Can. J. Math., Vol. XXVII, No. 1, 1975, pp. 11–24

PERFECT MAPS AND EPI-REFLECTIVE HULLS

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The main theorems concern the relation between the $\mathscr{A}$-compact spaces and the $\mathscr{A}$-regular spaces, and their analogues in uniform spaces. In either of the categories of Tychonoff spaces or uniform spaces, let $\mathscr{A}$ be a class of spaces, let $\mathscr{R}(\mathscr{A})$ be the epi-reflective hull of $\mathscr{A}$ (closed subspaces of products of members of $\mathscr{A}$), let $\mathscr{O}(\mathscr{A})$ be the "onto-reflective" hull of $\mathscr{A}$ (all subspaces of products of members of $\mathscr{A}$), and let r and o be the associated functors. Let $p_{\mathscr{K}}\mathscr{A}$ be the class of spaces which admit a perfect map into a member of $\mathscr{A}$. Then, $p_{\mathscr{K}}\mathscr{R}(\mathscr{A})$ is epi-reflective (and in Tych, $= \mathscr{R}(p_{\mathscr{K}}\mathscr{A})$; but in Unif, the equality fails); call the functor p.

The main theorems assert that (a) $\mathscr{O}(\mathscr{A}) \cap p_{\mathscr{K}}\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$, and (b) $op = r$. Theorem (b) applies, of course, to factor an arbitrary epi-reflector r, as the product of one (o) for which all reflection maps are onto and one (p) for which all reflection maps are homeomorphisms; this factorization includes as a special case one of Banaschewski's constructions of the maximal 0-dimensional compactification of a space with a basis of clopen sets.

Additionally, in order to handle perfect maps in Unif (on which there is no explicit literature of which I am aware), and to understand the role of compactness, the notion of perfect map is generalized. In terms of the generalization, Theorem (a) has a (stronger) analogue using complete uniform spaces; (b) does not.

Some generalizations of known theorems appear as by-products.

1. Epi-reflections. We take note of the following somewhat non-standard usages which shall prevail in the sequel. (a) Homeomorphisms are *not* understood to be onto. (b) If $\mathscr{R}$ is a subcategory of $\mathscr{C}$, then the notation "$f : X \to Y \in \mathscr{R}$" means that X is an object of $\mathscr{C}$, f is a morphism of $\mathscr{C}$, and the range-carrier (only) is an object of $\mathscr{R}$.

Let $\mathscr{R}$ be a full subcategory of the category $\mathscr{C}$. An $\mathscr{R}$-reflection of the object $X \in \mathscr{C}$ is a pair (rX, r_x), where $r_x : X \to rX \in \mathscr{R}$ (the reflection map) and to each $f : X \to R \in \mathscr{R}$ corresponds unique $f^r : rX \to R$ with $f = f^r \circ r_x$. It is easily seen that a reflection is essentially unique, and that if X and Y have reflections, and $f : X \to Y$, then there is unique $f^r : rX \to rY$ with $f^r \circ r_x = r_y \circ f$. If every object has a reflection, then $\mathscr{R}$ is said to be reflective; then r is a covariant functor. We shall suppose that each r_x is epic; then $\mathscr{R}$ is called epi-reflective.

Received March 6, 1973 and in revised form, August 21, 1974. Some of these results were announced in Notices Amer. Math. Soc. *19* (1972), p. A-612 (Abstract no. 72T-G143).

When the category is not stipulated in the sequel, it is understood to be either Tych: Tychonoff spaces (i.e., completely regular and Hausdorff) with continuous maps; or Unif: separated uniform spaces with uniformly continuous maps. Maps are understood to be morphisms of the category. In these categories, $f : X \to Y$ is epic if and only if $f(X)$ is dense in Y. We suppose that every class of spaces is neither $\emptyset$ nor $\{\emptyset\}$, and that every subcategory is full and replete.

Let $\mathscr{A}$ be a class of spaces. The space X is called $\mathscr{A}$-weak (or $\mathscr{A}$-regular) if there is a set $\mathscr{F}$ of maps to members of $\mathscr{A}$ such that X's structure is the weak one generated by $\mathscr{F}$. By means of the usual technique: X is $\mathscr{A}$-weak if and only if X can be embedded as a subspace of a product of members of $\mathscr{A}$.

The following is the basic result on epi-reflections.

1.1 THEOREM. *The following conditions on the subcategory $\mathscr{R}$ are equivalent.*

(a) *$\mathscr{R}$ is epi-reflective.*

(b) *If X is $\mathscr{R}$-weak, then X has $\mathscr{R}$-reflection (rX, r_x), with r_x an embedding.*

(c) *$\mathscr{R}$ is productive and closed-hereditary.*

The essentials of this are due to Isbell [**11**] and Kennison [**13**]. Herrlich and van der Slot [**10**] added condition (b) and formulated the result for the category of Hausdorff spaces exactly as above; one sees readily that their proof is valid in Tych or Unif.

Let $\mathscr{K}$ be the subcategory (of either Tych or Unif) of compact spaces, and Γ the subcategory of Unif of complete spaces. These are most familiar as epi-reflective. In Tych, the $\mathscr{K}$-reflection is βX, the Stone-Čech compactification, and in Unif it is the Samuel compactification; say sX. The complete reflection γX is, of course, the completion of X.

It sometimes happens that all reflection maps have a certain property. In the above examples: each reflection map $X \to \beta X$ is a homeomorphism (into), hence an embedding; each reflection map $X \to sX$ is a homeomorphism (but not an embedding); each reflection map $X \to \gamma X$ is a uniform isomorphism (into), hence an embedding. Other examples are the $\mathscr{A}$-weak spaces for any $\mathscr{A}$; these satisfy 1.1(c), clearly. With $\mathscr{A} = \{\{0, 1\}\}$, we get in Tych, the spaces with basis of clopen sets; with $\mathscr{A} = \mathscr{K}$, we get in Tych all spaces, and in Unif, precompact spaces. As we shall see below, for all these the reflection maps are onto.

The following classifies epi-reflections by properties of the reflection maps. The proofs are not difficult; some are easy from 1.1, and the rest will be given in application of the results of §4.

1.2 THEOREM. *Let $\mathscr{R}$ be epi-reflective.*

(a) [**10**] *r_x is embedding if and only if X is $\mathscr{R}$-weak. Hence, all reflection maps are embeddings if and only if all spaces are $\mathscr{R}$-weak.*

(b) [**10**; **15**] *All reflection maps are homeomorphisms if and only if $\mathscr{R} \supset \mathscr{K}$.*

(c) [**15**] *In Unif, all reflection maps are embeddings if and only if $\mathscr{R} \supset \Gamma$.*

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(d) [**13**; **10**] *These are equivalent.* (i) $\mathscr{R}$ *is hereditary.* (ii) $\mathscr{R}$ *contains all* $\mathscr{R}$*-weak spaces.* (iii) *All reflection maps are onto.*

Further examples in Tych: Of 1.2(b), $\mathscr{R}$ = realcompact spaces, $\mathscr{R}$ = topologically complete spaces. In Unif: Of 1.2(c), $\mathscr{R}$ = all spaces whose underlying topology is topologically complete; here, the rationals with the usual uniformity is in $\mathscr{R} - \Gamma$. Of 1.2(d), $\mathscr{R} = \mathscr{A}$-weak, where $\mathscr{A}$ = the real line with the usual uniformity, or all separable metric uniform spaces; the functors here are called c and e, and have received considerable attention [**12**].

It is evident that a class $\mathscr{A}$ determines at least two epi-reflective subcategories as follows: $\mathscr{R}(\mathscr{A})$ = all closed subspaces of products of members of $\mathscr{A}$; this is the least epi-reflective subcategory containing $\mathscr{A}$, and is called the epi-reflective hull of $\mathscr{A}$. $\mathscr{O}(\mathscr{A})$ = the $\mathscr{A}$-weak spaces; this is the least epi-reflective subcategory containing $\mathscr{A}$ for which all reflection maps are onto, and we call it the onto-reflective hull. (There are also the least epi-reflective subcategory containing $\mathscr{A}$ for which all reflection maps are homeomorphisms, or embeddings. With 1.2, these are $\mathscr{R}(\mathscr{A} \cup \mathscr{K})$, and $\mathscr{R}(\mathscr{A} \cup \mathscr{K})$ or $\mathscr{R}(\mathscr{A} \cup \Gamma)$. See 4.5.)

Two of our main theorems relate $\mathscr{O}(\mathscr{A})$ and $\mathscr{R}(\mathscr{A})$ (§5). The following will be useful.

1.3 PROPOSITION. *Given $\mathscr{A}$, let r and r' denote the functors for $\mathscr{R}(\mathscr{A})$ and $\mathscr{O}(\mathscr{A})$. For any X, $r_x : X \to r_x(X)(\subset rX)$ is the $\mathscr{O}(\mathscr{A})$-reflection; so r'_x is the "range-restriction" of r_x, and $r'X = r_x(X)$.*

Proof. Let $f : X \to Y \in \mathscr{O}(\mathscr{A})$. There is unique $g : rX \to rY$ with $g \circ r_x = r_y \circ f$. Now Y is $\mathscr{R}(\mathscr{A})$-weak, so r_y is an embedding, by 1.1. Let $i : r_x(X) \to rX$ be inclusion. Define $h : r_x(X) \to Y$ by $i : h \equiv r_y^{-1} \circ g \circ i$. Clearly, $h \circ r_x = f$; uniqueness is obvious.

See [**7**] and [**8**] for further results on epi-reflections in Tych.

2. $\mathscr{S}$-perfect maps.

Let $\mathscr{S}$ be an epi-reflective subcategory.

2.1 *Definition.* The map $f : X \to Y$ is $\mathscr{S}$-perfect if $f^s(sX - s_x(X)) \subset sY - s_y(Y)$.

In Tych, with $\mathscr{S} = \mathscr{K}$ and $sX = \beta X$, the condition of 2.1 is known to be equivalent to: f is closed and point-inverses are compact. (This follows from [**6**, 1.5].) This is the usual definition of "perfect map". It also follows from [**6**, 1.5] that for uniform spaces X and Y, and a function $f : X \to Y$, f is $\mathscr{K}$-perfect in Unif if and only if f is $\mathscr{K}$-perfect in Tych, and uniformly continuous.

Additionally, there are characterizations of $\mathscr{S}$-perfectness in terms of filters (which we shall not use). For example,

(a) In Tych, $f : X \to Y$ is $\mathscr{K}$-perfect if and only if whenever $\mathscr{U}$ is an ultrafilter in X (or ultrafilter in the zero-sets of X), then $\mathscr{U}$ converges if $f(\mathscr{U})$ does;

(b) In Tych, if $\mathscr{S}$ is realcompact spaces, then $f : X \to Y$ is $\mathscr{S}$-perfect if and only if whenever $\mathscr{U}$ is a zero-set ultrafilter with the countable intersection property, in X, then $\mathscr{U}$ converges if $f(\mathscr{U})$ does;

(c) In Unif, $f : X \to Y$ is Γ-perfect if and only if whenever $\mathscr{U}$ is a Cauchy filter in X, then $\mathscr{U}$ converges if $f(\mathscr{U})$ does.

Definition 2.1 is quite uninteresting if, for example, all $\mathscr{S}$-reflection maps are onto, for then every map is $\mathscr{S}$-perfect. Throughout the paper, *we restrict attention to subcategories $\mathscr{S}$ for which each reflection map is a homeomorphism* (i.e., $\mathscr{S} \supset \mathscr{K}$, by 1.2, though we do not use this). In Tych, this results in "substantial remainders" $sX - s_x(X)$, though not necessarily in Unif (e.g., $\mathscr{S}$ = precompact spaces; here all reflection maps are onto). In succeeding sections we restrict further.

Several other authors independently have considered variations and generalizations of the idea of a perfect map: For example, [**2**] considers (in Hausdorff spaces) maps which preserve remainder in the Katětov H-closed extension; [**9**] gives a categorical generalization and shows that in Tych, when $\mathscr{S} \supset \mathscr{K}$, the definition agrees with 2.1; [**16**] uses 2.1 (in Hausdorff spaces) and gives some applications. The only apparent overlap with [**9**] and [**16**] is in a few of the propositions of this section (which for perfect maps in Tych are all known). (The referee was kind enough to call [**2**] and [**16**] to my attention.)

2.2 Proposition. *If $f : X \to Y$ and $g : Y \to Z$ are $\mathscr{S}$-perfect, so is $g \circ f$.*

Proof. Since $(g \circ f)^s = g^s \circ f^s$, this is trivial.

2.3 Lemma. *If $s_x(X) \subset A \subset sX$, then $sA = sX$* (or, more exactly, the inclusion $i : A \to sX$ is the $\mathscr{S}$-reflection of A).

Proof. Let $f : A \to S \in \mathscr{S}$. Then there is $g : sX \to S$ with $g \circ s_x = f|s_x(X)$. Using density, and the Hausdorff property, this readily implies that $g \circ i = f$, and that g is unique.

2.4 Proposition. *If $f : X \to Y$, then $f^s|f^{s-1}(Y) : f^{s-1}(Y) \to Y$ is $\mathscr{S}$-perfect.*

Proof. Using 2.3, $sf^{s-1}(Y) = sX$, and therefore $(f^s|f^{s-1}(Y))^s = f^s$. The result is now clear.

2.5 Lemma. *If $h : X \to S \in \mathscr{S}$ is a homeomorphism, then $h^s(sX - s_x(X)) \subset S - h(X)$.*

Proof. This is a special case of [**5**, 6.11].

2.6 Proposition. *A homeomorphism with closed range is $\mathscr{S}$-perfect.*

Proof. Let $h : X \to Y$ be a homeomorphism with $h(X)$ closed. By continuity, $h^s(sX) \subset \overline{h^s(s_x(X))}$ (the closure in sY). Now $h^s \circ s_x = s_y \circ h$ is a homeomorphism, and $h^s(s_x(X)) = s_y(h(X))$ is closed in sY. Using 2.5 and the last fact, we have $h^s(sX - s_x(X)) \subset \overline{h^s(s_x(X))} - h^s(s_x(X)) \subset sY - s_y(Y)$.

2.7 Proposition. *If $S \in \mathscr{S}$, then the projection $\pi : S \times X \to X$ is $\mathscr{S}$-perfect.*

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Proof. Let $\pi^* : S \times sX \to sX$ be projection. Clearly, $\pi^*(S \times sX - S \times s_x(X)) \subset sX - s_x(X)$. Let $g : S \times X \to S \times sX$ be $g(s, x) = (s, s_x(x))$. By 2.5, g^s "preserves remainder". Since $\pi^s = \pi^* \circ g^s$, the result follows.

We consider two kinds of maps into products. First, suppose given $f_\lambda : X_\lambda \to Y_\lambda$ for each $\lambda \in \Lambda$; the product $\Pi f_\lambda : \Pi X_\lambda \to \Pi Y_\lambda$ is defined by $(\Pi f_\lambda)((x_\lambda)) = (f_\lambda(x_\lambda))$. Next, suppose given a set $\mathscr{F}$ of maps of a fixed space X; say, $f : X \to X_f$, for $f \in \mathscr{F}$; the evaluation $e : X \to \Pi X_f$ is defined by $\pi_f \circ e = f$, for each $f \in \mathscr{F}$.

2.8 Lemma. *Let $f : X \to \Pi Y_\lambda$ be a map into a product, let $h_1 \equiv \Pi s_{y_\lambda} : \Pi Y_\lambda \to \Pi s Y_\lambda$ and let h be the "range-restriction" of h_1. These are equivalent:*

(a) *f is $\mathscr{S}$-perfect.*

(b) *$h \circ f$ is $\mathscr{S}$-perfect.*

(c) $(h_1 \circ f)^s[sX - s_x(X)] \subset \Pi s Y_\lambda - \Pi s_{y_\lambda}(Y_\lambda)$.

Proof. Consider the commuting diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & \Pi Y_\lambda \arrow[r, "h"] \arrow[d] & \Pi s_{y_\lambda}(Y_\lambda) \arrow[d] \arrow[dr, "i"] & \\
sX \arrow[r, "f^s"] & s\Pi Y_\lambda \arrow[r, "h^s"] & s\Pi s_{y_\lambda}(Y) \arrow[r, "i^s"] & \Pi s Y_\lambda
\end{tikzcd}$$

where i is the inclusion, so that $h_1 = i \circ h$, and $(h_1 \circ f)^s = i^s \circ h^s \circ f^s$. Because h, i, and the reflection maps are homeomorphisms, 2.5 applies to show that h^s and i^s (and hence $i^s \circ h^s$) "preserve remainders". Since (a) says that f^s preserves remainder, the equivalence of the three conditions follows.

When f is a product or an evaluation, the conditions in 2.8(c) become the following.

2.9 Proposition. *A product Πf_λ is $\mathscr{S}$-perfect if and only if each f_λ is $\mathscr{S}$-perfect.*

2.10 Proposition. *The evaluation*

$$e : X \to \prod_{f \in \mathscr{F}} X_f$$

is $\mathscr{S}$-perfect if and only if each $x \in sX - s_x(X)$ there is $f \in \mathscr{F}$ with $f^s(x) \in sX_f - s_{x_f}(X_f)$; that is, $s_x(X) = \bigcap_{f \in \mathscr{F}} f^{s-1}(s_{x_f}(X_f))$.

3. $\mathscr{S}$-perfect and epi-reflective hulls, 1. Again, $\mathscr{S}$ is an epi-reflective subcategory for which all reflection maps are homeomorphisms.

3.1 *Definition.* If $\mathscr{A}$ is a class of spaces, $p_s\mathscr{A}$ consists of all X for which there is an $\mathscr{S}$-perfect map $X \to A \in \mathscr{A}$. We call $p_s\mathscr{A}$ the $\mathscr{S}$-perfect hull of $\mathscr{A}$ (because of 3.2(a), below).

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In Tych, $p_k\mathscr{A}$ has been called the left-fitting hull of $\mathscr{A}$ [**8**]. For this case, 3.2 is known; see [**10**; **4**; **8**].

3.2 PROPOSITION.
(a) $p_s(p_s\mathscr{A}) = p_s\mathscr{A}$.
(b) $p_s\mathscr{A}$ *is closed-hereditary.*
(c) $\mathscr{S} \subset p_s\mathscr{A}$.
(d) $X \in p_s\mathscr{A}$ *and* $S \in \mathscr{S}$ *imply* $S \times X \in p_s\mathscr{A}$.

Proof. (a) by 2.2. (b) by 2.6 and 2.2. For (c), if $S \in \mathscr{S}$, then $sS - s_s(S) = \emptyset$, and any map of S is $\mathscr{S}$-perfect; so map $S \to \{p\} \subset A \in \mathscr{A}$. (d) follows from 2.8 and 2.2.

3.3 PROPOSITION. *If $\mathscr{A}$ is productive, then so is $p_s\mathscr{A}$; hence $p_s\mathscr{A}$ is epi-reflective. In general, $p_s\mathscr{R}(\mathscr{A})$ and $p_s\{\prod\mathscr{A}' : \mathscr{A}'$ is a set contained in $\mathscr{A}\}$ coincide.*

Proof. 2.8 shows the first, and the second follows from this, 3.2(b), and 1.1. Clearly, $p_s\mathscr{R}(\mathscr{A}) \supset p_s\{\prod\mathscr{A}' : \mathscr{A}' \subset \mathscr{A}\}$. The latter is epi-reflective and contains $\mathscr{A}$, hence contains $\mathscr{R}(\mathscr{A})$, and with 3.2(a), contains $p_s\mathscr{R}(\mathscr{A})$.

3.4 PROPOSITION. *For any $\mathscr{A}$, $\mathscr{R}(\mathscr{S} \cup \mathscr{A}) \subset \mathscr{R}(p_s\mathscr{A}) \subset p_s\mathscr{R}(\mathscr{A})$.*

Proof. Clearly, $\mathscr{A} \subset p_s\mathscr{A}$, and $\mathscr{S} \subset p_s\mathscr{A}$ by 3.2(c). Thus, $\mathscr{R}(\mathscr{S} \cup \mathscr{A}) \subset \mathscr{R}(p_s\mathscr{A})$.

Evidently, $p_s\mathscr{A} \subset p_s\mathscr{R}(\mathscr{A})$. The latter is epi-reflective by 3.3, so $\mathscr{R}(p_s\mathscr{A}) \subset p_s\mathscr{R}(\mathscr{A})$.

As we shall see in §4, the inclusions in 3.4 become equalities if all $\mathscr{S}$-reflection maps are embeddings. The principal example not of this sort is $\mathscr{S} = \mathscr{K}$ in Unif, and here the inclusions can be proper. We give examples after further examination of $p_s\mathscr{R}(\mathscr{A})$ and some related classes.

Consider as before an evaluation $e : X \to \prod_{f\in\mathscr{F}} X_f$, and related maps as follows: The homeomorphism (onto) $h \equiv \prod s_{x_f} : \prod X_f \to \prod s_{x_f}(X_f)$; the product $F \equiv \prod(f^s|f^{s-1}(s_{x_f}(X_f))$; the diagonal embedding $\Delta : s_x(X) \to \prod f^{s-1}(s_{x_f}(X_f))$.

Evidently, $h \circ e = F \circ \Delta \circ s'_x$, where s'_x denotes the "range-restriction" of s_x.

We need the following standard lemma about Hausdorff spaces.

3.6 LEMMA. *Let $\mathscr{D}$ be a family of subspaces of Y, each containing Z. The diagonal embedding of Z into $\prod\mathscr{D}$ has closed range if and only if $Z = \bigcap \mathscr{D}$.*

3.7 PROPOSITION. *These conditions on $\mathscr{F}$ are equivalent:*
(a) *e is $\mathscr{S}$-perfect.*
(b) $s_x(X) = \bigcap_{f\in\mathscr{F}} f^{s-1}(s_{x_f}(X_f))$.
(c) *$\Delta(s_x(X))$ is closed.*
(d) *Δ is $\mathscr{S}$-perfect.*

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Proof. (a) $\Rightarrow$ (b) by **2.10**. (b) $\Rightarrow$ (c) by 3.6. (c) $\Rightarrow$ (d) by 2.6. (d) $\Rightarrow$ (a): The following are $\mathscr{S}$-perfect: s'_x, obviously; F, by 2.9; with (d), $F \circ \Delta \circ s'_x$, by 2.2; hence $h \circ e$; finally, e, by 2.8.

3.8. COROLLARY. *$X \in p_s\mathscr{R}(\mathscr{A})$ if and only if there is a set $\mathscr{F}$ of maps $f : X \to A_f \in \mathscr{A}$ for which $s_x(X) = \bigcap_{f\in\mathscr{F}} f^{s-1}(s_{A_f}(A_f))$.*

Proof. This follows by 3.7 (a) and (b), and 3.3.

From 1.3, we know that the "range-restrictions" s'_x are the reflection maps for $\mathscr{O}(\mathscr{S})$; as there, we let s' denote the functor. $s'(\mathscr{A})$ stands for the class of spaces $s'A\,(A \in \mathscr{A})$, and $(s')^{-1}\mathscr{A}$ stands for the class of spaces X for which $s'X \in \mathscr{A}$. Consideration of 3.7(c) yields the following.

3.9 COROLLARY. *If $X \in p_s\mathscr{R}(\mathscr{A})$, then $s'X \in \mathscr{R}(p_s s'(\mathscr{A}))$.*

Proof. By 3.8 and 3.7(c), there is a set $\mathscr{F}$ of maps to A_f's $\in \mathscr{A}$ such that $\Delta : s'X \to \prod f^{s-1}(s'A_f)$ is an embedding with closed range. Each $f^{s-1}(s'A_f) \in p_s s'(\mathscr{A})$, by 2.4.

The following, with the examples below, summarizes the relationships among the classes under consideration.

3.10 PROPOSITION. $\mathscr{R}(\mathscr{S} \cup \mathscr{A}) \subset \mathscr{R}(p_s\mathscr{A}) \cup (s')^{-1}\mathscr{R}(p_s\mathscr{A}) \subset p_s\mathscr{R}(\mathscr{A}) \subset (s')^{-1}\mathscr{R}(p_s s'(\mathscr{A})) \subset p_s\mathscr{R}(p_s s'(\mathscr{A})) = p_s\mathscr{R}(s'(\mathscr{A}))$.

Proof. There are two general facts: (a) $(s')^{-1}\mathscr{B} \subset p_s\mathscr{B}$, and (b) $p_s\mathscr{R}(p_s\mathscr{B}) = p_s\mathscr{R}(\mathscr{B})$, for any $\mathscr{B}$. The proofs are trivial.

$\mathscr{R}(\mathscr{S} \cup \mathscr{A}) \subset \mathscr{R}(p_s\mathscr{A}) \subset p_s\mathscr{R}(\mathscr{A})$, by 3.4; $(s')^{-1}\mathscr{R}(p_s\mathscr{A}) \subset p_s\mathscr{R}(\mathscr{A})$ by (a) and (b). This shows the first two inclusions. The third is 3.9, the fourth follows from (a), and the equality from (b).

I do not know at present if $\mathscr{R}(p_s\mathscr{A}) \subset (s')^{-1}\mathscr{R}(p_s\mathscr{A})$, or if the fourth inclusion is equality (i.e., if $p_s\mathscr{R}(s'(\mathscr{A})) \subset (s')^{-1}\mathscr{R}(p_s s'(\mathscr{A}))$). We present examples to dispose of the other questions.

3.11 LEMMA. *In Unif, each $\mathscr{K}$-perfect map is Γ-perfect. Hence $p_k\Gamma = \Gamma$.*

Proof. If $k_x : X \to kX$ is Samuel compactification of X, then since $\mathscr{K} \subset \Gamma$, we have $k_x^\gamma : \gamma X \to kX$, and this is Samuel compactification of γX. The results follow.

3.12 *Examples.* In Unif, let $\mathscr{S} = \mathscr{K}$.

(a) $\mathscr{R}(\mathscr{K} \cup \mathscr{E}) \not\supset \mathscr{R}(p_k\mathscr{E}) \cup (k')^{-1}\mathscr{R}(p_k\mathscr{E})$, where $\mathscr{E}$ is the class of all spaces which have no uncountable uniformly discrete set: Here, $\mathscr{E}$ is epireflective and $\mathscr{E} \supset \mathscr{K}$. The reflection eX is the set X with the uniformity generated from all countable uniform covers of the uniform space X, and the map $X \to eX$ is 1_x (so $\mathscr{E} = \mathscr{O}(\mathscr{E})$, in fact). The $\mathscr{K}$-reflections (Samuel compactifications) of X and eX coincide, so each reflection map $X \to eX$ is $\mathscr{K}$-perfect. (See [**12**, p. 52].)

Thus, $\mathscr{R}(\mathscr{K} \cup \mathscr{E}) = \mathscr{E}$, while $\mathscr{R}(p_k\mathscr{E}) = (k')^{-1}\mathscr{R}(p_k\mathscr{E}) =$ Unif.

(b) $\mathscr{R}(p_k\mathscr{L}) \not\supset p_k\mathscr{R}(\mathscr{L})$, where $\mathscr{L}$ is the class of all spaces whose underlying topology has the Lindelöf property: Clearly, $\mathscr{K} \subset \mathscr{L} \subset \mathscr{E}$, and $\mathscr{R}(\mathscr{L}) \subset \mathscr{E}$ (in fact, they are equal). Now, uniformly $\mathscr{K}$-perfect maps are topologically $\mathscr{K}$-perfect, and in Tych, p_k (Lindelöf spaces) = Lindelöf spaces [**6**, 2.2]. So, $\mathscr{R}(p_k\mathscr{L}) = \mathscr{R}(\mathscr{L})$.

Let D be a set of power $\aleph_1$ (or any nonmeasurable power) with the discrete uniformity: $D \notin \mathscr{E}$ so $D \notin \mathscr{R}(p_k\mathscr{L})$. But it is not hard to see that $eD \in \mathscr{R}(\mathscr{L})$; in fact, $eD \in \mathscr{R}(\{N\})$, where N is the countable discrete uniform space. Since the $\mathscr{E}$-reflection map $D \to eD$ is $\mathscr{K}$-perfect, $D \in p_k\mathscr{R}(\mathscr{L})$.

(c) $(k')^{-1}\mathscr{R}(p_k\Gamma) \not\supset p_k\mathscr{R}(\Gamma)$: Here, Γ is complete uniform spaces; so $\mathscr{R}(\Gamma) = \Gamma$, and by 3.11, $p_k\Gamma = \Gamma$. But $(k')^{-1}\Gamma = \mathscr{K}$.

(d) $\mathscr{R}(p_k\mathscr{A}) \cup (k')^{-1}\mathscr{R}(p_k\mathscr{A}) \not\supset p_k\mathscr{R}(\mathscr{A})$, for $\mathscr{A} = \mathscr{E} \cap \Gamma$: It is not hard to see that $\mathscr{R}(p_k\mathscr{A}) = \mathscr{E}$, $(k')^{-1}\mathscr{R}(p_k\mathscr{A}) = \mathscr{K}$, and $p_k\mathscr{R}(\mathscr{A}) = \Gamma$; so the space D in (b) works.

(e) $p_k\mathscr{R}(\Gamma) \not\supset (k')^{-1}\mathscr{R}(p_kk'(\Gamma))$: Let $Y \in \Gamma - \mathscr{K}$, and let $X = k'Y$.

4. $\mathscr{S}$-perfect and epi-reflective hulls, 2. In this section, $\mathscr{S}$ is an epi-reflective subcategory for which all reflection maps are *embeddings*. Thus, the $\mathscr{O}(\mathscr{S})$ reflector s' is the identity functor. We shall simplify notation by viewing X as a subspace of sX, and s_x as the inclusion $X \subset sX$, which will sometimes go unlabeled.

In this case, 3.7 becomes:

4.1 Proposition. *Let all $\mathscr{S}$-reflection maps be embeddings. These conditions on the set $\mathscr{F}$ of maps of X are equivalent.*

(a) *e is $\mathscr{S}$-perfect.*

(b) $X = \bigcap_{f\in\mathscr{F}} f^{s-1}(X_f)$.

(c) *Δ embeds X as a closed subspace of* $\prod_{f\in\mathscr{F}} f^{s-1}(X_f)$.

Now as 3.7 yielded 3.8 and 3.9, 4.1 implies that $X \in p_s\mathscr{R}(\mathscr{A})$ if and only if $X \in \mathscr{R}(p_s\mathscr{A})$ (from 4.1 (a) and (c)). This improves part of 3.4 (and 3.10; but 3.10 reduces to 3.4 when $s' = 1$). An alternative argument gives further improvement:

4.2 Theorem. *Let all $\mathscr{S}$-reflection maps be embeddings. Then,* $\mathscr{R}(\mathscr{S} \cup \mathscr{A}) = \mathscr{R}(p_s\mathscr{A}) = p_s\mathscr{R}(\mathscr{A})$.

4.3 Lemma. *If $f : X \to Y$, and $B \subset Y$, then $f^{-1}(B)$ embeds as a closed subset of $X \times B$.*

Proof. Define $h : f^{-1}(B) \to X \times B$ by $h(x) = (x, f(x))$; then h is an embedding. Define $g : X \times B \to Y \times B$ by $g(x, b) = (f(x), b)$; then g is a morphism. The set $D \equiv \{(b, b) : b \in B\}$ is closed in $Y \times B$ (since the topological spaces are Hausdorff). One checks that $g^{-1}(D) = h(f^{-1}(B))$, so the latter is closed.

https://doi.org/10.4153/CJM-1975-003-6 Published online by Cambridge University Press

4.4 PROPOSITION. *$X \in p_s\mathscr{A}$ if and only if X embeds as a closed subset of a space $S \times A$, for $S \in \mathscr{S}$ and $A \in \mathscr{A}$.*

Proof. Suppose X closed-embeds in $S \times A$. By 3.2(d), $S \times A \in p_s\mathscr{A}$; by 3.2(b), $X \in p_s\mathscr{A}$.

If $f : X \to A \in \mathscr{A}$ is $\mathscr{S}$-perfect, then we have $s_x(X) = f^{s-1}(s_A(A))$. Applying 4.3, $s_x(X)$ closed-embeds in $sX \times s_A(A)$. Since s_x and s_A are embeddings, X closed-embeds in $sX \times A$.

Remark. 4.3 slightly generalizes [**10**, Lemma 3]. 4.4 generalizes [**10**, Proposition 2] and [**8**, 3.3 and 3.5].

Proof of 4.2. With 3.4, it suffices that $p_s\mathscr{R}(\mathscr{A}) \subset \mathscr{R}(\mathscr{S} \cup \mathscr{A})$. If $X \in p_s\mathscr{R}(\mathscr{A})$, then by 4.4, X closed-embeds in an $S \times Y$, with $Y \in \mathscr{R}(\mathscr{A})$. Evidently, $S \times Y \in \mathscr{R}(\mathscr{S} \cup \mathscr{A})$, and so $X \in \mathscr{R}(\mathscr{S} \cup \mathscr{A})$ (both by 1.1).

4.2 has the following interpretation. Let $\mathscr{I}(\mathscr{A})$ denote the least epi-reflective subcategory containing $\mathscr{A}$ and for which all reflection maps are embeddings. Because of 1.2, $\mathscr{I}(\mathscr{A}) = \mathscr{R}(\mathscr{A} \cup \mathscr{K})$ in Tych, and $= \mathscr{R}(\mathscr{A} \cup \Gamma)$ in Unif. 4.2 yields:

4.5 COROLLARY. *In Tych, $\mathscr{I}(\mathscr{A}) = p_k\mathscr{R}(\mathscr{A})$. In Unif, $\mathscr{I}(\mathscr{A}) = p_\gamma\mathscr{R}(\mathscr{A})$.*

4.6 COROLLARY. *Let all $\mathscr{S}$-reflection maps be embeddings.*

(a) *$X \in p_s\mathscr{R}(\mathscr{A})$ if and only if there is a family of subsets of sX, each in $p_s\mathscr{A}$, with intersection X.*

(b) *Let e be the functor for $p_s\mathscr{R}(\mathscr{A})$. Then, $eX = \bigcap\{P : X \subset P \subset sX, P \in p_s\mathscr{A}\}$.*

Proof. (a) follows from 2.4 and 3.8.

(b) : Let $i : X \to eX$ be the reflection map. If $f : X \to S \in \mathscr{S}$, then there is $g : eX \to S$ with $g \circ i = f$ (since $S \subset p_s\mathscr{R}(\mathscr{A})$). Then there is $h : seX \to S$ extending g. This shows that $seX = sX$, and we write $eX \subset sX$.

Now, let aX stand for $\bigcap\{P'\text{s}\}$. Applying (a) to $eX \subset seX = sX$ shows that $eX \supset aX$. It follows that each $f : X \to Y \in p_s\mathscr{R}(\mathscr{A})$ extends over aX (since f extends over eX). Then $eX = aX$ will follow if $aX \in p_s\mathscr{R}(\mathscr{A})$. To see this note that the diagonal map embeds aX into $\Pi\{P'\text{s}\}$, and the image is closed because the spaces are Hausdorff; since $p_s\mathscr{A} \subset p_s\mathscr{R}(\mathscr{A})$, 1.1 shows that $aX \in p_s\mathscr{R}(\mathscr{A})$.

Franklin's theorem [**4**] is obtained from 4.6 by specializing to Tych, taking $\mathscr{S} = \mathscr{K}$, and supposing $\mathscr{A} = p_k\mathscr{A}$; then $\mathscr{R}(\mathscr{A}) = p_k\mathscr{R}(\mathscr{A})$ by 4.2. When Franklin presented his paper at the Pittsburgh conference of June 1970, he remarked that the role of compactness was not clear. The present development does not use compactness explicitly, for the sole hypothesis is that $\mathscr{S}$-reflection maps are embeddings; but the implicit role of compactness is crucial (by 1.2).

4.7 COROLLARY (part of 1.2). *Let $\mathscr{R}$ be epi-reflective in Tych* (respectively, Unif). *Then, all* $\mathscr{R}$*-reflection maps are embeddings if and only if* $\mathscr{R} \supset \mathscr{K}$ (respectively, $\mathscr{R} \supset \Gamma$).

Proof. Now, a dense embedding of a compact (respectively, complete) space is onto; if all reflection maps are embeddings, then $X \to rX$ is an isomorphism, for $X \in \mathscr{K}$ (respectively, Γ). Thus, $\mathscr{R} \supset \mathscr{K}$ (respectively, Γ).

If $\mathscr{R} \supset \mathscr{K}$ (respectively, Γ), then use 4.6 with $\mathscr{S} = \mathscr{K}$ (respectively, Γ); so $sX = \beta X$ (respectively, γX). Since by 4.2 $p_s\mathscr{R} = \mathscr{R}(\mathscr{S} \cup \mathscr{R}) = \mathscr{R}$, e in 4.6 is r. 4.6 shows that $rX \subset \beta X$ (respectively, γX), so $X \to rX$ is an embedding.

5. Splitting epi-reflections with $\mathscr{S}$. We return to assuming only that $\mathscr{S}$-reflection maps are homeomorphisms. We shall prove the main results, mentioned in the introduction. In 2.7, we noted that a closed embedding is $\mathscr{S}$-perfect. The first result requires for $\mathscr{S}$, roughly, that the converse hold.

5.1 PROPOSITION. *If each* $\mathscr{S}$*-perfect embedding into a product of members of* $\mathscr{A}$ *has closed range, then* $\mathscr{O}(\mathscr{A}) \cap p_s\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$.

Proof. Clearly, $\mathscr{R}(\mathscr{A}) \subset \mathscr{O}(\mathscr{A}) \cap p_s\mathscr{R}(\mathscr{A})$.

If $X \in \mathscr{O}(\mathscr{A})$, then there is a set $\mathscr{F}_1$ of maps of X to members of $\mathscr{A}$ for which the evaluation e_1 is an embedding. If $X \in p_s\mathscr{R}(\mathscr{A})$, then by 3.8 there is a set $\mathscr{F}_2$ of maps of X to members of $\mathscr{A}$ satisfying 3.7(b). For $\mathscr{F} \equiv \mathscr{F}_1 \cup \mathscr{F}_2$, we still have 3.7(b), hence 3.7(a), and the evaluation e is $\mathscr{S}$-perfect. Since $\mathscr{F} \supset \mathscr{F}_1$, e is an embedding. By hypothesis, $e(X)$ is closed; so $X \in \mathscr{R}(\mathscr{A})$.

5.2 PROPOSITION. *Each* $\mathscr{S}$*-perfect map* (respectively, embedding) *has closed range if* $\mathscr{S} = \mathscr{K}$ (respectively, $\mathscr{S} = \Gamma$ in Unif).

Proof. Let $f : X \to Y$ be $\mathscr{S}$-perfect and let $y \in Y - f(X)$. Then $y \notin f^s(sX)$. If $\mathscr{S} = \mathscr{K}$, then $f^s(sX)$ is compact, hence closed; if $\mathscr{S} = \Gamma$, and f is an embedding, then $f^s(sX)$ is complete, hence closed. In either case, there is a neighborhood U of y with $U \cap f^s(sX) = \emptyset$. Then $(U \cap Y) \cap f(X) = \emptyset$, showing $y \notin \overline{f(X)}^Y$.

5.1 and 5.2 immediately yield the first main theorem.

5.3 THEOREM. *If* $\mathscr{S} = \mathscr{K}$, *or if* $\mathscr{S} = \Gamma$ *in Unif, then for any* $\mathscr{A}$, $\mathscr{O}(\mathscr{A}) \cap p_s\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$.

5.4 *Remarks.* (a) In uniform spaces, 5.3 for Γ is clearly a better result than 5.3 for $\mathscr{K}$.

(b) Concerning the sharpness of 5.3, the following can be shown easily. Let $\mathscr{S}$ be epi-reflective, with all reflection maps homeomorphisms; so $\mathscr{S} \supset \mathscr{K}$ by 1.2. (1) In Tych, if $\mathscr{O}(\mathscr{K}) \cap p_s\mathscr{K} = \mathscr{K}$, then $\mathscr{S} = \mathscr{K}$. (2) In uniform spaces, if $\mathscr{O}(\Gamma) \cap p_s\Gamma = \Gamma$, then $\mathscr{S} \subset \Gamma$. Thus (generalizing 3.11) each $\mathscr{S}$-perfect map is Γ-perfect, and $p_s\mathscr{A} \subset p_\gamma\mathscr{A}$ follows; so any equality

$\mathscr{O}(\mathscr{A}) \cap p_s\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$ is a weaker statement than $\mathscr{O}(\mathscr{A}) \cap p_\gamma\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$.

(c) An interesting failure of the conclusion of 5.3 is this: In Tych, take $\mathscr{S}$ = realcompact spaces (i.e., $\mathscr{R}(\{R\})$, R = the real line), and $\mathscr{A} = \{N\}$, N = the countable discrete space. Then (as can be shown) $\mathscr{O}(\mathscr{A})$ consists of all spaces with a basis of clopen sets, $p_s\mathscr{R}(\mathscr{A}) = \mathscr{S}$, $\mathscr{R}(\mathscr{A})$ is the N-compact spaces of Engelking and Mrówka [3]. $\mathscr{O}(\mathscr{A}) \cap p_s\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$ would say that every realcompact space "with a clopen base" is N-compact. This was shown to be false (after ten years or so) by Nyikos [**14**], with the horrible metric space of P. Roy.

(d) 5.3 is closely related to a lemma of Zenor [**17**]. Zenor's result restricted to Tychonoff spaces reads: Let $X \in \mathscr{O}(\mathscr{A})$. Then $X \in \mathscr{R}(\mathscr{A})$ if and only if whenever $\mathscr{F}$ is a free ultrafilter of closed sets, there is $f : X \rightarrow A \in \mathscr{A}$ and an open cover $\mathscr{U}$ of A with $f^{-1}(\mathscr{U})$ refining the family of $\mathscr{F}$-complements. It can be shown that:

(1) For normal X, the condition on closed ultrafilters is equivalent to the same condition for zero-set ultrafilters;

(2) Let $\mathscr{Z}$ be a free zero-set ultrafilter, p the unique point of $\beta X - X$ associated with $\mathscr{Z}$ [**5**], and $f : X \rightarrow Y$ a continuous function. Then, there is an open cover $\mathscr{U}$ of Y with $f^{-1}(\mathscr{U})$ refining the family of $\mathscr{Z}$-complements if and only if $f^\beta(p) \in \beta Y - Y$. *Via* 3.7, one now easily sees the connection with 5.3.

The following is the main result of the paper.

5.5 THEOREM. *Let $\mathscr{A}$ be a class of spaces, and let o, p, r be the functors associated with $\mathscr{O}(\mathscr{A})$, $p_k\mathscr{R}(\mathscr{A})$, and $\mathscr{R}(\mathscr{A})$, respectively. Then $op = r$.*

5.6 *Remarks.* (a) By 1.2, all $\mathscr{O}(\mathscr{A})$-reflection maps are onto, and all $p_k\mathscr{R}(\mathscr{A})$-reflection maps are homeomorphisms (since $\mathscr{K} \subset p_k\mathscr{R}(\mathscr{A})$). Thus, 5.5 says that an $\mathscr{R}(\mathscr{A})$-reflection $r_x : X \rightarrow rX$ can be factored as

$$o_{pX} \circ p_x : X \rightarrow pX \rightarrow opX,$$

where p_x is a homeomorphism and o_{px} is onto.

5.5 applies, of course, to factor an arbitrary epi-reflector: just take $\mathscr{A}$ epi-reflective, so $\mathscr{R}(\mathscr{A}) = \mathscr{A}$.

(b) 5.5 is stronger than 5.3 for $\mathscr{S} = \mathscr{K}$: whenever epi-reflective functors s, t, r satisfy both $st = r$ and $s\mathscr{T} \subset \mathscr{T}$, then $\mathscr{S} \cap \mathscr{T} = \mathscr{R}$; this is not hard to show.

(c) In uniform spaces, let $\mathscr{A} = \mathscr{K}$, and let q be the functor associated with $p_\gamma\mathscr{R}(\mathscr{K})$. Then $oq = r$ can fail. For, $\mathscr{O}(\mathscr{K})$ is precompact spaces, $p_\gamma\mathscr{R}(\mathscr{K}) = \Gamma$, so $q = \gamma$, and $\mathscr{R}(\mathscr{K}) = \mathscr{K}$, so r is the Samuel compactification functor. With $X = R$ with the usual uniformity, $qX = X$, and $oqX = oX$, which is not compact. But rX is compact.

The "reason" for such examples is that Lemma 5.7 below fails for Γ.

Curiously, in this example we have $qo = r$. (This is a standard definition of Samuel compactification, in fact.)

(d) We consider 5.5 in Tych with $\mathscr{A}$ consisting of just the two-point (discrete) space 2. Then $\mathscr{O}(\mathscr{A})$ is the class of all spaces with a basis of the clopen sets (by an old theorem of Alexandroff), $p_k\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A} \cup \mathscr{K}) = \mathscr{K}$ (using 4.5) so that p is the Stone-Čech compactification functor. The equation $op = r$ thus says that to construct rX, first take βX (i.e., apply p), then form the quotient obtained by identifying each connected component to a point (i.e., apply o to βX). For $X \in \mathscr{O}(\mathscr{A})$, this reproduces one of Banaschewski's constructions of his maximal 0-dimensional compactification [**1**].

Note that even for $X \in \mathscr{O}(\mathscr{A})$, the applying of o to βX need not be a vacuous operation; i.e., there is $X \in \mathscr{O}(\mathscr{A})$ with $\beta X \notin \mathscr{O}(\mathscr{A})$ [**5**]. Also, such an X shows that $po = r$ need not obtain (as it does in 5.4(c)); for $poX = \beta X$, while $opX \neq \beta X$.

We turn to the proof of 5.5. Several lemmas are required. Notation is as in 5.5.

5.7 Lemma. *Let $f : X \to Y$ and $g : Y \to Z$. If $g \circ f$ is $\mathscr{K}$-perfect, and if f is onto, then g is $\mathscr{K}$-perfect.*

Proof. The $\mathscr{K}$-reflection of X is either βX or the Samuel compactification: denote it kX. Of course, $(g \circ f)^k = g^k \circ f^k$. The crucial fact is that f^k is onto if f is onto (because $f^k(kX)$ is dense and compact). So, if $y \in kY - k_y(Y)$, there is $x \in kX$, necessarily $x \notin k_x(X)$, with $f^k(x) = y$. Then, $g^k(y) = g^k(f^k(x)) \in kZ - k_Z(Z)$, because $g \circ f$ is $\mathscr{K}$-perfect.

5.8 Corollary. *If $g : Y \to Z \in \mathscr{R}(\mathscr{A})$ is $\mathscr{K}$-perfect, then so is the map $g^\circ : oY \to Z$ (with $g = g^\circ \circ o_y$; note that $Z \in \mathscr{O}(\mathscr{A})$).*

Proof. $o_y : Y \to oY$ is onto. Apply 5.7.

5.9 Proposition. *$o(p_k\mathscr{R}(\mathscr{A})) \subset p_k\mathscr{R}(\mathscr{A})$.*

Proof. Apply 5.8.

5.10 *Remark.* Beyond Theorem 5.3, 5.9 is the crucial technical step in proving 5.5. Toward further explanation (beyond Example 5.6(c)) of why 5.5 fails when $\mathscr{K}$ is replaced by large $\mathscr{S}$: (a) Suppose there is onto $f : S \to Y$ with $S \in \mathscr{S}$ and $Y \notin \mathscr{S}$. Let 1 be the one-point space, and $g : Y \to 1$. Then the analogue of 5.7 fails. (b) Suppose there is $S \in \mathscr{S}$ with $oS \notin \mathscr{K}$. Now $1 \in \mathscr{R}(\mathscr{A})$, so with $g : S \to 1$, the analogue of 5.8 fails; hence 5.9 fails.

5.11 Lemma. *Let $\mathscr{M}$ and $\mathscr{N}$ be epi-reflective subcategories, with functors m and n. Suppose that $m\mathscr{N} \subset \mathscr{N}$. Then $m\mathscr{N} = \mathscr{M} \cap \mathscr{N}$, and this subcategory is epi-reflective, with functor mn.*

Proof. Let $N \in \mathscr{N}$. Then $mN \in \mathscr{M} \cap \mathscr{N}$. If $X \in \mathscr{M} \cap \mathscr{N}$, then $X \in \mathscr{N}$ and $mX = X$. Now, mn is a covariant functor, with range $\mathscr{M} \cap \mathscr{N}$, and $mn|\mathscr{M} \cap \mathscr{N}$ is the identity functor (up to isomorphism). It suffices that mn

have the reflective property. Given $f : X \to Y \in \mathscr{M} \cap \mathscr{N}$, consider the commuting diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & nX \arrow[r] \arrow[dl, dashed, "(1)"'] & mnX \arrow[dll, dashed, "(2)"] \\ Y & & \end{tikzcd}$$

where the unique lift (1) exists because $Y \in \mathscr{N}$, then the unique lift (2) of (1) exists because $Y \in \mathscr{M}$.

5.12 COROLLARY (5.5). (a) *op epi-reflects onto* $\mathscr{O}(\mathscr{A}) \cap p_k\mathscr{R}(\mathscr{A})$.
(b) $\mathscr{O}(\mathscr{A}) \cap p_k\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$, *so that* $op = r$.

Proof. (a) follows from 5.9 and 5.11. The first part of (b) is just 5.3; $op = r$ follows by uniqueness of reflections.

In view of 5.3 and 5.5, this question is of interest: When is $\mathscr{O}(\mathscr{A})$ all spaces? Clearly, if so, then in Tych (respectively, Unif) $p_k\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$ (respectively, $p_\gamma\mathscr{R}(\mathscr{A}) = \mathscr{R}(\mathscr{A})$), by 5.3, and since $\mathscr{S} \subset p_s\mathscr{A}$ always holds, $\mathscr{K} \subset \mathscr{R}(\mathscr{A})$ (respectively, $\Gamma \subset \mathscr{R}(\mathscr{A})$). Conversely, since $\mathscr{O}(\mathscr{K})$ (respectively, $\mathscr{O}(\Gamma)$) is all spaces, it is clear that $\mathscr{K} \subset \mathscr{R}(\mathscr{A})$ (respectively, $\Gamma \subset \mathscr{R}(\mathscr{A})$) implies that $\mathscr{O}(\mathscr{A}) = \mathscr{O}(\mathscr{R}(\mathscr{A}))$ is all spaces. The following appears in [**7**, 19.1.1] without proof.

5.13 THEOREM. *In Tych,* $\mathscr{K} \subset \mathscr{R}(\mathscr{A})$ *if and only if* $[0, 1]$ *is a subspace of some* $A \in \mathscr{A}$.

Proof. If $[0, 1] \subset A \in \mathscr{A}$ then any closed subset of any Tychonoff cube $\in \mathscr{R}(\mathscr{A})$, i.e., $\mathscr{K} \subset \mathscr{R}(\mathscr{A})$.

If $\mathscr{K} \subset \mathscr{R}(\mathscr{A})$, then $[0, 1]$ closed-embeds in some $\prod A_\alpha$ $(A_\alpha \in \mathscr{A})$, say $[0, 1] \subset \prod A_\alpha$. Let $P_\alpha = \pi_\alpha[0, 1] \subset A_\alpha$. By the Hahn-Mazurkiewicz Theorem, P_α is arcwise-connected and hence will contain a copy of $[0, 1]$ if P_α has two points. But since $[0, 1] \subset \prod P_\alpha$, it is not possible that every P_α is a singleton.

Thus, in Tychonoff spaces, $\mathscr{O}(\{[0, 1]\})$ is all spaces. It is readily seen that there is no uniform space X with $\mathscr{O}(\{X\})$ all spaces.

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https://doi.org/10.4153/CJM-1975-003-6 Published online by Cambridge University Press