omega(\mathcal{A}_R)'',\mathrm{Ad}_{\Gamma_\omega},\hat{\alpha}_\omega)$ and the endomorphism $\rho$ of Lemma 5.1. Then the Hilbert space $\mathcal{K}\otimes\mathbb{C}^2$, self-adjoint unitary $\Gamma_\mathcal{K}$, finite type I factor $\iota\circ\pi_\omega(\mathcal{A}_{\{0\}})$ with matrix units $\{\iota\circ\pi_\omega\circ\big(E^{(0)}_{\mu,\nu}\big)\}_{\mu,\nu\in\mathcal{P}}\subset\mathcal{B}(\mathcal{K}\otimes\mathbb{C}^2)$ and $\rho$ satisfy the hypothesis of Proposition 5.4. Applying Proposition 5.4, we obtain the isometries $\{B_\mu\}$ such that $B^*_\mu B_\nu=\delta_{\mu,\nu}\mathbb{I}$ and that satisfy (5.8) and (5.9) from the statement of the theorem.

To show (5.10), set $\Gamma_0:=\iota\circ\pi_\omega\big(\Gamma(-\mathbb{I})\big)=\sum_\mu(-1)^{|\mu|}\iota\circ\pi_\omega\big(E^{(0)}_{\mu,\mu}\big)$. We claim for a homogeneous $x\in\mathcal{R}_{0,\mathcal{K}}$ and $N\in\mathbb{N}$ that

$$T^N_{\mathbf{B}}(x)=\Gamma_0^{\partial x}\rho(\Gamma_0^{\partial x})\cdots\rho^{N-1}(\Gamma_0^{\partial x})\rho^N(x).\tag{5.38}$$

First set $\Gamma_1:=\sum_\mu\Gamma_\mathcal{K}^{|\mu|}\iota\circ\pi_\omega\big(E^{(0)}_{\mu,\mu}\big)$, which is a self-adjoint unitary. Because of (5.9) with $N=0$, we have $\iota\circ\pi_\omega\big(E^{(0)}_{\mu\mu}\big)=B_\mu B^*_\mu$. Therefore, we have

$$\rho\circ\iota\circ\pi_\omega(A)=\sum_{\mu\in\mathcal{P}}\mathrm{Ad}_{\Gamma_\mathcal{K}^{|\mu|}B_\mu}\circ\iota\circ\pi_\omega(A)=\mathrm{Ad}_{\Gamma_1}\circ T_{\mathbf{B}}\circ\iota\circ\pi_\omega(A),\quad A\in\mathcal{A}_R.\tag{5.39}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Hence, we obtain for any homogeneous $x \in \mathcal{R}_{0,\mathcal{K}}$,

$$\begin{aligned} T_{\mathbf{B}}(x) = \mathrm{Ad}_{\Gamma_1} \circ \rho(x) &= \sum_{\mu,\nu} \Gamma_{\mathcal{K}}^{|\mu|} \left(\iota \circ \pi_\omega(E_{\mu,\mu}^{(0)})\right) \rho(x) \Gamma_{\mathcal{K}}^{|\nu|} \left(\iota \circ \pi_\omega(E_{\nu,\nu}^{(0)})\right) \\ &= \sum_{\mu} \iota \circ \pi_\omega(E_{\mu,\mu}^{(0)}) \Gamma_{\mathcal{K}}^{|\mu|} \rho(x) \Gamma_{\mathcal{K}}^{|\mu|}. \\ &= \sum_{\mu} \iota \circ \pi_\omega(E_{\mu,\mu}^{(0)}) \rho \circ \mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\mu|}}(x). \\ &= \sum_{\mu} \iota \circ \pi_\omega(E_{\mu,\mu}^{(0)}) (-1)^{|\mu| \partial x} \rho(x) = \Gamma_0^{\partial x} \rho(x), \end{aligned} \tag{5.40}$$

where in the third equality we used that $\iota \circ \pi_\omega(E_{\nu,\nu}^{(0)})$ commutes with $\Gamma_{\mathcal{K}}$ and elements from $\rho(\mathcal{R}_{0,\mathcal{K}})$. This proves (5.38) for the case $N = 1$. Now we proceed by induction and suppose that (5.38) holds for $N$. Then using (5.40) and the induction assumption, for any homogeneous $x \in \mathcal{R}_{0,\mathcal{K}}$,

$$\begin{aligned} T_{\mathbf{B}}^{N+1}(x) &= T_{\mathbf{B}} \left(\Gamma_0^{\partial x} \rho(\Gamma_0^{\partial x}) \cdots \rho^{N-1}(\Gamma_0^{\partial x}) \rho^N(x)\right) \\ &= \Gamma_0^{\partial\left(\Gamma_0^{\partial x} \rho(\Gamma_0^{\partial x}) \cdots \rho^{N-1}(\Gamma_0^{\partial x}) \rho^N(x)\right)} \rho(\Gamma_0^{\partial x}) \rho^2(\Gamma_0^{\partial x}) \cdots \rho^N(\Gamma_0^{\partial x}) \rho^{N+1}(x) \\ &= \Gamma_0^{\partial x} \rho(\Gamma_0^{\partial x}) \rho^2(\Gamma_0^{\partial x}) \cdots \rho^N(\Gamma_0^{\partial x}) \rho^{N+1}(x). \end{aligned} \tag{5.41}$$

Hence, (5.38) holds for $N + 1$ and proves the claim.

Now we show (5.10). Because $\kappa_\omega = 0$, $\pi_\omega(\mathcal{A}_R)''$ is a factor. Therefore, for any homogeneous $x \in \pi_\omega(\mathcal{A}_R)''$, the sequence

$$\begin{aligned} T_{\mathbf{B}}^N \circ \iota(x) &= \Gamma_0^{\partial x} \rho(\Gamma_0^{\partial x}) \cdots \rho^{N-1}(\Gamma_0^{\partial x}) \rho^N \circ \iota(x) \\ &= \iota \circ \left(\pi_\omega \left(\Gamma(-\mathbb{I})^{\partial x} \beta_{S_1}(\Gamma(-\mathbb{I})^{\partial x}) \cdots \beta_{S_{N-1}}(\Gamma(-\mathbb{I})^{\partial x})\right) \hat{\beta}_{S_N}(x)\right) \end{aligned} \tag{5.42}$$

converges to $\langle \Omega_\omega, x\Omega_\omega \rangle \mathbb{I}_{\mathcal{K} \otimes \mathbb{C}^2}$ in the $\sigma$-weak topology by Lemma 5.2. This proves (5.10).

To prove (5.11) set

$$T_\nu := \sum_{\lambda \in \mathcal{P}} \overline{\langle \psi_\lambda, \Gamma(U_g)^* \psi_\nu \rangle}^{\mathfrak{p}(g)} V_g B_\lambda V_g^*, \quad \nu \in \mathcal{P}. \tag{5.43}$$

Recall that for $c \in \mathbb{C}$, $\overline{c}^{\mathfrak{p}(g)} = c$ for $\mathfrak{p}(g) = 0$ and $\bar{c}$ if $\mathfrak{p}(g) = 1$. We claim that $\{T_\mu\}_{\mu \in \mathcal{P}}$ satisfies (5.16) with $E_{\mu\nu}$ and $\Gamma$ replaced by $\iota \circ \pi_\omega(E_{\mu\nu}^{(0)})$ and $\Gamma_{\mathcal{K}}$ respectively. We compute

$$\begin{aligned} T_\mu^* T_\nu &= \sum_{\lambda,\zeta} \overline{\langle \psi_\lambda, \Gamma(U_g)^* \psi_\mu \rangle}^{\mathfrak{p}(g)+1} \overline{\langle \psi_\zeta, \Gamma(U_g)^* \psi_\nu \rangle}^{\mathfrak{p}(g)} V_g B_\lambda^* B_\zeta V_g^* \\ &= \sum_{\lambda} \overline{\langle \Gamma(U_g)^* \psi_\mu, \psi_\lambda \rangle \langle \psi_\lambda, \Gamma(U_g)^* \psi_\nu \rangle}^{\mathfrak{p}(g)} \mathbb{I} = \delta_{\mu,\nu} \mathbb{I}. \end{aligned}$$

To see the second property of (5.16), note that

$$\Gamma(U_g)^* E_{\mu,\nu}^{(0)} \Gamma(U_g) = \sum_{\lambda,\zeta \in \mathcal{P}} \overline{\langle \psi_\nu, \Gamma(U_g) \psi_\zeta \rangle}^{\mathfrak{p}(g)} \langle \psi_\lambda, \Gamma(U_g)^* \psi_\mu \rangle E_{\lambda,\zeta}^{(0)}.$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Using this, we obtain

$$\begin{aligned}
T_\mu T_\nu^* &= \sum_{\lambda,\zeta} \overline{\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle\langle\Gamma(U_g)^*\psi_\nu, \psi_\zeta\rangle}^{\mathfrak{p}(g)} V_g B_\lambda B_\zeta^* V_g^* \\
&= \sum_{\lambda,\zeta} \overline{\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle\langle\Gamma(U_g)^*\psi_\nu, \psi_\zeta\rangle}^{\mathfrak{p}(g)} \iota\circ\pi_\omega\big(\Gamma(U_g)E^{(0)}_{\lambda,\zeta}\Gamma(U_g)^*\big) \\
&= \iota\circ\pi_\omega\circ \mathrm{Ad}_{\Gamma(U_g)}\Big(\sum_{\lambda,\zeta}\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle\langle\Gamma(U_g)^*\psi_\nu, \psi_\zeta\rangle E^{(0)}_{\lambda,\zeta}\Big) \\
&= \iota\circ\pi_\omega\circ \mathrm{Ad}_{\Gamma(U_g)}\big(\Gamma(U_g)^* E^{(0)}_{\mu,\nu}\Gamma(U_g)\big) = \iota\circ\pi_\omega(E^{(0)}_{\mu,\nu}).
\end{aligned}$$

To check the third property of (5.16), note that $\langle\psi_\mu, \Gamma(U_g)^*\psi_\nu\rangle = 0$ if $|\mu| \neq |\nu|$, because $\Gamma(U_g)$ commutes with $\Gamma(-\mathbb{I}_{\mathbb{C}^d})$. Using this, we check that

$$\begin{aligned}
&\sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{T_\mu}\circ \mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\mu|}}(\iota\circ\pi_\omega(A)) \\
&= \sum_{\mu}\sum_{\lambda,\zeta}\Big(\overline{\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle}^{\mathfrak{p}(g)}\overline{\langle\psi_\zeta, \Gamma(U_g)^*\psi_\mu\rangle}^{\mathfrak{p}(g)+1} \\
&\qquad\qquad \times \delta_{|\mu|,|\lambda|} V_g B_\lambda V_g^* \Gamma_{\mathcal{K}}^{|\mu|}(\iota\circ\pi_\omega)(A)\Gamma_{\mathcal{K}}^{|\mu|} V_g B_\zeta^* V_g^*\Big) \\
&= \sum_{\lambda,\zeta}\sum_{\mu} \overline{\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle\langle\Gamma(U_g)^*\psi_\mu, \psi_\zeta\rangle}^{\mathfrak{p}(g)} V_g B_\lambda V_g^* \Gamma_{\mathcal{K}}^{|\lambda|}(\iota\circ\pi_\omega)(A)\Gamma_{\mathcal{K}}^{|\lambda|} V_g B_\zeta^* V_g^* \\
&= \sum_{\lambda} V_g B_\lambda V_g^* \Gamma_{\mathcal{K}}^{|\lambda|}(\iota\circ\pi_\omega)(A)\Gamma_{\mathcal{K}}^{|\lambda|} V_g B_\lambda^* V_g^* \\
&= \sum_{\lambda} V_g\big(\mathrm{Ad}_{B_\lambda}\circ \mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\lambda|}}\big)\big(\iota\circ\pi_\omega\circ\alpha_g^{-1}(A)\big)V_g^*
\end{aligned}$$

and, recalling (5.14),

$$\begin{aligned}
\sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{T_\mu}\circ \mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\mu|}}(\iota\circ\pi_\omega(A)) &= \mathrm{Ad}_{V_g}\circ\rho\big(\iota\circ\pi_\omega\circ\alpha_g^{-1}(A)\big) \\
&= \mathrm{Ad}_{V_g}\circ\iota\circ\pi_\omega\big(\beta_{S_1}\circ\alpha_g^{-1}(A)\big) \\
&= \iota\circ\pi_\omega\circ\alpha_g\circ\beta_{S_1}\circ\alpha_g^{-1}(A) = \rho\circ\iota\circ\pi_\omega(A),
\end{aligned}$$

for all $A \in \mathcal{A}_R$. Hence, we have proven that $\{T_\mu\}_{\mu\in\mathcal{P}}$ satisfies (5.16). Applying Proposition 5.4, there is some $c_g \in \mathbb{T}$ such that $B_\mu = c_g T_\mu$ for all $\mu \in \mathcal{P}$. Therefore,

$$\begin{aligned}
\sum_{\mu}\overline{c_g}\langle\psi_\mu, \Gamma(U_g)\psi_\nu\rangle B_\mu &= \sum_{\mu,\lambda}\langle\psi_\mu, \Gamma(U_g)\psi_\nu\rangle\overline{\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle}^{\mathfrak{p}(g)} V_g B_\lambda V_g^* \\
&= \sum_{\lambda,\mu}\overline{\langle\Gamma(U_g)^*\psi_\mu, \psi_\nu\rangle\langle\psi_\lambda, \Gamma(U_g)^*\psi_\mu\rangle}^{\mathfrak{p}(g)} V_g B_\lambda V_g^* \\
&= V_g B_\nu V_g^*.
\end{aligned}$$

Hence, $\sum_\mu\langle\psi_\mu, \Gamma(U_g)\psi_\nu\rangle B_\mu = c_g V_g B_\nu V_g^*$, which completes the proof. □

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

### 5.2. Case: $\kappa_\omega = 1$

We now consider endomorphisms on $W^*$-$(G, \mathfrak{p})$-dynamical systems that are equivalent to $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_1$ from Example 2.5. Recall that $\Gamma(U_g)$ denotes the second quantisation of $U_g$ on $\mathcal{F}(\mathbb{C}^d)$. Our aim is to prove the following.

**Theorem 5.9.** *Let $\omega$ be a pure $\alpha$-invariant and translation-invariant split state on $\mathcal{A}$. Suppose that the graded $W^*$-$(G, \mathfrak{p})$-dynamical system $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ associated to $\omega$ is equivalent to $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_1$ via a $*$-isomorphism $\iota : \pi(\mathcal{A}_R)'' \to \mathcal{R}_{1,\mathcal{K}}$. Let $\rho$ be the $*$-endomorphism on $\mathcal{R}_{1,\mathcal{K}}$ given in Lemma 5.1. Then there is some $\sigma_0 \in \{0,1\}$ such that $\rho(\mathbb{I}_\mathcal{K} \otimes \sigma_x) = (-1)^{\sigma_0}\iota \circ \pi_\omega(\Gamma(-\mathbb{I}))(\mathbb{I}_\mathcal{K} \otimes \sigma_x)$ and a set of isometries $\{S_\mu\}_{\mu\in\mathcal{P}}$ on $\mathcal{K}$ such that $S_\nu^* S_\mu = \delta_{\mu,\nu}\mathbb{I}_\mathcal{K}$,*

$$\rho \circ \iota \circ \pi_\omega(A) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{\hat{S}_\mu} \circ \iota \circ \pi_\omega(A), \quad A \in \mathcal{A}_R, \tag{5.44}$$

*with $\hat{S}_\mu := S_\mu \otimes \sigma_z^{\sigma_0+|\mu|}$ and*

$$\iota \circ \pi_\omega\left(E^{(0)}_{\mu_0,\nu_0} E^{(1)}_{\mu_1,\nu_1} \cdots E^{(N)}_{\mu_N,\mu_N}\right) = (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}(\sigma_0+|\nu_j|)} S_{\mu_0}\cdots S_{\mu_N} S_{\nu_N}^* \cdots S_{\nu_0}^* \otimes \sigma_x^{\sum_{i=0}^{N}|\mu_i|+|\nu_i|} \tag{5.45}$$

*for all $N \in \mathbb{N} \cup \{0\}$ and $\mu_0, \ldots \mu_N, \nu_0, \ldots, \nu_N \in \mathcal{P}$. Furthermore, we have*

$$\sigma\text{-weak} \lim_{N\to\infty} T_{\hat{\mathbf{S}}}^N \circ \iota(x) = \langle \Omega_\omega, x\Omega_\omega\rangle \mathbb{I}_{\mathcal{K}\otimes\mathbb{C}^2}, \quad x \in \pi_\omega(\mathcal{A}_R)''. \tag{5.46}$$

*For each $g \in G$, there is some $c_g \in \mathbb{T}$ such that*

$$(-1)^{\mathfrak{q}(g)|\nu|} \sum_{\mu\in\mathcal{P}} \langle \psi_\mu, \Gamma(U_g)\psi_\nu\rangle S_\mu = c_g V_g^{(0)} S_\nu (V_g^{(0)})^*, \tag{5.47}$$

*where $V_g^{(0)}$ is given in Lemma 2.7.*

We again will prove this theorem in several steps. Parts of the proof follow the same argument as the case $\kappa_\omega = 0$, so some details will be omitted.

**Proposition 5.10.** *Let $\mathcal{K}$ be a Hilbert space and set $\Gamma_\mathcal{K} := \mathbb{I}_\mathcal{K} \otimes \sigma_z$ on $\mathcal{K} \otimes \mathbb{C}^2$. We give a grading to $\mathcal{R}_{1,\mathcal{K}} = \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$ by $\mathrm{Ad}_{\Gamma_\mathcal{K}}$. Suppose that $\mathcal{N}$ is a type I subfactor of $\mathcal{R}_{1,\mathcal{K}}$ with matrix units $\{E_{\mu,\nu}\}_{\mu,\nu\in\mathcal{P}} \subset \mathcal{N}$ spanning $\mathcal{N}$. Assume that*

$$\mathrm{Ad}_{\Gamma_\mathcal{K}}(E_{\mu,\nu}) = (-1)^{|\mu|+|\nu|} E_{\mu,\nu}, \quad \text{for} \quad \mu, \nu \in \mathcal{P}. \tag{5.48}$$

*Set $\Gamma_0 := \sum_{\mu\in\mathcal{P}} (-1)^{|\mu|} E_{\mu\mu}$. Let $\rho : \mathcal{R}_{1,\mathcal{K}} \to \mathcal{R}_{1,\mathcal{K}}$ be an injective graded, unital $*$-endomorphism such that $\rho(a)b - (-1)^{\partial a \partial b} b\rho(a) = 0$ for $b \in \mathcal{N}$, $a \in \mathcal{R}_{1,\mathcal{K}}$ with homogeneous grading. Suppose further that $\mathcal{R}_{1,\mathcal{K}} = \rho(\mathcal{R}_{1,\mathcal{K}}) \vee \mathcal{N}$.*

*Then there is some $\sigma_0 \in \{0,1\}$ such that $\rho(\mathbb{I}_\mathcal{K} \otimes \sigma_x) = (-1)^{\sigma_0}\Gamma_0(\mathbb{I}_\mathcal{K} \otimes \sigma_x)$ and there exist isometries $\{S_\mu\}_{\mu\in\mathcal{P}}$ on $\mathcal{K}$ with the property that*

$$S_\nu^* S_\mu = \delta_{\mu,\nu}\,\mathbb{I}_\mathcal{K}, \qquad \rho(b) = \sum_\mu \mathrm{Ad}_{(S_\mu\otimes\sigma_z^{\sigma_0+|\mu|})}(b) \tag{5.49}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

*for all $\mu, \nu \in \mathcal{P}$ and $b \in \mathcal{R}_{1,\mathcal{K}}$. Furthermore, for $N \in \mathbb{N}$, $\mu_0, \ldots, \mu_{N-1}, \nu_0, \ldots, \nu_{N-1} \in \mathcal{P}$, the identity*

$$E_{\mu_0,\nu_0}\rho(E_{\mu_1,\nu_1})\rho^2(E_{\mu_2,\nu_2})\cdots\rho^{N-1}(E_{\mu_{N-1},\nu_{N-1}}) = (-1)^{\sum_{j=1}^{N-1}\left(\sum_{k=0}^{j-1}(\sigma_0+|\nu_k|)\right)(|\mu_j|+|\nu_j|)} S_{\mu_0}\cdots S_{\mu_{N-1}}S^*_{\nu_{N-1}}\cdots S^*_{\nu_0}\otimes\sigma_x^{\sum_{i=0}^{N-1}|\mu_i|+|\nu_i|} \tag{5.50}$$

*holds.*

*If there are isometries $\{T_\mu\}_{\mu\in\mathcal{P}}$ on $\mathcal{K}$ such that*

$$T^*_\nu T_\mu = \delta_{\mu,\nu}\mathbb{I}_{\mathcal{K}}, \quad T_\mu T^*_\nu \otimes \sigma_x^{|\mu|+|\nu|} = E_{\mu,\nu}, \quad \rho(b) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{T_\mu\otimes\sigma_z^{\sigma_0+|\mu|}}(b), \quad b \in \mathcal{R}_{1,\mathcal{K}}, \tag{5.51}$$

*then there is some $c \in \mathbb{T}$ such that $T_\mu = cS_\mu$, for all $\mu \in \mathcal{P}$.*

To study the situation, we note the following general property.

**Lemma 5.11.** *Let $\mathcal{K}$ be a Hilbert space and set $\Gamma_{\mathcal{K}} := \mathbb{I}_{\mathcal{K}} \otimes \sigma_z$ on $\mathcal{K}\otimes\mathbb{C}^2$. We give a grading to $\mathcal{R}_{1,\mathcal{K}} = \mathcal{B}(\mathcal{K})\otimes\mathfrak{C}$ by $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}$. Let $\mathcal{N}$ and $\mathcal{M}$ be $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}$-invariant von Neumann subalgebras of $\mathcal{R}_{1,\mathcal{K}} = \mathcal{B}(\mathcal{K})\otimes\mathfrak{C}$ satisfying*

$$ab - (-1)^{\partial a \partial b} ba = 0, \quad \text{for homogeneous} \quad a \in \mathcal{N}, \quad b \in \mathcal{M}. \tag{5.52}$$

*Suppose that $\mathcal{N}$ is a type I factor with a self-adjoint unitary $\Gamma_1 \in \mathcal{N}$ satisfying $\mathrm{Ad}_{\Gamma_1}(a) = \mathrm{Ad}_{\Gamma_{\mathcal{K}}}(a)$, for all $a \in \mathcal{N}$. Suppose that $Z(\mathcal{M})^{(1)} \neq \{0\}$ and $\mathcal{N}\vee\mathcal{M} = \mathcal{B}(\mathcal{K})\otimes\mathfrak{C}$. Then the following hold:*

(i) *There are Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, a unitary $U : \mathcal{K}\otimes\mathbb{C}^2 \to \mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathbb{C}^2$ and a self-adjoint unitary $\tilde{\Gamma}_1$ on $\mathcal{H}_1$ such that*

$$\begin{aligned}
&\mathrm{Ad}_U(\mathcal{N}) = \mathcal{B}(\mathcal{H}_1)\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}, && \mathrm{Ad}_U(\mathcal{B}(\mathcal{K})\otimes\mathfrak{C}) = \mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)\otimes\mathfrak{C},\\
&\mathrm{Ad}_U(\Gamma_{\mathcal{K}}) = \tilde{\Gamma}_1\otimes\mathbb{I}_{\mathcal{H}_2}\otimes\sigma_z, && \mathrm{Ad}_U(\Gamma_1) = \tilde{\Gamma}_1\otimes\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{I}_{\mathbb{C}^2},\\
&\mathrm{Ad}_U(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x) = \mathbb{I}_{\mathcal{H}_1}\otimes\mathbb{I}_{\mathcal{H}_2}\otimes\sigma_x
\end{aligned} \tag{5.53}$$

*and*

$$\mathrm{Ad}_U(\mathcal{M}) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2} + \mathbb{C}\tilde{\Gamma}_1\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\sigma_x. \tag{5.54}$$

(ii) $\mathcal{M}' = \mathcal{N}^{(0)}(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C}) + \mathcal{N}^{(1)}(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C})\Gamma_1\Gamma_{\mathcal{K}}$.

(iii) *For any minimal projection $p$ of $\mathcal{N}$ that is even, we have $\mathcal{M}\cdot p = \mathcal{B}(q\mathcal{K})\otimes\mathfrak{C}$ with $q$ a projection on $\mathcal{K}$ satisfying $p = q\otimes\mathbb{I}_{\mathbb{C}^2}$. (Note that even $p$ is always of this form.)*

(iv) $Z(\mathcal{M}) = \mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathbb{I}_{\mathbb{C}^2} + \mathbb{C}\Gamma_1(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)$.

*Proof.* (i) Because $\mathcal{N}$ is a type I factor, there are Hilbert spaces $\mathcal{H}_1, \tilde{\mathcal{H}}_2$ and a unitary $\tilde{U} : \mathcal{K}\otimes\mathbb{C}^2 \to \mathcal{H}_1\otimes\tilde{\mathcal{H}}_2$ such that $\mathrm{Ad}_{\tilde{U}}(\mathcal{N}) = \mathcal{B}(\mathcal{H}_1)\otimes\mathbb{C}\mathbb{I}_{\tilde{\mathcal{H}}_2}$. Because $\Gamma_1 \in \mathcal{N}$, there is a self-adjoint unitary $\tilde{\Gamma}_1$ on $\mathcal{H}_1$ such that $\mathrm{Ad}_{\tilde{U}}(\Gamma_1) = \tilde{\Gamma}_1\otimes\mathbb{I}_{\tilde{\mathcal{H}}_2}$. Let $\mathcal{D} := \mathrm{span}_{\mathbb{C}}\{\mathbb{I}, \Gamma_1\Gamma_{\mathcal{K}}, (\mathbb{I}_{\mathcal{K}}\otimes\sigma_x), \Gamma_1\Gamma_{\mathcal{K}}(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\}$, a $*$-subalgebra of $\mathcal{N}'$. Let $\Gamma_1\Gamma_{\mathcal{K}} = e_{00} - e_{11}$ be a spectral decomposition of the self-adjoint unitary $\Gamma_1\Gamma_{\mathcal{K}}$. Set $e_{i,1-i} := e_{ii}(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)e_{1-i,1-i}$, $i = 0, 1$. Then because $\Gamma_1\Gamma_{\mathcal{K}}$ and $\mathbb{I}_{\mathcal{K}}\otimes\sigma_x$ anti-commute, we can check that $\{e_{ij}\}_{i,j=0,1}$ are matrix units in $\mathcal{D}$ spanning $\mathcal{D}$. Hence, $\mathcal{D}$ is a type $\mathrm{I}_2$ factor in $\mathcal{N}'$ generated by the matrix units $\{e_{ij}\}_{i,j=0,1}$. Therefore, there is a type $\mathrm{I}_2$ factor $\mathcal{D}_1$ on $\tilde{\mathcal{H}}_2$ such that $\mathrm{Ad}_{\tilde{U}}(\mathcal{D}) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\mathcal{D}_1$ and the generating matrix units $\{f_{ij}\}_{i,j=0,1}$ such that $\mathrm{Ad}_{\tilde{U}}(e_{ij}) = \mathbb{I}_{\mathcal{H}_1}\otimes f_{ij}$. Then there is a Hilbert space $\mathcal{H}_2$ and a unitary $W : \tilde{\mathcal{H}}_2 \to \mathcal{H}_2\otimes\mathbb{C}^2$ such that

$$\mathrm{Ad}_W(f_{ij}) = \mathbb{I}_{\mathcal{H}_2}\otimes\hat{e}_{ij}, \quad \mathrm{Ad}_W(\mathcal{D}_1) = \mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathrm{M}_2\,. \tag{5.55}$$

Here $\hat{e}_{ij}$ denotes the matrix unit of $2\times 2$ matrices $\mathrm{M}_2$ with respect to the standard basis of $\mathbb{C}^2$. Setting $U := \left(\mathbb{I}_{\mathcal{H}_1}\otimes W\right)\tilde{U} : \mathcal{K}\otimes\mathbb{C}^2 \to \mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathbb{C}^2$, we may check directly that $U, \mathcal{H}_1, \mathcal{H}_2, \tilde{\Gamma}_1$ satisfy (5.53).

We now prove (5.54). Because $\mathcal{M}^{(0)}$ is a von Neumann subalgebra of $\mathcal{N}'\cap(\mathcal{B}(\mathcal{K})\otimes\mathfrak{C})$, $\mathrm{Ad}_U\left(\mathcal{M}^{(0)}\right)$ is a von Neumann subalgebra of

$$\mathrm{Ad}_U(\mathcal{N}')\cap\mathrm{Ad}_U\left(\mathcal{B}(\mathcal{K})\otimes\mathfrak{C}\right) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathfrak{C}\,. \tag{5.56}$$

Furthermore, because elements in $\mathcal{M}^{(0)}$ are even with respect to $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}$, elements in $\mathrm{Ad}_U\left(\mathcal{M}^{(0)}\right)$ are even with respect to $\mathrm{Ad}_{\mathrm{Ad}_U(\Gamma_{\mathcal{K}})} = \mathrm{Ad}_{\tilde{\Gamma}_1\otimes\mathbb{I}_{\mathcal{H}_2}\otimes\sigma_z}$. Therefore, we have $\mathrm{Ad}_U\left(\mathcal{M}^{(0)}\right)\subset\mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}$. Hence, there is a von Neumann subalgebra $\tilde{\mathcal{M}}$ of $\mathcal{B}(\mathcal{H}_2)$ such that

$$\mathrm{Ad}_U\left(\mathcal{M}^{(0)}\right) = \mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\tilde{\mathcal{M}}\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}. \tag{5.57}$$

Next we consider $\mathrm{Ad}_U\left(\mathcal{M}^{(1)}\right)$. We claim $(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\Gamma_1\in Z(\mathcal{M})^{(1)}$. To see this, let $b\in Z(\mathcal{M})^{(1)}$ be a nonzero element, which exists because of the assumption, and set $\tilde{b} = (\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\Gamma_1 b$. Because $b\in Z(\mathcal{M})^{(1)}$, $\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\in Z(\mathcal{B}(\mathcal{K})\otimes\mathfrak{C})$ and $\Gamma_1$ is an even element in $\mathcal{N}$ implementing the grading on $\mathcal{N}$, we see that

$$\tilde{b}\in\mathcal{N}'\cap\mathcal{M}'\cap\{\Gamma_{\mathcal{K}}\}' = (\mathcal{B}(\mathcal{K})\otimes\mathfrak{C})'\cap\{\Gamma_{\mathcal{K}}\}' = \mathbb{C}\mathbb{I}_{\mathcal{K}\otimes\mathbb{C}^2}. \tag{5.58}$$

Hence, $(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\Gamma_1$ is proportional to $b\in Z(\mathcal{M})^{(1)}$; that is, it belongs to $Z(\mathcal{M})^{(1)}$, proving the claim. From this and (5.57) we have

$$\mathrm{Ad}_U(\mathcal{M}^{(1)}) = \mathrm{Ad}_U\left(\mathcal{M}^{(0)}(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\Gamma_1\right) = \mathbb{C}\tilde{\Gamma}_1\otimes\tilde{\mathcal{M}}\otimes\mathbb{C}\sigma_x \tag{5.59}$$

for $\tilde{\mathcal{M}}$ in (5.57). From (5.57) and (5.59), to show (5.54), it suffices to show that $\tilde{\mathcal{M}} = \mathcal{B}(\mathcal{H}_2)$. For any $a\in\tilde{\mathcal{M}}'$,

$$\begin{aligned}\mathrm{Ad}_{U^*}\left(\mathbb{I}_{\mathcal{H}_1}\otimes a\otimes\mathbb{I}_{\mathbb{C}^2}\right) &\in \left(\mathcal{M}^{(0)}\right)'\cap\left(\mathcal{M}^{(1)}\right)'\cap\mathcal{N}'\cap\{\Gamma_{\mathcal{K}}\}' \\ &= (\mathcal{B}(\mathcal{K})\otimes\mathfrak{C})'\cap\{\Gamma_{\mathcal{K}}\}' = \mathbb{C}\mathbb{I}_{\mathcal{K}\otimes\mathbb{C}^2}.\end{aligned}$$

Hence, we obtain $a\in\mathbb{C}\mathbb{I}_{\mathcal{H}_2}$. This proves that $\tilde{\mathcal{M}} = \mathcal{B}(\mathcal{H}_2)$.

(ii) We associate a spatial grading to $\mathbb{C}\mathbb{I}_{\mathcal{H}_1}$ and $\mathcal{B}(\mathcal{H}_2)\otimes\mathfrak{C}$ by $\tilde{\Gamma}_1$ and $\mathbb{I}_{\mathcal{H}_2}\otimes\sigma_z$, respectively. From (5.54), we see that $\mathrm{Ad}_U(\mathcal{M})$ is equal to the graded tensor product $\mathbb{C}\mathbb{I}_{\mathcal{H}_1}\hat{\otimes}\left(\mathcal{B}(\mathcal{H}_2)\otimes\mathfrak{C}\right)$ of $(\mathbb{C}\mathbb{I}_{\mathcal{H}_1}, \mathcal{H}_1, \tilde{\Gamma}_1)$ and $(\mathcal{B}(\mathcal{H}_2)\otimes\mathfrak{C}, \mathcal{H}_2\otimes\mathbb{C}^2, \mathbb{I}_{\mathcal{H}_2}\otimes\sigma_z)$. By Lemma A.4, its commutant $\mathrm{Ad}_U(\mathcal{M}')$ is equal to

$$\begin{aligned}\mathrm{Ad}_U(\mathcal{M}') &= \mathcal{B}(\mathcal{H}_1)^{(0)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathfrak{C}+\mathcal{B}(\mathcal{H}_1)^{(1)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathfrak{C}\,\sigma_z \\ &= \mathrm{Ad}_U\left(\mathcal{N}^{(0)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C}\right)+\mathcal{N}^{(1)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C}\right)\Gamma_1\Gamma_{\mathcal{K}}\right),\end{aligned} \tag{5.60}$$

where $\mathcal{B}(\mathcal{H}_1)$ is given a grading by $\tilde{\Gamma}_1$. This proves the claim.

(iii) Let $p$ be a minimal projection $\mathcal{N}$ that is even and hence of the form $p = q\otimes\mathbb{I}_{\mathbb{C}^2}$ with $q$ a projection on $\mathcal{K}$. Then because $p\in\mathcal{N}$ is minimal, we have $\mathrm{Ad}_U(p) = r\otimes\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{I}_{\mathbb{C}^2}$ with a rank 1 projection $r$ on $\mathcal{H}_1$. Because $p$ is even, $r$ is even with respect to $\mathrm{Ad}_{\tilde{\Gamma}_1}$. Therefore, there is a $\sigma\in\{0,1\}$ such that $\tilde{\Gamma}_1 r = (-1)^\sigma r$. Substituting (5.54), we then obtain

$$\begin{aligned}\mathrm{Ad}_U(\mathcal{M}p) &= \mathbb{C}r\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}+\mathbb{C}\tilde{\Gamma}_1 r\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\sigma_x \\ &= \mathbb{C}r\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}+\mathbb{C}(-1)^\sigma r\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\sigma_x \\ &= \mathrm{Ad}_U\left(p\left(\mathcal{B}(\mathcal{K})\otimes\mathfrak{C}\right)p\right) = \mathrm{Ad}_U\left(\mathcal{B}(q\mathcal{K})\otimes\mathfrak{C}\right)\end{aligned} \tag{5.61}$$

as required.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

(iv) From (5.54) and (5.60), we have

$$
\begin{aligned}
&\mathrm{Ad}_U\left(Z(\mathcal{M})^{(0)}\right)\\
&=\left(\mathbb{C}\mathbb{I}_{\mathcal{H}_1}\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}\right)\cap\left(\mathcal{B}(\mathcal{H}_1)^{(0)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}+\mathcal{B}(\mathcal{H}_1)^{(1)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\sigma_x\sigma_z\right)\\
&=\mathbb{C}\mathbb{I}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Ad}_U\left(Z(\mathcal{M})^{(1)}\right)\\
&=\left(\mathbb{C}\tilde{\Gamma}_1\otimes\mathcal{B}(\mathcal{H}_2)\otimes\mathbb{C}\sigma_x\right)\cap\left(\mathcal{B}(\mathcal{H}_1)^{(0)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\sigma_x+\mathcal{B}(\mathcal{H}_1)^{(1)}\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\sigma_z\right)\\
&=\mathbb{C}\tilde{\Gamma}_1\otimes\mathbb{C}\mathbb{I}_{\mathcal{H}_2}\otimes\mathbb{C}\sigma_x=\mathrm{Ad}_U\left(\mathbb{C}\Gamma_1(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)\right).
\end{aligned}
$$

This proves the claim. ◻

We introduce some notation. Given a self-adjoint unitary $T$ on some Hilbert space, we write the $\pm 1$ eigenspace projections as

$$\mathbb{P}_\varepsilon(T)=\frac{\mathbb{I}+(-1)^\varepsilon T}{2},\quad \varepsilon\in\{0,1\}. \tag{5.62}$$

Note that because we use the presentation of $\mathbb{Z}_2$ as an additive group, $\mathbb{P}_1(T)$ is the projection onto the *negative* eigenspace. We also have that $T\mathbb{P}_\varepsilon(T)=(-1)^\varepsilon\mathbb{P}_\varepsilon(T)=\mathbb{P}_\varepsilon(T)T$.

*Proof of Proposition 5.10.* Because $\mathbb{I}_{\mathcal{K}}\otimes\sigma_x$ belongs to $Z(\mathcal{R}_{1,\mathcal{K}})^{(1)}$ and $\rho$ is graded, $\rho(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)$ belongs to $Z(\rho(\mathcal{R}_{1,\mathcal{K}}))^{(1)}$. In particular, because $\rho$ is injective, $Z(\rho(\mathcal{R}_{1,\mathcal{K}}))^{(1)}$ is not zero. Therefore, we satisfy the hypothesis of Lemma 5.11 with $\mathcal{M}$ and $\Gamma_1$ replaced by $\rho(\mathcal{R}_{1,\mathcal{K}})$ and $\Gamma_0$, respectively. Applying the lemma, we have that

(i) $Z\left(\rho(\mathcal{R}_{1,\mathcal{K}})\right)=\mathbb{C}\mathbb{I}+\mathbb{C}\Gamma_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)$.
(ii) For any $\mu\in\mathcal{P}$, $E_{\mu\mu}=e_{\mu\mu}\otimes\mathbb{I}_{\mathbb{C}^2}$ with $e_{\mu\mu}$ a projection on $\mathcal{K}$, $\rho\left(\mathcal{R}_{1,\mathcal{K}}\right)E_{\mu\mu}=\mathcal{B}(e_{\mu\mu}\mathcal{K})\otimes\mathfrak{C}$.
(iii) $\rho\left(\mathcal{R}_{1,\mathcal{K}}\right)'=\mathcal{N}^{(0)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C}\right)+\mathcal{N}^{(1)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}}\otimes\mathfrak{C}\right)\Gamma_0\Gamma_{\mathcal{K}}$.

Because of (i), $\rho(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x)$, an odd self-adjoint unitary in $Z\left(\rho(\mathcal{R}_{1,\mathcal{K}})\right)$ should be either $\Gamma_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)$ or $-\Gamma_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)$. Therefore, there is $\sigma_0\in\{0,1\}$ such that

$$\rho\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)=(-1)^{\sigma_0}\Gamma_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right). \tag{5.63}$$

By (ii), (5.63) and the fact that $E_{\mu\mu}\in\mathcal{N}^{(0)}$ commutes with $\rho(\mathcal{R}_{1,\mathcal{K}})$, for each $\mu\in\mathcal{P}$ we have

$$
\begin{aligned}
\rho\left(\left(\mathcal{B}(\mathcal{K})\otimes\mathbb{C}\mathbb{I}_{\mathbb{C}^2}\right)\cdot\mathbb{P}_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)\right)E_{\mu\mu}&=\rho\left(\mathcal{R}_{1,\mathcal{K}}\mathbb{P}_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)\right)E_{\mu\mu}\\
&=\mathcal{B}(e_{\mu\mu}\mathcal{K})\otimes\mathbb{C}\mathbb{P}_{\sigma_0+|\mu|}\left(\sigma_x\right).
\end{aligned} \tag{5.64}
$$

Therefore, there is a $*$-isomorphism $\rho_\mu:\mathcal{B}(\mathcal{K})\to\mathcal{B}(e_{\mu\mu}\mathcal{K})$ such that

$$\rho\left((a\otimes\mathbb{I})\cdot\mathbb{P}_0\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)\right)E_{\mu\mu}=\rho_\mu(a)\otimes\mathbb{P}_{\sigma_0+|\mu|}(\sigma_x),\quad a\in\mathcal{B}(\mathcal{K}). \tag{5.65}$$

Applying $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}$, we also get that

$$\rho\left((a\otimes\mathbb{I})\cdot\mathbb{P}_1\left(\mathbb{I}_{\mathcal{K}}\otimes\sigma_x\right)\right)E_{\mu\mu}=\rho_\mu(a)\otimes\mathbb{P}_{\sigma_0+|\mu|+1}(\sigma_x),\quad a\in\mathcal{B}(\mathcal{K}). \tag{5.66}$$

From (5.65) and (5.66), we obtain

$$\rho\left(a\otimes\mathbb{I}\right)E_{\mu\mu}=\rho_\mu(a)\otimes\mathbb{I}_{\mathbb{C}^2},\quad a\in\mathcal{B}(\mathcal{K}). \tag{5.67}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Furthermore, by (5.63), we have

$$\rho\left(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x\right) E_{\mu\mu} = (-1)^{\sigma_0+|\mu|}\left(e_{\mu\mu} \otimes \sigma_x\right). \tag{5.68}$$

By Wigner's theorem, for each $\mu \in \mathcal{P}$, there is a unitary $T_\mu : \mathcal{K} \to e_{\mu\mu}\mathcal{K}$ such that

$$T_\mu^* T_\nu = \delta_{\mu,\nu}\mathbb{I}_{\mathcal{K}}, \quad T_\mu T_\mu^* = e_{\mu\mu}, \quad \mu, \nu \in \mathcal{P}, \quad \mathrm{Ad}_{T_\mu}(a) = \rho_\mu(a), \quad a \in \mathcal{B}(\mathcal{K}). \tag{5.69}$$

From this, (5.67) and (5.68), we obtain

$$\rho(b)E_{\mu\mu} = \mathrm{Ad}_{T_\mu \otimes \sigma_z^{\sigma_0+|\mu|}}(b), \quad b \in \mathcal{R}_{1,\mathcal{K}}. \tag{5.70}$$

Summing this over $\mu$, we obtain

$$\rho(b) = \sum_{\mu\in\mathcal{P}} \mathrm{Ad}_{T_\mu \otimes \sigma_z^{\sigma_0+|\mu|}}(b), \quad b \in \mathcal{R}_{1,\mathcal{K}}. \tag{5.71}$$

Multiplying $T_\nu T_\mu^* \otimes \sigma_z^{|\mu|+|\nu|}$ from the left or right of (5.71), we obtain the same value for any $b \in \mathcal{R}_{1,\mathcal{K}}$. Therefore, $T_\nu T_\mu^* \otimes \sigma_z^{|\mu|+|\nu|}$ belongs to $\rho(\mathcal{R}_{1,\mathcal{K}})'$. By (iii), we then have

$$T_\nu T_\mu^* \otimes \sigma_z^{|\mu|+|\nu|} \in \rho(\mathcal{R}_{1,\mathcal{K}})' = \mathcal{N}^{(0)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}} \otimes \mathfrak{C}\right) + \mathcal{N}^{(1)}\left(\mathbb{C}\mathbb{I}_{\mathcal{K}} \otimes \mathfrak{C}\right)\Gamma_0\Gamma_{\mathcal{K}}. \tag{5.72}$$

Hence, if $|\mu| = |\nu|$, $T_\nu T_\mu^* \otimes \mathbb{I}_{\mathbb{C}^2} \in \mathcal{N}^{(0)}$, and if $|\mu| \neq |\nu|$, this means $T_\nu T_\mu^* \otimes \mathbb{I}_{\mathbb{C}^2} \in \mathcal{N}^{(1)}\left(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x\right)$. From (5.69), $\{T_\mu T_\nu^* \otimes \mathbb{P}_0(\sigma_x)\}_{\mu,\nu\in\mathcal{P}}$ are matrix units in $\mathcal{N}\left(\mathbb{I}_{\mathcal{K}} \otimes \mathbb{P}_0(\sigma_x)\right)$ with $e_{\mu\mu}T_\mu T_\nu^* e_{\nu\nu} \otimes \mathbb{P}_0(\sigma_x) = T_\mu T_\nu^* \otimes \mathbb{P}_0(\sigma_x)$. Then as in the proof of Proposition 5.4, there are $c_\mu \in \mathbb{T}$ such that $S_\mu S_\nu^* \otimes \mathbb{P}_0(\sigma_x) = E_{\mu\nu}\mathbb{P}_0(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x)$ for $S_\mu = c_\mu T_\mu$. Applying $\mathrm{Ad}_{\Gamma_{\mathcal{K}}}$, we also obtain $S_\mu S_\nu^* \otimes \mathbb{P}_1(\sigma_x) = (-1)^{|\mu|+|\nu|}E_{\mu\nu}\mathbb{P}_1(\mathbb{I}_{\mathcal{K}} \otimes \sigma_x)$, which then implies that

$$\begin{aligned}\left(S_\mu \otimes \sigma_x^{|\mu|}\right)\left(S_\nu \otimes \sigma_x^{|\nu|}\right)^* &= S_\mu S_\nu^* \otimes \sigma_x^{|\mu|+|\nu|} \\ &= S_\mu S_\nu^* \otimes \left(\mathbb{P}_0(\sigma_x) + (-1)^{|\mu|+|\nu|}\mathbb{P}_1(\sigma_x)\right) = E_{\mu\nu}.\end{aligned} \tag{5.73}$$

It is clear that $\{S_\mu\}_{\mu\in\mathcal{P}}$ are isometries satisfying (5.49). The proof of (5.50) comes from an induction argument using (5.49) and (5.73). Because the argument is the same as in the proof of Proposition 5.4, we omit the details. Similarly, the proof that the isometries $\{S_\mu\}_{\mu\in\mathcal{P}}$ are unique up to scalar multiplication in $\mathbb{T}$ is the same as in Proposition 5.4. □

*Proof of Theorem 5.9.* The Hilbert space $\mathcal{K}$, finite type I factor $\iota \circ \pi_\omega(\mathcal{A}_{\{0\}})$ with matrix units $\{\iota \circ \pi_\omega \circ (E_{\mu,\nu}^{(0)})\}_{\mu,\nu\in\mathcal{P}} \subset \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$ and $\rho$ satisfy the conditions of Proposition 5.10. Applying the proposition, we obtain $\sigma_0 \in \{0, 1\}$ and $\{S_\mu\}$ satisfying (5.44) and (5.45) from the statement of the theorem. The property (5.46) follows from (5.44) and parts (i) and (iii) of Lemma 5.2. For the proof of (5.47), we set

$$T_\nu := (-1)^{\mathfrak{q}(g)|\nu|} \sum_{\mu\in\mathcal{P}} \overline{\langle \psi_\mu, \Gamma(U_g)\psi_\nu \rangle}^{\mathfrak{p}(g)} \left(V_g^{(0)}\right)^* S_\mu V_g^{(0)}. \tag{5.74}$$

As in the proof of Theorem 5.3, we then can check that $T_\mu$ satisfies (5.51) for $E_{\mu\nu}$ replaced by $\iota \circ \pi_\omega(E_{\mu\nu}^{(0)})$. Applying the last statement of Proposition 5.10, there is some $c_g \in \mathbb{T}$ such that $S_\mu = c_g T_\mu$ for all $\mu \in \mathcal{P}$. The proof of (5.47) is given by the same argument as in the proof of Theorem 5.3. □

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

## 6. Fermionic matrix product states

Using our results from Section 5, in this section we consider a translation-invariant split state $\omega$ of $\mathcal{A}$ whose density matrices have uniformly bounded rank on finite intervals. Our main result is that such states can be written as the thermodynamic limit of an even or odd fermionic MPS depending on the value $\kappa_\omega \in \mathbb{Z}_2$. See [11, 20] for the basic properties of fermionic MPS in the finite setting. The idea of the proof is the same as quantum spin case (cf. [6, 21, 32]), although anti-commutativity results in richer structures. We start with some preliminary results.

The following lemma is immediate because each $\mathcal{A}_{[0,N-1]}$ is isomorphic to a matrix algebra.

**Lemma 6.1.** *Let $\omega$ be a $\Theta$-invariant state of $\mathcal{A}$. For each $N \in \mathbb{N}$, let $Q_N$ be the support projection of the density matrix of $\omega|_{\mathcal{A}_{[0,N-1]}}$, the restriction of $\omega$ to $\mathcal{A}_{[0,N-1]}$. Then $Q_N$ is even.*

We consider the situation where the matrices $Q_N$ have uniformly bounded rank.

**Lemma 6.2.** *Let $\{Q_N\}$ be a sequence of orthogonal projections with $Q_N \in \mathcal{A}^{(0)}_{[0,N-1]}$. We suppose that the rank of $Q_N$ is uniformly bounded; that is, $\sup_{N\in\mathbb{N}} \operatorname{rank}(Q_N) < \infty$. Let $\pi$ be an irreducible representation of $\mathcal{A}_R$ or $\mathcal{A}^{(0)}_R$ on a Hilbert space $\mathcal{H}$. Set $\mathcal{H}_0 = \bigcap_{N=1}^{\infty} \big(\pi(Q_N)\mathcal{H}\big)$. Then $\dim \mathcal{H}_0 < \infty$.*

*Proof.* Because the statement is trivial if $\mathcal{H}_0 = \{0\}$, assume that $\mathcal{H}_0 \neq \{0\}$. We fix a unit vector $\eta \in \mathcal{H}_0$ and let $\{\xi_j\}_{j=1}^l \subset \mathcal{H}_0$ be an orthonormal system. We let $\mathfrak{A}$ denote either $\mathcal{A}_R$ or $\mathcal{A}^{(0)}_R$ with $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ irreducible and let $\mathfrak{A}_{\mathrm{loc}}$ denote local elements in $\mathfrak{A}$. We similarly write $\mathfrak{A}_{[0,N-1]}$ to denote either $\mathcal{A}_{[0,N-1]}$ or its even subalgebra. Note that the $l \times l$ matrix $(\langle \xi_i, \xi_j\rangle)_{i,j=1,\dots,l}$ is an identity. Because $\pi$ is irreducible, approximating $\xi_i$ with elements in $\pi(\mathfrak{A}_{\mathrm{loc}})\eta$, there exists an $N \in \mathbb{N}$ and elements $a_{j,N} \in Q_N \mathfrak{A}_{[0,N-1]} Q_N$ such that for the $l \times l$-matrix $X_N = (\langle \pi(a_{i,N})\eta, \pi(a_{j,N})\eta\rangle)_{i,j=1,\dots,l}$,

$$\left\| X_N - \mathbb{I}_{\mathrm{M}_l} \right\| < \frac{1}{2} \tag{6.1}$$

holds.

We now claim that $\{a_{j,N}\}_{j=1}^l$ are linearly independent within $Q_N \mathfrak{A}_{[0,N-1]} Q_N$. So we suppose that $\sum_j d_j a_{j,N} = 0$ for $\{d_j\}_{j=1}^l \subset \mathbb{C}$. Then taking the vector $d = (d_1, \dots, d_l)$,

$$\langle d, X_N d\rangle = \sum_{i,j=1}^{l} \langle \pi(a_{i,N})\eta, \pi(a_{j,N})\eta\rangle\, \overline{d_i} d_j = \left\| \pi\Big(\sum_{j=1}^{l} d_j a_{j,N}\Big)\eta \right\|^2 = 0.$$

Therefore,

$$0 = \langle d, X_N d\rangle = \|d\|^2 + \langle d, (X_N - \mathbb{I})d\rangle \geq \|d\|^2 - \frac{1}{2}\|d\|^2 = \frac{1}{2}\|d\|^2$$

and so $d = 0$ and $\{a_{j,N}\}_{j=1}^l$ are linearly independent.

By the assumption, we have $\dim\big(Q_N \mathfrak{A}_{[0,N-1]} Q_N\big) \leq C^2$, for $C := \sup_{N\in\mathbb{N}} \operatorname{rank}(Q_N) < \infty$. This tells us that $l \leq C^2$ and so $\dim \mathcal{H}_0 \leq C^2$. □

We now consider the case of even and odd fermionic MPS separately.

### 6.1. *Case:* $\kappa_\omega = 0$ *(even fermionic MPS)*

**Theorem 6.3.** *Let $\omega$ be a pure, split, translation-invariant and $\alpha$-invariant state on $\mathcal{A}$ with index $\operatorname{Ind}(\omega) = (0, \mathfrak{q}, [v])$. For each $N \in \mathbb{N}$, let $Q_N$ be the support projection of the density matrix of $\omega|_{\mathcal{A}_{[0,N-1]}}$ and assume $\sup_{N\in\mathbb{N}} \operatorname{rank}(Q_N) < \infty$. Then there is some $m \in \mathbb{N}$, a faithful density matrix $D \in \mathrm{M}_m$, a self-adjoint unitary $\Theta \in \mathrm{M}_m$ and a set of matrices $\{v_\mu\}_{\mu\in\mathcal{P}}$ in $\mathrm{M}_m$ satisfying the following:*

(i) *For all* $x \in \mathrm{M}_m$, $\lim_{N\to\infty} T_{\mathbf{v}}^N(x) = \mathrm{Tr}\,(Dx)\,\mathbb{I}_{\mathrm{M}_m}$ *in the norm topology.*
(ii) *There is some* $\sigma_0 = 0, 1$ *such that* $\mathrm{Ad}_\Theta\left(v_\mu\right) = (-1)^{|\mu|+\sigma_0} v_\mu$ *for all* $\mu \in \mathcal{P}$.
(iii) $\mathrm{Ad}_\Theta(D) = D$.
(iv) *For any* $l \in \mathbb{N} \cup \{0\}$ *and* $\mu_0, \dots \mu_l, \nu_0, \dots \nu_l \in \mathcal{P}$,

$$\omega\big(E^{(0)}_{\mu_0,\nu_0} E^{(1)}_{\mu_1,\nu_1} \cdots E^{(l)}_{\mu_l,\nu_l}\big) = (-1)^{\sum_{k=1}^{l} (|\mu_k|+|\nu_k|) \sum_{j=0}^{k-1} |\nu_j|}\, \mathrm{Tr}\left(D v_{\mu_0} \cdots v_{\mu_l} v^*_{\nu_l} \cdots v^*_{\nu_0}\right). \tag{6.2}$$

(v) *There is a projective unitary/anti-unitary representation* $W$ *on* $\mathbb{C}^m$ *relative to* $\mathfrak{p}$ *and* $c_g \in \mathbb{T}$ *such that*

$$\sum_{\mu\in\mathcal{P}} \langle \psi_\mu, \Gamma(U_g)\psi_\nu\rangle v_\mu = c_g W_g v_\nu W_g^*. \tag{6.3}$$

*The second cohomology class associated to* $W$ *is* $[\upsilon]$ *and*

$$\mathrm{Ad}_{W_g^*}(D) = D, \quad \mathrm{Ad}_{W_g}(\Theta) = (-1)^{\mathfrak{q}(g)}\Theta, \quad g \in G. \tag{6.4}$$

**Remark 6.4** (Comparison with index for even fermionic MPS)**.** Given an even fermionic MPS with on-site $G$-symmetry, $H^1(G,\mathbb{Z}_2) \times H^2(G, U(1)_\mathfrak{p})$-valued indices are defined in [11, 20, 39]. Briefly, an irreducible even fermionic MPS is specified by matrices $\{a_\mu\}_{\mu\in\mathcal{P}} \subset \mathrm{M}_m$ spanning a simple algebra that is $\mathbb{Z}_2$-graded by the adjoint action of a self-adjoint unitary $\Theta \in \mathrm{M}_m$. The on-site group action is given by $\mathrm{Ad}_{\tilde{W}_g}$ on the generators up to a $U(1)$-phase, where $\tilde{W}$ is a projective unitary/anti-unitary representation of $G$. The indices $(\tilde{\mathfrak{q}}, [\tilde{\upsilon}])$ defined in [11, 20, 39] are given by the grading of the representation and its second cohomology class,

$$\mathrm{Ad}_{\tilde{W}_g}(\Theta) = (-1)^{\tilde{\mathfrak{q}}(g)}\Theta, \qquad \tilde{W}_g\tilde{W}_h = \tilde{\upsilon}(g,h)\tilde{W}_{gh}.$$

It is therefore clear from part (v) of Theorem 6.3 that the the indices $(\mathfrak{q}, [\upsilon])$ defined for $\omega$ coincide with the indices defined from the corresponding fermionic MPS.

To prove Theorem 6.3 we start with a preparatory lemma.

**Lemma 6.5.** *Consider the setting of Theorem 6.3. Suppose that the graded* $W^*$*-*$(G,\mathfrak{p})$*-dynamical system* $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ *associated to* $\omega$ *is equivalent to* $(\mathcal{R}_{0,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g}) \in \mathcal{S}_0$*, via a* $*$*-isomorphism* $\iota : \pi_\omega(\mathcal{A}_R)'' \to \mathcal{B}(\mathcal{K}\otimes\mathbb{C}^2)$*. Then the following hold:*

(i) *There is a finite rank density operator* $D$ *on* $\mathcal{K}\otimes\mathbb{C}^2$ *such that*

$$\mathrm{Ad}_{\Gamma_\mathcal{K}}(D) = D, \quad \textit{and} \quad \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega(A))\right) = \omega(A) \tag{6.5}$$

*for all* $A \in \mathcal{A}_R$*. For* $P_{\mathrm{Supp}(D)}$*, the support projection of* $D$*,* $\mathrm{Ad}_{\Gamma_\mathcal{K}}\left(P_{\mathrm{Supp}(D)}\right) = P_{\mathrm{Supp}(D)}$.
(ii) *Let* $\{B_\mu\}_{\mu\in\mathcal{P}}$ *be the set of isometries given in Theorem 5.3. Then we have*

$$v_\mu := P_{\mathrm{Supp}(D)} B_\mu = P_{\mathrm{Supp}(D)} B_\mu P_{\mathrm{Supp}(D)}, \quad \mu \in \mathcal{P}. \tag{6.6}$$

(iii) $P_{\mathrm{Supp}(D)} V_g = V_g P_{\mathrm{Supp}(D)}$ *and* $DV_g = V_g D$ *for any* $g \in G$.

*Proof.* (i) Given the cyclic vector $\Omega_\omega$, $\langle \Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle$ defines a normal state on $\mathcal{B}(\mathcal{K}\otimes\mathbb{C}^2)$. Let $D$ be a density operator on $\mathcal{K}\otimes\mathbb{C}^2$ such that $\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(Dx) = \langle\Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle$. We then see that

$$\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)(A)\right) = \langle\Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle = \omega(A), \quad A \in \mathcal{A}_R.$$

Because $\omega\circ\Theta = \omega$ and $\iota\circ\pi_\omega\circ\Theta|_{\mathcal{A}_R} = \mathrm{Ad}_{\Gamma_\mathcal{K}}\circ\iota\circ\pi_\omega|_{\mathcal{A}_R}$, it follows that $\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(\mathrm{Ad}_{\Gamma_\mathcal{K}}(D)(\iota\circ\pi_\omega)(A)) = \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(D(\iota\circ\pi_\omega)(A))$ for all $A \in \mathcal{A}_R$. As such, $\mathrm{Ad}_{\Gamma_\mathcal{K}}(D) = D$. From this, we have $\mathrm{Ad}_{\Gamma_\mathcal{K}}\left(P_{\mathrm{Supp}(D)}\right) = P_{\mathrm{Supp}(D)}$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Let $\mathcal{H}_0 = \bigcap_{N=1}^{\infty} (\iota \circ \pi_\omega(Q_N)) \left(\mathcal{K} \otimes \mathbb{C}^2\right)$. Because $\iota \circ \pi_\omega$ is an irreducible representation of $\mathcal{A}_R$, from Lemma 6.2, $\mathcal{H}_0$ is finite-dimensional. Because $\omega(\mathbb{I}-Q_N) = 0$, we have $\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)(\mathbb{I}-Q_N)\right) = \omega(\mathbb{I}-Q_N) = 0$. This means that $P_{\mathrm{Supp}(D)}$, the support projection of $D$, satisfies $P_{\mathrm{Supp}(D)} \le \iota\circ\pi_\omega(Q_N)$ for all $N \in \mathbb{N}$. Hence, we have $P_{\mathrm{Supp}(D)} \left(\mathcal{K} \otimes \mathbb{C}^2\right) \subset \mathcal{H}_0$. Therefore, $D$ is finite rank.

(ii) Recall the endomorphism $\rho$ satisfying (5.1) from Lemma 5.1. Because $\omega(A) = \omega(\beta_{S_1}(A))$ for all $A \in \mathcal{A}_R$, the set of isometries $\{B_\mu\}_{\mu\in\mathcal{P}}$ given in Theorem 5.3 are such that

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)(A)\right) &= \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\rho\circ\iota\circ\pi_\omega)(A)\right)\\ &= \sum_\mu \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(\mathrm{Ad}_{B_\mu^*}\circ\mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\mu|}}(D)(\iota\circ\pi_\omega)(A)\right)\end{aligned}$$

for all $A \in \mathcal{A}_R$. This implies that $D = \sum_\mu \mathrm{Ad}_{B_\mu^*}\circ\mathrm{Ad}_{\Gamma_{\mathcal{K}}^{|\mu|}}(D) = \sum_\mu \mathrm{Ad}_{B_\mu^*}(D)$ and so

$$\sum_\mu \left(\mathbb{I}-P_{\mathrm{Supp}(D)}\right) B_\mu^* D B_\mu \left(\mathbb{I}-P_{\mathrm{Supp}(D)}\right) = \left(\mathbb{I}-P_{\mathrm{Supp}(D)}\right) D \left(\mathbb{I}-P_{\mathrm{Supp}(D)}\right) = 0.$$

Hence, we obtain $P_{\mathrm{Supp}(D)} B_\mu \left(\mathbb{I}-P_{\mathrm{Supp}(D)}\right) = 0$.

(iii) For an element $A \in \mathcal{A}_R$ and $\mathfrak{p}(g) \in \mathbb{Z}_2$, we set $A^{\mathfrak{p}(g)*}$ as $A$ if $\mathfrak{p}(g) = 0$ and $A^*$ if $\mathfrak{p}(g) = 1$. Because $\omega(\alpha_g(A^{\mathfrak{p}(g)*})) = \omega(A) = \mathrm{Tr}(D(\iota\circ\pi_\omega)(A))$, $A \in \mathcal{A}_R$, we have that

$$\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)(A)\right) = \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)\left(\alpha_g(A^{\mathfrak{p}(g)*})\right)\right) = \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(DV_g\left((\iota\circ\pi_\omega)(A^{\mathfrak{p}(g)*})\right)V_g^*\right).$$

Given an orthonormal basis $\{\xi_j\}_j$ of $\mathcal{K}\otimes\mathbb{C}^2$, we see that for any $A \in \mathcal{A}_R$,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D(\iota\circ\pi_\omega)(A)\right) &= \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(DV_g\left((\iota\circ\pi_\omega)(A^{\mathfrak{p}(g)*})\right)V_g^*\right)\\ &= \sum_j \langle V_g\xi_j, DV_g(\iota\circ\pi_\omega)(A^{\mathfrak{p}(g)*})\xi_j\rangle\\ &= \sum_j \overline{\langle \xi_j, V_g^* DV_g(\iota\circ\pi_\omega)(A^{\mathfrak{p}(g)*})\xi_j\rangle}^{\mathfrak{p}(g)}\\ &= \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(V_g^* DV_g(\iota\circ\pi_\omega)(A)\right),\end{aligned}$$

where for the second equality we used that $\{V_g\xi_j\}_j$ is an orthonormal basis of $\mathcal{K}\otimes\mathbb{C}^2$. Therefore, $V_g^* DV_g = D$ and so $P_{\mathrm{Supp}(D)}V_g = V_g P_{\mathrm{Supp}(D)}$. □

*Proof of Theorem 6.3.* We use the notation of Theorem 5.3 and Lemma 6.5. Let $m \in \mathbb{N}$ be the rank of $D$ from Lemma 6.5. We naturally identify $P_{\mathrm{Supp}(D)}\mathcal{B}(\mathcal{K}\otimes\mathbb{C}^2)P_{\mathrm{Supp}(D)}$ and $\mathrm{M}_m$. Then we may regard $D$ as a faithful density matrix in $\mathrm{M}_m$ and $\{v_\mu\}_{\mu\in\mathcal{P}}$ matrices in $\mathrm{M}_m$. Because $\Gamma_{\mathcal{K}}$ commutes with $P_{\mathrm{Supp}(D)}$, $\Theta := \Gamma_{\mathcal{K}} P_{\mathrm{Supp}(D)}$ defines a self-adjoint unitary in $\mathrm{M}_m$. Similarly, because of (iii) of Lemma 6.5, $W_g := V_g P_{\mathrm{Supp}(D)}$ defines a projective unitary/anti-unitary representation of $G$ on $P_{\mathrm{Supp}(D)}$ relative to $\mathfrak{p}$. Clearly, the second cohomology class associated to $W$ is the same of that of $V$; that is, $[v]$. From $\mathrm{Ad}_{V_g}(\Gamma_{\mathcal{K}}) = (-1)^{\mathfrak{q}(g)}\Gamma_{\mathcal{K}}$, we have that $\mathrm{Ad}_{W_g}(\Theta) = (-1)^{\mathfrak{q}(g)}\Theta$.

Now we check the properties (i)–(v).

Parts (ii) and (v) are immediate from the definition of $v_\mu$, $\Theta$, $W_g$ and the corresponding properties of $B_\mu, \Gamma_{\mathcal{K}}, V_g$. Part (iii) follows from Lemma 6.5 (i), (iii). For part (i), using (5.10), (6.6) and that $P_{\mathrm{Supp}(D)}$ is of finite rank, we have

$$T_{\mathbf{v}}^N(x) = P_{\mathrm{Supp}(D)}\, T_{\mathbf{B}}^N(x)\, P_{\mathrm{Supp}(D)} \xrightarrow[N\to\infty]{} \langle\Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle\, P_{\mathrm{Supp}(D)} = \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(Dx\right)P_{\mathrm{Supp}(D)} \quad (6.7)$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

for $x \in P_{\mathrm{Supp}(D)}\mathcal{R}_{0,\mathcal{K}}P_{\mathrm{Supp}(D)} = \mathrm{M}_m$ and convergence in the norm topology. For part (iv), (5.9) and (6.6) imply that

$$\begin{aligned}\omega\left(E^{(0)}_{\mu_0,\nu_0}E^{(1)}_{\mu_1,\nu_1}\cdots E^{(N)}_{\mu_N,\mu_N}\right) &= \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(D\left(\iota\circ\pi_\omega\left(E^{(0)}_{\mu_0,\nu_0}E^{(1)}_{\mu_1,\nu_1}\cdots E^{(N)}_{\mu_N,\mu_N}\right)\right)\right)\\ &= (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(DB_{\mu_0}\cdots B_{\mu_N}B^*_{\nu_N}\cdots B^*_{\nu_0}\right)\\ &= (-1)^{\sum_{k=1}^{N}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}|\nu_j|}\mathrm{Tr}_{\mathrm{M}_m}\left(Dv_{\mu_0}\cdots v_{\mu_N}v^*_{\nu_N}\cdots v^*_{\nu_0}\right) \qquad (6.8)\end{aligned}$$

for all $N \in \mathbb{N}\cup\{0\}$ and $\mu_0,\dots\mu_N,\nu_0,\dots,\nu_N \in \mathcal{P}$. This proves (iv). □

### 6.2. *Case:* $\kappa_\omega = 1$ *(odd fermionic MPS)*

**Theorem 6.6.** *Let $\omega$ be a pure, split, translation-invariant and $\alpha$-invariant state on $\mathcal{A}$ with index $\mathrm{Ind}(\omega) = (1,\mathfrak{q},[v])$. For each $N\in\mathbb{N}$, let $Q_N$ be the support projection of the density matrix of $\omega|_{\mathcal{A}_{[0,N-1]}}$ and assume $\sup_{N\in\mathbb{N}}\mathrm{rank}(Q_N) < \infty$. Then there is some $m\in\mathbb{N}$, a faithful density matrix $D\in\mathrm{M}_m$, a set of matrices $\{v_\mu\}_{\mu\in\mathcal{P}}$ in $\mathrm{M}_m$ and $\sigma_0\in\{0,1\}$ satisfying the following:*

(i) *Set $\hat{v}_\mu := v_\mu\otimes\sigma_z^{\sigma_0+|\mu|}$ on $\mathbb{C}^m\otimes\mathbb{C}^2$. Then $\lim_{N\to\infty}T^N_{\hat{\mathbf{v}}}(b) = \mathrm{Tr}\left(\left(D\otimes\frac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right)b\right)\mathbb{I}_{\mathrm{M}_m}\otimes\mathbb{I}_{\mathbb{C}^2}$ in norm for all $b\in\mathrm{M}_m\otimes\mathfrak{C}$.*

(ii) *For any $l\in\mathbb{N}\cup\{0\}$ and $\mu_0,\dots\mu_l,\nu_0,\dots\nu_l\in\mathcal{P}$,*

$$\begin{aligned}&\omega\big(E^{(0)}_{\mu_0,\nu_0}E^{(1)}_{\mu_1,\nu_1}\cdots E^{(l)}_{\mu_l,\nu_l}\big)\\ &\quad= (-1)^{\sum_{k=1}^{l}(|\mu_k|+|\nu_k|)\sum_{j=0}^{k-1}(\sigma_0+|\nu_j|)}\delta_{\sum_{i=0}^{l}(|\mu_i|+|\nu_i|),0}\,\mathrm{Tr}\left(D\left(v_{\mu_0}\cdots v_{\mu_l}v^*_{\nu_l}\cdots v^*_{\nu_0}\right)\right). \qquad (6.9)\end{aligned}$$

(iii) *There is a projective unitary/anti-unitary representation $W$ of $G$ on $\mathbb{C}^m$ relative to $\mathfrak{p}$ and $c_g\in\mathbb{T}$ such that for all $g\in G$ and $\nu\in\mathcal{P}$,*

$$(-1)^{\mathfrak{q}(g)|\nu|}\sum_{\mu\in\mathcal{P}}\langle\psi_\mu,\Gamma(U_g)\psi_\nu\rangle v_\mu = c_gW_gv_\nu W_g^*, \qquad \mathrm{Ad}_{W_g}(D) = D. \qquad (6.10)$$

*The second cohomology class associated to $W$ is $[v]$.*

**Remark 6.7** (Comparison with index for fermionic MPS)**.** Like Remark 6.4, we briefly compare our results with the $H^1(G,\mathbb{Z}_2)\times H^2(G,U(1)_\mathfrak{p})$-valued indices for fermionic MPS in [11, 20, 39]. An irreducible odd fermionic MPS is specified by matrices spanning a simple $\mathbb{Z}_2$-graded algebra with an odd central element. Like the even case, the group action is implemented by the adjoint action on generators by a projective unitary/anti-unitary representation, giving a second cohomology class. The representation will commute or anti-commute with the odd central element, giving a homomorphism $G\to\mathbb{Z}_2$. Considering $\omega$ as a fermionic MPS, part (iii) of Theorem 6.6 shows that the second cohomology classes coincide and (6.10) shows that the commutation of the projective unitary/anti-unitary representation with the odd central element is specified by $\mathfrak{q}$. Hence, in this setting the indices for fermionic MPS agree with the indices defined in Section 2.

**Lemma 6.8.** *Consider the setting of Theorem 6.6. Suppose that the graded $W^*$-$(G,\mathfrak{p})$-dynamical system $(\pi_\omega(\mathcal{A}_R)'', \mathrm{Ad}_{\Gamma_\omega}, \hat{\alpha}_\omega)$ associated to $\omega$ is equivalent to $(\mathcal{R}_{1,\mathcal{K}}, \mathrm{Ad}_{\Gamma_\mathcal{K}}, \mathrm{Ad}_{V_g})\in\mathcal{S}_1$, via a $*$-isomorphism $\iota:\pi_\omega(\mathcal{A}_R)''\to\mathcal{R}_{1,\mathcal{K}}$. Then the following hold:*

(i) *There is a finite rank density operator $D$ on $\mathcal{K}$ such that for all $A \in \mathcal{A}_R$,*

$$\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(\left(D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right)\left(\iota \circ \pi_\omega(A)\right)\right) = \omega(A). \tag{6.11}$$

(ii) *Let $\{S_\mu\}_{\mu\in\mathcal{P}}$ be the set of isometries given in Theorem 5.9. Then we have*

$$v_\mu := P_{\mathrm{Supp}(D)} S_\mu = P_{\mathrm{Supp}(D)} S_\mu P_{\mathrm{Supp}(D)}, \quad \mu \in \mathcal{P}. \tag{6.12}$$

(iii) $P_{\mathrm{Supp}(D)} V_g^{(0)} = V_g^{(0)} P_{\mathrm{Supp}(D)}$ *and* $\mathrm{Ad}_{V_g^{(0)}}(D) = D$ *for any* $g \in G$.

*Proof.* (i) Given the cyclic vector $\Omega_\omega$, $\langle \Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle$, $x \in \mathcal{R}_{1,\mathcal{K}}$, defines a normal state on $\mathcal{R}_{1,\mathcal{K}}$. Let $\tilde{D}$ be a density operator on $\mathcal{K} \otimes \mathbb{C}^2$ such that $\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(\tilde{D}x) = \langle \Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle$ for $x \in \mathcal{R}_{1,\mathcal{K}}$. Because $\mathcal{R}_{1,\mathcal{K}} = \mathcal{B}(\mathcal{K}) \otimes \mathfrak{C}$ and recalling the notation $\mathbb{P}_\varepsilon$ from (5.62), we may assume that $\tilde{D}$ is of the form $\tilde{D} = D_0 \otimes \mathbb{P}_0(\sigma_x) + D_1 \otimes \mathbb{P}_1(\sigma_x)$. Because $\omega \circ \Theta = \omega$ and $\iota \circ \pi_\omega \circ \Theta|_{\mathcal{A}_R} = \mathrm{Ad}_{\Gamma_\mathcal{K}} \circ \iota \circ \pi_\omega|_{\mathcal{A}_R}$, it follows that $\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(\mathrm{Ad}_{\Gamma_\mathcal{K}}(\tilde{D})(\iota\circ\pi_\omega)(A)) = \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}(\tilde{D}(\iota\circ\pi_\omega)(A))$ for all $A \in \mathcal{A}_R$. Therefore, we have $\mathrm{Ad}_{\Gamma_\mathcal{K}}(\tilde{D}) = \tilde{D}$, which implies $D_0 = D_1$. We set $D := 2D_0$ and see that $D$ is a density operator on $\mathcal{K}$ satisfying (6.11).

Let $\pi_0$ be the irreducible representation of $\mathcal{A}_R^{(0)}$ on $\mathcal{K}$ given by

$$\iota \circ \pi_\omega(a) = \pi_0(a) \otimes \mathbb{I}_{\mathbb{C}^2}, \quad a \in \mathcal{A}_R^{(0)}. \tag{6.13}$$

Let $\mathcal{H}_0 = \bigcap_{N=1}^{\infty} (\pi_0(Q_N)\mathcal{K})$. Because $\pi_0$ is an irreducible representation of $\mathcal{A}_R^{(0)}$, $\mathcal{H}_0$ is finite-dimensional by Lemma 6.2. Because $\omega(\mathbb{I} - Q_N) = 0$, we have

$$\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left((D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2})(\pi_0(\mathbb{I} - Q_N) \otimes \mathbb{I}_{\mathbb{C}^2})\right) = \omega(\mathbb{I} - Q_N) = 0.$$

This means that $P_{\mathrm{Supp}(D)}$ satisfies $P_{\mathrm{Supp}(D)} \le \pi_0(Q_N)$ for all $N \in \mathbb{N}$. Hence, we have $P_{\mathrm{Supp}(D)}\mathcal{K} \subset \mathcal{H}_0$ and $D$ is finite rank.

(ii) Recall the endomorphism $\rho$ satisfying (5.1) from Lemma 5.1. Because $\omega(A) = \omega(\beta_{S_1}(A))$ for all $A \in \mathcal{A}_R$, the set of isometries $\{S_\mu\}_{\mu\in\mathcal{P}}$ given in Theorem 5.9 and $\sigma_0$, (5.44) gives that

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(\left(D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right)(\iota\circ\pi_\omega)(A)\right) &= \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(\left(D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right)(\rho\circ\iota\circ\pi_\omega)(A)\right)\\ &= \sum_\mu \mathrm{Tr}_{\mathcal{K}\otimes\mathbb{C}^2}\left(\mathrm{Ad}_{(S_\mu^*\otimes\sigma_z^{\sigma_0+|\mu|})}\left(D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right)(\iota\circ\pi_\omega)(A)\right),\end{aligned} \tag{6.14}$$

which implies that $D = \sum_\mu \mathrm{Ad}_{S_\mu^*}(D)$. We then obtain (6.12) by the same proof as in Lemma 6.5.

(iii) By the same argument as in the proof of Lemma 6.5, we obtain $(V_g^{(0)})^* D V_g^{(0)} = D$ and so $P_{\mathrm{Supp}(D)} V_g^{(0)} = V_g^{(0)} P_{\mathrm{Supp}(D)}$. □

*Proof of Theorem 6.6.* We use the notation of Theorem 5.9 and Lemma 6.8. Let $m \in \mathbb{N}$ be the rank of $D$ from Lemma 6.8. We naturally identify $P_{\mathrm{Supp}(D)}\mathcal{B}(\mathcal{K})P_{\mathrm{Supp}(D)}$ and $\mathrm{M}_m$. Then we may regard $D$ as a faithful density matrix in $\mathrm{M}_m$ and $\{v_\mu\}_{\mu\in\mathcal{P}}$ matrices in $\mathrm{M}_m$. Because of part (iii) of Lemma 6.8, $W_g := V_g^{(0)} P_{\mathrm{Supp}(D)}$ defines a projective unitary/anti-unitary representation of $G$ on $P_{\mathrm{Supp}(D)}\mathcal{K}$ relative to $\mathfrak{p}$ whose cohomology class is the same as $V^{(0)}$; that is, $[v]$. Now we check the properties (i)–(iii) of Theorem 6.6.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Part (iii) is immediate from the definition of $v_\mu$, $W_g$ and the corresponding properties of $S_\mu$ and $V_g^{(0)}$. For part (i), using (5.46), (6.12) and that $P_{\mathrm{Supp}(D)}$ is finite rank, we have

$$T_{\hat{\mathbf{v}}}^N(x) = P_{\mathrm{Supp}(D)} T_{\hat{\mathbf{S}}}^N(x) P_{\mathrm{Supp}(D)} \xrightarrow[N\to\infty]{} \langle \Omega_\omega, \iota^{-1}(x)\Omega_\omega\rangle P_{\mathrm{Supp}(D)} = \mathrm{Tr}\left(\left(D \otimes \tfrac{1}{2}\mathbb{I}_{\mathbb{C}^2}\right) x\right) P_{\mathrm{Supp}(D)}$$

for $x \in \left(P_{\mathrm{Supp}(D)} \otimes \mathbb{I}\right) \mathcal{R}_{1,\mathcal{K}} \left(P_{\mathrm{Supp}(D)} \otimes \mathbb{I}\right) = \mathrm{M}_m \otimes \mathfrak{C}$ and convergence in the norm topology.

Part (ii) follows from (5.45) and (6.12), as in the proof of Theorem 6.3. □

## Appendix A. Graded von Neumann algebras

For convenience, we collect some facts about graded von Neumann algebras and linear/anti-linear group actions. See Subsections 2.1 and 4.1 for basic definitions.

**Lemma A.1.** *Let $(\mathcal{M}, \theta)$ be a balanced graded von Neumann algebra. Assume that $\mathcal{M}$ is of type $\mu$ and $\mathcal{M}^{(0)}$ is of type $\lambda$, with some $\mu, \lambda = \mathrm{I}, \mathrm{II}, \mathrm{III}$, and that both of $\mathcal{M}$ and $\mathcal{M}^{(0)}$ have finite-dimensional centers. Then $\lambda = \mu$.*

*Proof.* Let $U \in \mathcal{M}^{(1)}$ be a self-adjoint unitary. Let $\mathbb{E} : \mathcal{M} \to \mathcal{M}^{(0)}$ be the conditional expectation

$$\mathbb{E}(x) := \frac{1}{2}(x + \theta(x)), \quad x \in \mathcal{M}. \tag{A.1}$$

If $\mathcal{M}^{(0)}$ has a faithful normal semifinite trace $\tau_0$ (i.e., $\mathcal{M}^{(0)}$ is semifinite), then $\tau := (\tau_0 + \tau_0 \circ \mathrm{Ad}_U) \circ \mathbb{E}$ defines a faithful normal semifinite trace on $\mathcal{M}$. Hence, if $\mathcal{M}^{(0)}$ is semifinite, then $\mathcal{M}$ is semifinite.

Let us denote by $\mathcal{P}(\mathcal{M}), \mathcal{P}(\mathcal{M}^{(0)})$ the set of all orthogonal projections in $\mathcal{M}, \mathcal{M}^{(0)}$. Because $\tau|_{\mathcal{M}^{(0)}}$ is a faithful normal semifinite trace on $\mathcal{M}^{(0)}$, if $\lambda = \mathrm{II}$, then we have $\tau\left(\mathcal{P}(\mathcal{M}^{(0)})\right) = [0, \tau(1)]$. Because $\tau(\mathcal{P}(\mathcal{M}))$ contains $\tau\left(\mathcal{P}(\mathcal{M}^{(0)})\right)$ and $\mathcal{M}$ is a finite direct sum of type $\mu$-factors, this means that $\mu = \mathrm{II}$.

If $\lambda = \mathrm{I}$, then there is a nonzero abelian projection $p$ of $\mathcal{M}^{(0)}$. We claim that there is a nonzero abelian projection $r$ in $\mathcal{M}$ such that $r \le p$. If $p\mathcal{M}^{(1)}p = \{0\}$, then $p\mathcal{M}p = \mathbb{C}p$ and $p$ itself is abelian in $\mathcal{M}$. If $p\mathcal{M}^{(1)}p \neq \{0\}$, then there is a self-adjoint odd element $b \in \mathcal{M}^{(1)}$ such that $pbp \neq 0$. Because $(pbp)^2 = pbpbp \in p\mathcal{M}^{(0)}p = \mathbb{C}p$, we may assume that $pbp$ is a nonzero self-adjoint unitary in $p\mathcal{M}p$. For any $x \in \mathcal{M}^{(1)}$, we also have $pxppbp \in p\mathcal{M}^{(0)}p = \mathbb{C}p$. By the unitarity of $pbp$, we have $pxp \in \mathbb{C}pbp$ and $p\mathcal{M}^{(1)}p = \mathbb{C}pbp$. Because $pbp$ is self-adjoint unitary, we have a spectral decomposition $pbp = r_+ - r_-$, with mutually orthogonal projections $r_\pm$ in $\mathcal{M}$ and at least one of $r_\pm$ is nonzero. From $p\mathcal{M}^{(1)}p = \mathbb{C}pbp = \mathbb{C}(r_+ - r_-)$ and $p\mathcal{M}^{(0)}p = \mathbb{C}p$, $r_\pm$ are abelian in $\mathcal{M}$ and $r_\pm \le p$, proving the claim. Hence, $\mathcal{M}$ is type I as well, $\mu = \mathrm{I}$.

Conversely, if $\mathcal{M}$ has a faithful normal semifinite trace $\tau$ (i.e., if $\mathcal{M}$ is semifinite), then $\tau|_{\mathcal{M}^{(0)}}$ is a faithful normal semfinite trace on $\mathcal{M}^{(0)}$. Therefore, $\mu = \mathrm{III}$ if and only if $\lambda = \mathrm{III}$.

If $\mu = \mathrm{I}$, then $\lambda$ cannot be II or III and so is type I. If $\mu = \mathrm{II}$, then $\lambda$ cannot be I or III and so is type II. □

**Lemma A.2.** *Let $(\mathcal{M}, \theta)$ be a central graded von Neumann algebra. Then either $Z(\mathcal{M}) = \mathbb{C}\mathbb{I}$ or $Z(\mathcal{M})$ has a self-adjoint unitary $b \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$ such that*

$$Z(\mathcal{M}) \cap \mathcal{M}^{(1)} = \mathbb{C}b. \tag{A.2}$$

*Proof.* Let us assume that $\mathcal{M}$ is not a factor. By the condition of centrality, $Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}\mathbb{I}$, there is a nonzero self-adjoint element $b \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$. Because $b^2 \in Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}\mathbb{I}$, we may assume that $b$ is unitary. For any $x \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$, $xb$ also belongs to $Z(\mathcal{M}) \cap \mathcal{M}^{(0)} = \mathbb{C}\mathbb{I}$, and by the unitarity of $b$, we obtain (A.2). □

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

When $(\mathcal{M}, \theta)$ is spatially graded, an analogous result holds for $\mathcal{M} \cap \mathcal{M}'\Gamma$.

**Lemma A.3.** *Let* $(\mathcal{M}, \mathrm{Ad}_\Gamma)$ *be a central graded von Neumann algebra on* $\mathcal{H}$*, spatially graded by a self-adjoint untiary* $\Gamma$*. Then the following hold:*

(i) *If* $\mathcal{M}$ *is not a factor,* $\mathcal{M} \cap \mathcal{M}'\Gamma = \{0\}$.
(ii) *If* $\mathcal{M} \cap \mathcal{M}'\Gamma \neq \{0\}$*, then there is a self-adjoint unitary* $b \in \mathcal{M} \cap \mathcal{M}'\Gamma$ *such that* $\mathcal{M} \cap \mathcal{M}'\Gamma = \mathbb{C}b$*. In particular, if* $\Gamma \in \mathcal{M}$*, then* $\mathcal{M} \cap \mathcal{M}'\Gamma = \mathbb{C}\Gamma$.

*Proof.*

(i) If $\mathcal{M}$ is not a factor, from Lemma A.2, $Z(\mathcal{M})$ has a self-adjoint unitary $b \in Z(\mathcal{M}) \cap \mathcal{M}^{(1)}$ such that $Z(\mathcal{M}) \cap \mathcal{M}^{(1)} = \mathbb{C}b$. For any $a \in \mathcal{M} \cap \mathcal{M}'\Gamma$, we have

$$ba = ab = a\Gamma\Gamma b\Gamma\Gamma = a\Gamma(-b)\Gamma = -(a\Gamma)b\Gamma = -b(a\Gamma)\Gamma = -ba. \tag{A.3}$$

The first equality is because $b \in Z(\mathcal{M})$, and the fifth equality is because $a\Gamma \in \mathcal{M}'$. Because $b$ is unitary, this means $a = 0$.
(ii) Note that for any $a, b \in \mathcal{M} \cap \mathcal{M}'\Gamma$, $ab \in Z(\mathcal{M})$. From this observation and (i), the same proof as Lemma A.2 gives the claim. If $\Gamma \in \mathcal{M}$, as $\Gamma = \mathbb{I}\Gamma$, we have $\Gamma \in \mathcal{M} \cap \mathcal{M}'\Gamma$.

□

Recall the graded tensor product product defined in Subsection 4.1.

**Lemma A.4.** *For* $i = 1, 2$*, let* $(\mathcal{M}_i, \mathrm{Ad}_{\Gamma_i})$ *be a graded von Neumann algebra on* $\mathcal{H}_i$ *spatially graded by a self-adjoint unitary* $\Gamma_i$ *on* $\mathcal{H}_i$*. Let* $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ *be the graded tensor product of* $(\mathcal{M}_1, \mathcal{H}_1, \Gamma_1)$ *and* $(\mathcal{M}_2, \mathcal{H}_2, \Gamma_2)$*. Then commutant of the graded tensor product* $(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2)'$ *is generated by*

$$(\mathcal{M}_1')^{(0)} \odot \mathcal{M}_2', \qquad (\mathcal{M}_1')^{(1)} \odot \mathcal{M}_2'\Gamma_2. \tag{A.4}$$

*Proof.* The proof is given by a modification of the corresponding result for ungraded tensor products. Let $\mathcal{M} := \mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ and $\mathcal{N}$ be a von Neumann algebra generated by (A.4). We would like to show $\mathcal{N} = \mathcal{M}'$. A brief computation gives the inclusion $\mathcal{M} \subset \mathcal{N}'$.

We let $\sigma \in \{0, 1\}$ and denote by $\mathcal{R}^{h,(\sigma)}$ the set of all self-adjoint elements with grading $\sigma$ in a graded von Neumann algebra $\mathcal{R}$. For a complex Hilbert space $\mathcal{K}$ and its real subspace $\mathcal{V}$, $\mathcal{V}_\mathbb{R}^\perp$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{K}$ regarding $\mathcal{K}$ as a real Hilbert space, with respect to the inner product $\langle\cdot,\cdot\rangle_\mathbb{R} := \Re\langle\cdot,\cdot\rangle$.

First we assume that $\mathcal{M}_j$, $j = 1, 2$, has a cyclic vector $\Omega_j$ which is homogeneous in the sense that $\Gamma_j\Omega_j = (-1)^{\epsilon_j}\Omega_j$ for some $\epsilon_j \in \{0, 1\}$.

Because $\Omega := \Omega_1 \otimes \Omega_2$ is cyclic for $\mathcal{M}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$, to show $\mathcal{M}' = \mathcal{N}$, it suffices to show that $\mathcal{M}^h\Omega + i\mathcal{N}^h\Omega$ is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$ by [38, Chapter IV, Lemma 5.7]. For $\sigma_j = 0, 1$, $j = 1, 2$, set $\mathcal{L}_{\sigma_j}^{(j)} := (\mathbb{I} + (-1)^{\sigma_j}\Gamma_j)\mathcal{H}_j$, $j = 1, 2$. Then by the cyclicity of $\Omega_j$ and $\Gamma_j\Omega_j = (-1)^{\epsilon_j}\Omega_j$, $\mathcal{M}_j^{(\sigma_j)}\Omega_j$ is a dense subspace of $\mathcal{L}_{\sigma_j+\epsilon_j}^{(j)}$. We also note $(\mathcal{M}_j')^{(\sigma_j)}\Omega_j \subset \mathcal{L}_{\sigma_j+\epsilon_j}^{(j)}$. By [38, Chapter IV, Lemma 5.7], $i(\mathcal{M}_j')^h\Omega_j$ is dense in $(\mathcal{M}_j^h\Omega_j)_\mathbb{R}^\perp$. Therefore, $i(\mathcal{M}_j')^{h,(\sigma_j+\epsilon_j)}\Omega_j$ is dense in $((\mathcal{M}_j)^{h,(\sigma_j+\epsilon_j)}\Omega_j)_\mathbb{R}^\perp \cap \mathcal{L}_{\sigma_j}^{(j)}$. Set $Y_{\sigma_1} := (\mathcal{M}_1)^{h,(\sigma_1+\epsilon_1)}\Omega_1$ and $Z_{\sigma_2} := (\mathcal{M}_2)^{h,(\sigma_2+\epsilon_2)}\Omega_2$. By the above observation, $i(\mathcal{M}_1')^{h,(\sigma_1+\epsilon_1)}\Omega_1$ is dense in $(Y_{\sigma_1})_\mathbb{R}^\perp \cap \mathcal{L}_{\sigma_1}^{(1)}$ and $i(\mathcal{M}_2')^{h,(\sigma_2+\epsilon_2)}\Omega_2$ is dense in $(Z_{\sigma_2})_\mathbb{R}^\perp \cap \mathcal{L}_{\sigma_2}^{(2)}$. Because $Y_{\sigma_1} + iY_{\sigma_1}$ and $Z_{\sigma_2} + iZ_{\sigma_2}$ are dense in $\mathcal{L}_{\sigma_1}^{(1)}$ and $\mathcal{L}_{\sigma_2}^{(2)}$, respectively, by [38, Chapter IV, Lemma 5.8], $Y_{\sigma_1} \odot Z_{\sigma_2} + i((Y_{\sigma_1})_\mathbb{R}^\perp \cap \mathcal{L}_{\sigma_1}^{(1)}) \odot ((Z_{\sigma_2})_\mathbb{R}^\perp \cap \mathcal{L}_{\sigma_2}^{(2)})$ is dense in $\mathcal{L}_{\sigma_1}^{(1)} \otimes \mathcal{L}_{\sigma_2}^{(2)}$. Hence, we conclude that

$$(\mathcal{M}_1)^{h,(\sigma_1+\epsilon_1)}\Omega_1 \odot (\mathcal{M}_2)^{h,(\sigma_2+\epsilon_2)}\Omega_2 + i(\mathcal{M}_1')^{h,(\sigma_1+\epsilon_1)}\Omega_1 \odot (\mathcal{M}_2')^{h,(\sigma_2+\epsilon_2)}\Omega_2 =: \mathcal{V}_{\sigma_1,\sigma_2} \tag{A.5}$$

is dense in $\mathcal{L}_{\sigma_1}^{(1)} \otimes \mathcal{L}_{\sigma_2}^{(2)}$. Using the homogeneity of $\Omega_j$, $\Gamma_j\Omega_j = (-1)^{\epsilon_j}\Omega_j$, we can prove that $\mathcal{M}^h\Omega + i\mathcal{N}^h\Omega$

includes

$$\sum_{\sigma_1,\sigma_2=0,1} i^{(\sigma_1+\epsilon_1)(\sigma_2+\epsilon_2)}\mathcal{V}_{\sigma_1,\sigma_2}. \tag{A.6}$$

By the density of $\mathcal{V}_{\sigma_1,\sigma_2}$ in $\mathcal{L}^{(1)}_{\sigma_1}\otimes\mathcal{L}^{(2)}_{\sigma_2}$, $\mathcal{M}^h\Omega+i\mathcal{N}^h\Omega$ is dense in $\mathcal{H}_1\otimes\mathcal{H}_2$ and this completes the proof for the case with cyclic vectors.

Now we drop the assumption of the existence of the cyclic vectors. Let $\{E'_a\}_a$ be a family of mutually orthogonal projections in $\mathcal{M}'_1$ such that each $E'_a$ is an orthogonal projection onto $\overline{\mathcal{M}_1\xi_a}$, with a homogeneous $\xi_a\in\mathcal{H}_1$, and $\sum_a E'_a=\mathbb{I}_{\mathcal{H}_1}$. Let $\{F'_b\}_b$ be a family of mutually orthogonal projections in $\mathcal{M}'_2$ such that each $F'_b$ is an orthogonal projection onto $\overline{\mathcal{M}_2\eta_b}$, with a homogeneous $\eta_b\in\mathcal{H}_2$, and $\sum_b F'_b=\mathbb{I}_{\mathcal{H}_2}$. Note that because $\xi_a,\eta_b$ are homogeneous, $E'_a$ and $F'_b$ are even with respect to $\mathrm{Ad}_{\Gamma_1},\mathrm{Ad}_{\Gamma_2}$, respectively. Because $E'_a$ and $F'_b$ are even, the argument in [18, Lemma 11.2.14] shows that the central support of $E'_a\otimes F'_b\in\mathcal{N}\subset\mathcal{M}'$ with respect to $\mathcal{N}$ and the central support of $E'_a\otimes F'_b\in\mathcal{N}\subset\mathcal{M}'$ with respect to $\mathcal{M}'$ coincide. We denote the common central support by $P_{a,b}$. By the first part of the proof, we know that $\left(E'_a\otimes F'_b\right)\mathcal{N}\left(E'_a\otimes F'_b\right)=\left(E'_a\otimes F'_b\right)\mathcal{M}'\left(E'_a\otimes F'_b\right)$. We also have $\sum_{a,b}E'_a\otimes F'_b=\mathbb{I}_{\mathcal{H}_1\otimes\mathcal{H}_2}$. Therefore, applying [18, Lemma 11.2.15], we get $\mathcal{N}=\mathcal{M}'$. □

**Lemma A.5.** *Let $(\mathcal{M}_i,\mathrm{Ad}_{\Gamma_i})$ , $(\mathcal{N}_i,\mathrm{Ad}_{W_i})$, $i=1,2$, be spatially graded von Neumann algebras on $\mathcal{H}_i$ and $\mathcal{K}_i$, respectively, with grading operators $\Gamma_i$ and $W_i$. Let $\alpha_i:\mathcal{M}_i\to\mathcal{N}_i$, $i=1,2$ be graded $*$-isomorphisms. Suppose that $\mathcal{M}_2$ (hence $\mathcal{N}_2$ as well) is either balanced or trivially graded. Let $\mathcal{M}_1\hat{\otimes}\mathcal{M}_2$ be the graded tensor product of $(\mathcal{M}_1,\mathcal{H}_1,\Gamma_1)$ and $(\mathcal{M}_2,\mathcal{H}_2,\Gamma_2)$. Let $\mathcal{N}_1\hat{\otimes}\mathcal{N}_2$ be the graded tensor product of $(\mathcal{N}_1,\mathcal{K}_1,W_1)$ and $(\mathcal{N}_2,\mathcal{K}_2,W_2)$. Then there exists a unique $*$-isomorphism $\alpha_1\hat{\otimes}\alpha_2:\mathcal{M}_1\hat{\otimes}\mathcal{M}_2\to\mathcal{N}_1\hat{\otimes}\mathcal{N}_2$ such that*

$$\left(\alpha_1\hat{\otimes}\alpha_2\right)\left(a\hat{\otimes}b\right)=\alpha_1(a)\hat{\otimes}\alpha_2(b), \tag{A.7}$$

*for all $a\in\mathcal{M}_1$ and homogeneous $b\in\mathcal{M}_2$.*

*Proof.* Because $\alpha_2^{(0)}:=\alpha_2|_{\mathcal{M}_2^{(0)}}$ is a normal $*$-isomorphism from $\mathcal{M}_2^{(0)}$ onto $\mathcal{N}_2^{(0)}$, by [38, Chapter IV, Corollary 5.3] there is a unique $*$-isomorphism $\alpha^{(0)}$ from $\mathcal{M}_1\otimes\mathcal{M}_2^{(0)}$ onto $\mathcal{N}_1\otimes\mathcal{N}_2^{(0)}$ such that

$$\alpha^{(0)}(a\otimes b)=\alpha_1(a)\otimes\alpha_2(b),\quad a\in\mathcal{M}_1,\quad b\in\mathcal{M}_2^{(0)}. \tag{A.8}$$

If $\mathcal{M}_2$ is trivially graded, then we set $\alpha_1\hat{\otimes}\alpha_2:=\alpha^{(0)}$. If $\mathcal{M}_2$ is balanced, let $U$ be a self-adjoint unitary element in $\mathcal{M}_2^{(1)}$. Because we have $\mathcal{M}=\left(\mathcal{M}_1\otimes\mathcal{M}_2^{(0)}\right)\oplus\left(\mathcal{M}_1\otimes\mathcal{M}_2^{(0)}\right)(\Gamma_1\otimes U)$, we may define a linear map $\alpha_1\hat{\otimes}\alpha_2:\mathcal{M}\to\mathcal{N}$ by

$$(\alpha_1\hat{\otimes}\alpha_2)(x+y(\Gamma_1\otimes U))=\alpha^{(0)}(x)+\alpha^{(0)}(y)\left(W_1\otimes\alpha_2(U)\right),\quad x,y\in\mathcal{M}_1\otimes\mathcal{M}_2^{(0)}. \tag{A.9}$$

It is straightforward to check that $\alpha_1\hat{\otimes}\alpha_2$ is a normal $*$-homomorphism. Similarly, we may define a normal $*$-homomorphism $(\alpha_1)^{-1}\hat{\otimes}(\alpha_2)^{-1}:\mathcal{N}\to\mathcal{M}$, which turns out to be the inverse of $\alpha_1\hat{\otimes}\alpha_2$. Hence, $\alpha_1\hat{\otimes}\alpha_2$ is a $*$-isomorphism satisfying (A.7). The uniqueness is trivial from (A.7). □

**Lemma A.6.** *Let $(\mathcal{M}_i,\mathrm{Ad}_{\Gamma_i})$, $i=1,2$, be balanced and spatially graded von Neumann algebras on $\mathcal{H}_i$ with a grading operator $\Gamma_i$. Let $\mathcal{M}_1\hat{\otimes}\mathcal{M}_2$ be the graded tensor product of $(\mathcal{M}_1,\mathcal{H}_1,\Gamma_1)$ and $(\mathcal{M}_2,\mathcal{H}_2,\Gamma_2)$. For any graded $*$-automorphism $\beta_i$ on $\mathcal{M}_i$ implemented by a unitary $V_i$ on $\mathcal{H}_i$ satisfying $V_i\Gamma_i=(-1)^{\nu_i}\Gamma_iV_i$, $\nu_i\in\{0,1\}$ for each $i=1,2$, the automorphism $\beta_1\hat{\otimes}\beta_2$ on $\mathcal{M}_1\hat{\otimes}\mathcal{M}_2$ defined in*

*Lemma A.5 satisfies*

$$\left(\beta_1 \hat{\otimes} \beta_2\right)(a \hat{\otimes} b) = \mathrm{Ad}_{\left(V_1 \otimes V_2 \Gamma_2^{\nu_1}\right)}\left(a \hat{\otimes} b\right), \tag{A.10}$$

*for all* $a \in \mathcal{M}_1$ *and homogeneous* $b \in \mathcal{M}_2$.

*Proof.* We compute that

$$\begin{aligned}\left(\beta_1 \hat{\otimes} \beta_2\right)(a \hat{\otimes} b) &= \beta_1(a)\Gamma_1^{\partial b} \otimes \beta_2(b) = \mathrm{Ad}_{(V_1 \otimes V_2)}\left(a\Gamma_1^{\partial b}(-1)^{\partial b \cdot \nu_1} \otimes b\right)\\ &= \mathrm{Ad}_{(V_1 \otimes V_2)}\,\mathrm{Ad}_{\left(\mathbb{I} \otimes \Gamma_2^{\nu_1}\right)}\left(a\Gamma_1^{\partial b} \otimes b\right),\end{aligned}$$

from which (A.10) follows. □

We also consider anti-linear $*$-automorphisms.

**Lemma A.7.** *Let* $(\mathcal{M}_i, \mathrm{Ad}_{\Gamma_i})$, $i = 1, 2$, *be balanced and spatially graded von Neumann algebras on* $\mathcal{H}_i$ *with a grading operator* $\Gamma_i$. *Let* $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ *be the graded tensor product of* $(\mathcal{M}_1, \mathcal{H}_1, \Gamma_1)$ *and* $(\mathcal{M}_2, \mathcal{H}_2, \Gamma_2)$ *Suppose that* $\mathcal{M}_i$ *has a faithful normal representation* $(\mathcal{K}_i, \pi_i)$ *with a self-adjoint unitary* $W_i$ *on* $\mathcal{K}_i$ *satisfying* $\mathrm{Ad}_{W_i} \circ \pi_i(x) = \pi_i \circ \mathrm{Ad}_{\Gamma_i}(x)$, $x \in \mathcal{M}_i$ *and a complex conjugation* $\mathcal{C}_i$ *on* $\mathcal{K}_i$ *satisfying* $\mathrm{Ad}_{\mathcal{C}_i}(\pi_i(\mathcal{M}_i)) = \pi_i(\mathcal{M}_i)$ *and* $\mathcal{C}_i W_i = W_i \mathcal{C}_i$, *for* $i = 1, 2$. *Then for any graded anti-linear* $*$-*automorphism* $\beta_i$ *on* $\mathcal{M}_i$, $i = 1, 2$, *there exists a unique anti-linear* $*$-*automorphism* $\beta_1 \hat{\otimes} \beta_2$ *on* $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ *such that*

$$\left(\beta_1 \hat{\otimes} \beta_2\right)\left(a \hat{\otimes} b\right) = \beta_1(a) \hat{\otimes} \beta_2(b), \tag{A.11}$$

*for all* $a \in \mathcal{M}_1$ *and homogeneous* $b \in \mathcal{M}_2$.

*If* $\beta_i$ *is implemented by an anti-unitary* $V_i$ *on* $\mathcal{H}_i$ *satisfying* $V_i\Gamma_i = (-1)^{\nu_i}\Gamma_i V_i$, $\nu_i \in \{0, 1\}$ *for each* $i = 1, 2$, *then*

$$\left(\beta_1 \hat{\otimes} \beta_2\right)(a \hat{\otimes} b) = \mathrm{Ad}_{\left(V_1 \otimes V_2 \Gamma_2^{\nu_1}\right)}\left(a \hat{\otimes} b\right). \tag{A.12}$$

*Proof.* Let $\pi_1(\mathcal{M}_1) \hat{\otimes} \pi_2(\mathcal{M}_2)$ be the graded tensor product of the $(\pi_1(\mathcal{M}_1), \mathcal{K}_1, W_1)$ and $(\pi_2(\mathcal{M}_2), \mathcal{K}_2, W_2)$. By Lemma A.5, there is a $*$-isomorphism $\pi := \pi_1 \hat{\otimes} \pi_2$ from $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ onto $\pi_1(\mathcal{M}_1) \hat{\otimes} \pi_2(\mathcal{M}_2)$ satisfying $\left(\pi_1 \hat{\otimes} \pi_2\right)(a \hat{\otimes} b) = \pi_1(a) \hat{\otimes} \pi_2(b)$ for $a \in \mathcal{M}_1$ and homogeneous $b \in \mathcal{M}_2$. Because $\beta_i$, $\mathrm{Ad}_{\mathcal{C}_i}$ and $\pi_i$ preserve the grading, $\alpha_i := \mathrm{Ad}_{\mathcal{C}_i} \circ \pi_i \circ \beta_i \circ \pi_i^{-1}$ is a graded (linear) $*$-automorphism on $\pi_i(\mathcal{M}_i)$. By Lemma A.5, there is a $*$-automorphism $\alpha := \alpha_1 \hat{\otimes} \alpha_2$ on $\pi_1(\mathcal{M}_1) \hat{\otimes} \pi_2(\mathcal{M}_2)$ such that $\left(\alpha_1 \hat{\otimes} \alpha_2\right)(a \hat{\otimes} b) = \alpha_1(a) \hat{\otimes} \alpha_2(b)$ for $a \in \pi_1(\mathcal{M}_1)$ and homogeneous $b \in \pi_2(\mathcal{M}_2)$. Furthermore, for $\mathcal{C} := \mathcal{C}_1 \otimes \mathcal{C}_2$, $\mathrm{Ad}_{\mathcal{C}}$ preserves $\pi_1(\mathcal{M}_1) \hat{\otimes} \pi_2(\mathcal{M}_2)$. Therefore, $\beta_1 \hat{\otimes} \beta_1 := \pi^{-1} \circ \mathrm{Ad}_{\mathcal{C}} \circ \alpha \circ \pi$ defines an anti-linear $*$-automorphism on $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ and it satisfies (A.11).

The proof for the second half of the lemma is the same as in Lemma A.6. □

**Lemma A.8.** *Let* $G$ *be a finite group and* $\mathfrak{p} : G \to \mathbb{Z}_2$ *be a group homomorphism. Let* $(\mathcal{M}_1, \mathrm{Ad}_{\Gamma_1}, \alpha_1)$, $(\mathcal{M}_2, \mathrm{Ad}_{\Gamma_2}, \alpha_2)$ *be graded* $W^*$-$(G, \mathfrak{p})$-*dynamical systems such that, for* $i = 1, 2$, $\mathcal{M}_i$ *is a balanced, central, spatially graded and type I von Neumann algebra with grading operator* $\Gamma_i$. *Let* $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ *be the graded tensor product of* $(\mathcal{M}_1, \mathcal{H}_1, \Gamma_1)$ *and* $(\mathcal{M}_2, \mathcal{H}_2, \Gamma_2)$. *Then for every* $g \in G$, *there exists a linear* $*$-*automorphism* ($\mathfrak{p}(g) = 0$) *or anti-linear automorphism* ($\mathfrak{p}(g) = 1$), $(\alpha_1 \hat{\otimes} \alpha_2)_g$ *on* $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ *such that*

$$\left(\alpha_1 \hat{\otimes} \alpha_2\right)_g (a \hat{\otimes} b) = \alpha_{1,g}(a) \hat{\otimes} \alpha_{2,g}(b), \tag{A.13}$$

*for all homogeneous* $a \in \mathcal{M}_1$ *and* $b \in \mathcal{M}_2$.

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

*Proof.* By Lemma 2.9, there are graded $*$-isomorphisms $\iota_i : \mathcal{M}_i \to \mathcal{R}_{\kappa_i,\mathcal{K}_i}$ with some $(\mathcal{R}_{\kappa_i,\mathcal{K}_i}, \mathrm{Ad}_{\Gamma_{\mathcal{K}_i}}, \mathrm{Ad}_{V_{i,g}}) \in \mathcal{S}_{\kappa_i}$ for each $i = 1, 2$. Hence, $(\mathcal{K}_i \otimes \mathbb{C}^2, \iota_i)$ is a faithful normal representation with a self-adjoint unitary $\Gamma_{\mathcal{K}_i}$ implementing $\mathrm{Ad}_{\Gamma_i}$ on $\mathcal{K}_i \otimes \mathbb{C}^2$. Let $C$ be a complex conjugation with respect to the standard basis of $\mathbb{C}^2$ and $\mathcal{C}_i$ be any complex conjugation on $\mathcal{K}_i$. Then $\mathcal{C}_i \otimes C$ is a complex conjugation on $\mathcal{K}_i \otimes \mathbb{C}^2$ commuting with $\Gamma_{\mathcal{K}_i} = \mathbb{I}_{\mathcal{K}_i} \otimes \sigma_z$, preserving $\mathcal{R}_{\kappa_i,\mathcal{K}_i} = \iota_i(\mathcal{M}_i)$. Hence, we may apply Lemma A.5 and Lemma A.7, which gives the result. □

## Appendix B. Lieb-Robinson bound for lattice fermion systems

In this section, prove the Lieb-Robinson bound for one-dimensional lattice fermion systems. Though this result is not new (see [10, 27]), our method of using an odd self-adjoint unitary to derive the Lieb-Robinson bound for odd elements from even elements is new.

The result holds for more general metric graphs, but to avoid the introduction of further notation, we restrict ourselves to the one-dimensional case. Let us recall the basic setting for the Lieb-Robinson bound; see [3, 27, 28] for details.

**Definition B.1.** An $F$-function $F$ on $\mathbb{Z}$ is a non-increasing function $F : [0, \infty) \to (0, \infty)$ such that

(i) $\|F\| := \sup_{x\in\mathbb{Z}} \left(\sum_{y\in\mathbb{Z}} F(d(x, y))\right) < \infty$ and
(ii) $C_F := \sup_{x,y\in\mathbb{Z}} \left(\sum_{z\in\mathbb{Z}} \frac{F(d(x,z))F(d(z,y))}{F(d(x,y))}\right) < \infty$.

**Definition B.2.** Let $F$ be an $F$-function on $\mathbb{Z}$ and $I$ an interval in $\mathbb{R}$. We denote by $\mathcal{B}^e_F(I)$ the set of all norm-continuous paths of even interactions on $\mathcal{A}$ defined on an interval $I$ such that the function $\|\Phi\|_F : I \to \mathbb{R}$ defined by

$$\|\Phi\|_F(t) := \sup_{x,y\in\mathbb{Z}} \frac{1}{F(d(x, y))} \sum_{Z\in\mathfrak{S}_\mathbb{Z}, Z\ni x,y} \|\Phi(Z; t)\|, \quad t \in I, \tag{B.1}$$

is uniformly bounded; that is, $\sup_{t\in I} \|\Phi\|_F(t) < \infty$.

For the rest of this Appendix, we fix some $\Phi \in \mathcal{B}^e_F(I)$. For each $s \in I$, we define a local Hamiltonian by (1.6). We denote by $U_{\Lambda,\Phi}(t; s)$ the solution of

$$\frac{d}{dt} U_{\Lambda,\Phi}(t; s) = -iH_{\Lambda,\Phi}(t)U_{\Lambda,\Phi}(t; s), \quad t, s \in I, \quad U_{\Lambda,\Phi}(s; s) = \mathbb{I}. \tag{B.2}$$

We define the corresponding automorphisms $\tau^{(\Lambda),\Phi}_{t,s}$ on $\mathcal{A}_\mathbb{Z}$ by

$$\tau^{(\Lambda),\Phi}_{t,s}(A) := U_{\Lambda,\Phi}(t; s)^* A U_{\Lambda,\Phi}(t; s) \tag{B.3}$$

with $A \in \mathcal{A}_\mathbb{Z}$. Note that $\tau^{(\Lambda),\Phi}_{s,t}$ is the inverse of $\tau^{(\Lambda),\Phi}_{t,s}$. Because $\Phi(s)$ is even, the proof of [28, Theorem 3.1] gives the following.

**Lemma B.3.** *Let $X, Y \in \mathfrak{S}_\mathbb{Z}$ with $X \cap Y = \emptyset$. If either $A \in \mathcal{A}_X$ or $B \in \mathcal{A}_Y$ is even, then*

$$\left\|\left[\tau^{(\Lambda),\Phi}_{t,s}(A), B\right]\right\| \le \frac{2\,\|A\|\,\|B\|}{C_F} \left(e^{v|t-s|} - 1\right) D_0(X, Y), \tag{B.4}$$

*where $v > 0$ is some constant and*

$$D_0(X, Y) := \sum_{x\in X} \sum_{y\in Y} F(|x - y|). \tag{B.5}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

Using this lemma and because $\Phi$ is even, the proof of [28, Theorem 3.4] guarantees the existence of the limit

$$\tau_{t,s}^{\Phi}(A) := \lim_{\Lambda \nearrow \mathbb{Z}} \tau_{t,s}^{(\Lambda),\Phi}(A), \quad A \in \mathcal{A}, \quad t, s \in [0,1]. \tag{B.6}$$

Clearly, the limit dynamics $\tau_{t,s}^{\Phi}$ satisfy the same Lieb-Robinson bound as in Lemma B.3. We would like to have an analogous bound as Lemma B.3 for odd $A, B$. To do this, fix an odd self-adjoint unitary $U_0 \in \mathcal{A}_{\{0\}}$. For each $m \in \mathbb{Z}$, $\beta_{S_m}(U_0)$ is a self-adjoint unitary in $\mathcal{A}_{\{m\}}$. Define an interaction $\tilde{\Phi}_m(s)$ by

$$\tilde{\Phi}_m(Z;s) := \mathrm{Ad}_{\beta_{S_m}(U_0)}\left(\Phi(Z;s)\right), \quad Z \in \mathfrak{S}_{\mathbb{Z}}, \quad s \in I, \quad m \in \mathbb{N}. \tag{B.7}$$

Note that $\tilde{\Phi}_m(Z;s) = \Phi(Z;s)$ if $Z$ does not include $m$. Because $\tilde{\Phi}_m$ and $\Phi$ are even, Lemma B.3 and the proof of [28, Theorem 3.4] imply the bound

$$\begin{aligned}\left\|\tau_{t,s}^{\Phi}(A) - \tau_{t,s}^{\tilde{\Phi}_m}(A)\right\| &\le \frac{4\|A\|}{C_F} \sum_{Z \ni m} \int_{[s,t]} dr\, \|\Phi(Z;r)\|\, D_0(X,Z)\left(e^{v|t-r|} - 1\right) \\ &\le 4\|A\| \int_{[s,t]} \left(e^{v|t-r|} - 1\right) \|\Phi\|_F(r) \sum_{x \in X} F(|x-m|) =: g(m),\end{aligned} \tag{B.8}$$

for any $A \in \mathcal{A}_X^{(1)}$, where the last inequality uses (i) and (ii) of Definition B.1 as well as Equation (B.1). Note that $\lim_{m\to\infty} g(m) = 0$. Therefore, we have

$$\left\|\left\{\tau_{t,s}^{\Phi}(A), \beta_{S_m}(U_0)\right\}\right\| = \left\|\tau_{t,s}^{\Phi}(A) - \tau_{t,s}^{\tilde{\Phi}_m}(A)\right\| \le g(m), \tag{B.9}$$

for any $A \in \mathcal{A}_X^{(1)}$ and $X \in \mathfrak{S}_{\mathbb{Z}}$ with $m \notin X$. Let $X, Y \in \mathfrak{S}_{\mathbb{Z}}$ with $X \cap Y = \emptyset$, $A \in \mathcal{A}_X^{(1)}$, $B \in \mathcal{A}_Y^{(1)}$ and $m \notin X$. Because $B\beta_{S_m}(U_0) \in \mathcal{A}_{Y\cup\{m\}}^{(0)}$, Lemma B.3 and (B.9) imply

$$\begin{aligned}\left\|\left\{\tau_{t,s}^{\Phi}(A), B\right\}\right\| &= \left\|\left[\tau_{t,s}^{\Phi}(A), B\beta_{S_m}(U_0)\right]\beta_{S_m}(U_0) + B\beta_{S_m}(U_0)\left\{\tau_{t,s}^{\Phi}(A), \beta_{S_m}(U_0)\right\}\right\| \\ &\le \frac{2\|A\|\,\|B\|}{C_F}\left(e^{v|t-s|} - 1\right) D_0(X, Y \cup \{m\}) + g(m)\|B\|.\end{aligned} \tag{B.10}$$

Taking the limit $m \to \infty$ and using Lemma B.3, we obtain the following.

**Lemma B.4.** *Let $X, Y \in \mathfrak{S}_{\mathbb{Z}}$ with $X \cap Y = \emptyset$. For homogeneous $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, we have*

$$\left\|\tau_{t,s}^{\Phi}(A)B - (-1)^{\partial A \partial B} B\tau_{t,s}^{\Phi}(A)\right\| \le \frac{2\|A\|\,\|B\|}{C_F}\left(e^{v|t-s|} - 1\right) D_0(X,Y). \tag{B.11}$$

As in quantum spin systems, we can estimate the locality of the time-evolved observables from Lieb-Robinson bounds. To do this, let $\{\mathbb{E}_N : \mathcal{A} \to \mathcal{A}_{\Lambda_N} \mid N \in \mathbb{N}\}$ be the family of conditional expectations with respect to the trace on $\mathcal{A}$; see [2]. By the same argument as [28, Corollary 4.4], if $A \in \mathcal{A}^{(0)}$ is such that

$$\left\|[A,B]\right\| \le C\|B\|, \tag{B.12}$$

for all $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap [-N,N] = \emptyset}} \mathcal{A}_X$, then $\|A - \mathbb{E}_N(A)\| \le C$. We extend this bound to odd elements.

Suppose that $A \in \mathcal{A}^{(1)}$ is such that

$$\left\|AB - (-1)^{\partial B} BA\right\| \le C\|B\| \tag{B.13}$$

https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press

for all homogeneous $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap [-N,N] = \emptyset}} \mathcal{A}_X$. Let $U_0 \in \mathcal{A}^{(1)}_{\{0\}}$ be a self-adjoint unitary. Then we have $AU_0 \in \mathcal{A}^{(0)}$ and

$$\left\| [AU_0, B] \right\| = \left\| \left( AB - (-1)^{\partial B} BA \right) U_0 \right\| \le C \, \|B\| \tag{B.14}$$

for all homogeneous $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap [-N,N] = \emptyset}} \mathcal{A}_X$. Hence, we have that $\|[AU_0, B]\| \le 2C \, \|B\|$ for any $B \in \bigcup_{\substack{X \in \mathfrak{S}_{\mathbb{Z}^\nu} \\ X \cap [-N,N] = \emptyset}} \mathcal{A}_X$. Therefore, by the even case, we obtain that

$$\|A - \mathbb{E}_N(A)\| = \|(A - \mathbb{E}_N(A)) \, U_0\| = \|AU_0 - \mathbb{E}_N(AU_0)\| \le 2C, \tag{B.15}$$

where we used the fact that $U_0 \in \mathcal{A}_{\Lambda_N}$. From this and Lemma B.4, we have shown the following.

**Lemma B.5.** *For any $N \in \mathbb{N}$, $X \in \mathfrak{S}_{\mathbb{Z}}$ with $X \subset [-N, N]$ and $A \in \mathcal{A}_X$, we have*

$$\left\| \mathbb{E}_N \left( \tau^{\Phi}_{t,s}(A) \right) - \tau^{\Phi}_{t,s}(A) \right\| \le \frac{8 \, \|A\|}{C_F} \left( e^{\nu |t-s|} - 1 \right) D_0(X, [-N, N]^c). \tag{B.16}$$

Having Lemma B.4 and Lemma B.5 as input, we can carry out all of the arguments in [25, Theorem 1.3] and [29, Proposition 3.5].

**Acknowledgements.** YO thanks Y. Kubota and T. Matsui for discussion. CB is supported by a JSPS Grant-in-Aid for Early Career Scientists (No. 19K14548). YO is supported by JSPS Grants-in-Aid for Scientific Research (Nos. 16K05171 and 19K03534) and JST CREST Grant No. JPMJCR19T2.

**Conflict of Interest:** None.

## References

[1] I. Affleck, T. Kennedy, E. H. Lieb and H. Tasaki, 'Valence bond ground states in isotropic quantum antiferromagnets', *Commun. Math. Phys.* **115** (1988), 477–528.

[2] H. Araki and H. Moriya, 'Equilibrium statistical mechanics of fermion lattice systems', *Rev. Math. Phys.* **15**(2) (2003), 93–198.

[3] S. Bachmann, S. Michalakis, B. Nachtergaele and R. Sims, 'Automorphic equivalence of gapped phases of quantum lattice systems', *Commun. Math. Phys.* **309**(3) (2012), 835–871.

[4] C. Bourne and H. Schulz-Baldes, 'On $\mathbb{Z}_2$-indices for ground states of fermionic chains', *Rev. Math. Phys.* **32** (2020), 2050028.

[5] O. Bratelli and D. R. Robinson, *Operators Algebras and Quantum Statistical Mechanics 1*, 2nd ed. (Springer, Berlin, 1997).

[6] O. Bratelli and D. R. Robinson, *Operators Algebras and Quantum Statistical Mechanics 2*, 2nd ed. (Springer, Berlin, 1997).

[7] O. Bratteli, P. Jorgensen and G. Price, *Endomorphisms of $\mathcal{B}(\mathcal{H})$ Quantization, nonlinear partial differential equations, and operator algebra (Cambridge, MA, 1994)*, 93–138, Proc. Sympos. Pure Math., 59, *Amer. Math. Soc., Providence, RI*, 1996.

[8] O. Bratteli and P. E. T. Jorgensen, 'Endomorphisms of B (H)II. Finitely correlated states on $\mathrm{O_n}$', *J. Funct. Anal.* **145** (1997), 323–373.

[9] O. Bratteli, P. E. T. Jorgensen, A. Kishimoto and R. F. Werner, 'Pure states on $\mathcal{O}_d$', *J. Operator Theory* **43** (2000), 97–143.

[10] J.-B. Bru and W. de Siqueira Pedra, *Lieb-Robinson Bounds for Multi-Commutators and Applications to Response Theory*. Vol. **13** of Springer Briefs in Mathematical Physics (Springer, Berlin, 2017).

[11] N. Bultinck, D. J. Williamson, J. Haegeman and F. Verstraete, 'Fermionic matrix product states and one-dimensional topological phases', *Phys. Rev. B* **95** (2017), 075108.

[12] X. Chen, Z.-C. Gu and X.-G. Wen, 'Classification of gapped symmetric phases in one-dimensional spin systems', *Phys. Rev. B* **83** (2011), 035107.

[13] M. Fannes, B. Nachtergaele and R. F. Werner, 'Finitely correlated states on quantum spin chains', *Commun. Math. Phys.* **144** (1992), 443–490.

[14] L. Fidkowski and A. Kitaev, 'The effects of interactions on the topological classification of free fermion systems', *Phys. Rev. B* **81** (2009), 134509.

[15] L. Fidkowski and A. Kitaev, 'Topological phases of fermions in one dimension', *Phys. Rev. B* **83** (2011), 075103.

[16] Z.-C. Gu and X.-G. Wen, 'Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order', *Phys. Rev. B* **80** (2009), 155131.

[17] M. Hastings, 'An area law for one-dimensional quantum systems', *J. Stat. Mech. Theory Exp.* **2007**, no. 8, P08024, 14 pp.

[18] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras II*. Vol. **16** of *Graduate Studies in Mathematics* (American Mathematical Society, Providence, RI, 1997).

[19] A. Kapustin and R. Thorngren, 'Fermionic SPT phases in higher dimensions and bosonization', *J. High Energy Phys.* **2017**, no. 10, 080, front matter+48 pp.

[20] A. Kapustin, A. Turzillo and M. You, 'Spin topological field theory and fermionic matrix product states', *Phys. Rev. B* **98** (2018), 125101.

[21] T. Matsui, 'A characterization of pure finitely correlated states', *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1**(4) (1998), 647–661.

[22] T. Matsui, 'The split property and the symmetry breaking of the quantum spin chain', *Commun. Math. Phys.* **218** (2001), 393–416.

[23] T. Matsui, 'Boundedness of entanglement entropy and split property of quantum spin chains', *Rev. Math. Phys.* **26**(9) (2013), 1350017.

[24] T. Matsui, 'Split property and fermionic string order', 2020, arXiv:2003.13778.

[25] A. Moon and Y. Ogata, 'Automorphic equivalence within gapped phases in the bulk', *J. Funct. Anal.* **278**(8) (2020), 108422.

[26] M. E. Moutuou, 'Graded Brauer groups of a groupoid with involution', *J. Funct. Anal.* **266**(5) (2014), 2689–2739.

[27] B. Nachtergaele, R. Sims and A. Young, 'Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems', in *Mathematical Problems in Quantum Physics*, Vol. **717** of *Contemp. Math.* (American Mathematical Society, Providence, RI, 2018).

[28] B. Nachtergaele, R. Sims and A. Young, 'Quasi-locality bounds for quantum lattice systems. Part 1. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms', *J. Math. Phys.* **60** (2019), 061101.

[29] Y. Ogata, 'A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization II', *Commun. Math. Phys.* **348** (2016), 897–957.

[30] Y. Ogata, 'A $\mathbb{Z}_2$-index of symmetry protected topological phases with reflection symmetry for quantum spin chains', 2019, arXiv:1904.01669.

[31] Y. Ogata, 'A classification of pure states on quantum spin chains satisfying the split property with on-site finite group symmetries', 2019, arXiv:1908.08621.

[32] Y. Ogata, 'A $\mathbb{Z}_2$-index of symmetry protected topological phases with time reversal symmetry for quantum spin chains', *Commun. Math. Phys.* **374**( *2*) (2020), 705–734.

[33] Y. Ogata, Y. Tachikawa and H. Tasaki, 'General Lieb-Schultz-Mattis type theorems for quantum spin chains', 2020, arXiv:2004.06458.

[34] Y. Ogata and H. Tasaki, 'Lieb-Schultz-Mattis type theorems for quantum spin chains without continuous symmetry', *Commun. Math. Phys.* **372**(3)(2019), 951–962.

[35] D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete and J. I. Cirac, 'String order and symmetries in quantum spin lattices', *Phys. Rev. Lett.* **100** (2008), 167202.

[36] F. Pollmann, A. Turner, E. Berg and M. Oshikawa, 'Entanglement spectrum of a topological phase in one dimension', *Phys. Rev. B* **81** (2010), 064439.

[37] D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete and J. I. Cirac, 'String order and symmetries in quantum spin lattices', *Phys. Rev. Lett.* **100** (2008), 167202.

[38] M. Takesaki, *Theory of Operator Algebras*. I. *Encyclopaedia of Mathematical Sciences* (Springer-Verlag, Berlin, 2002).

[39] A. Turzillo and M. You, 'Fermionic matrix product states and one-dimensional short-range entangled phases with antiunitary symmetries', *Phys. Rev. B* **99** (2019), 035103.

[40] C. T. C. Wall, 'Graded Brauer groups', *J. Reine Angew. Math.* **213** (1963–1964), 187–199.


https://doi.org/10.1017/fms.2021.19 Published online by Cambridge University Press