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# THE $SP$-HULL OF A LATTICE-ORDERED GROUP

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There have been several recent papers on the subject of the $P$-hull and the $SP$-hull of an $l$-group (lattice-ordered group). The most natural formulation of the concepts was given by P. Conrad in [**6**]. T. Speed studied $P$-groups extensively in [**11**]; his work was motivated by earlier work by H. Nakano and I. Amemiya in a vector lattice setting. A. Vecksler [**12**] produced the $SP$-hull for $f$-rings. The ortho-completion of S. Bernau [**2**] is a related concept.

The best construction of the $P$-hull and $SP$-hull thus far has been given by D. Chambless [**4**]. However, his direct limit construction does not leave the reader with a "concrete" feeling for these hulls. K. Keimel [**10**] has given a nice sheaf-theoretic interpretation of the $SP$-hull.

In this paper we give a construction of the $SP$-hull and the $P$-hull which is substantially different from those previously given. If $G$ is represented as an $l$-subgroup of a cardinal product of totally-ordered groups indexed by $X$, then we construct these hulls out of $G$ and the index set $X$. Section 1 lays the foundation for the succeeding sections. In Section 2 we construct the $SP$-hull and obtain various corollaries from our construction. In Section 3 it is shown that each $l$-homomorphism of $G$ onto $H$ whose kernel is a polar extends to an $l$-homomorphism of the $SP$-hull of $G$ onto the $SP$-hull of $H$. Section 4 treats generalizations and the $P$-hull. A very nice description of the $P$-hull of the free vector lattice on two generators is given.

We briefly review the portion of $l$-group theory that we will be using. (We follow Conrad in our terminology. The reader is referred to [**8**] for the basic theory of $l$-groups.)

Let $S$ be a subset of an $l$-group $G$. Then

$$S' = \{g \in G \mid |g| \wedge |s| = 0 \quad \text{for all } s \in S\}$$

is called the polar of $S$ in $G$. $S'$ is a convex $l$-subgroup of $G$. The collection $\mathscr{P}(G)$ of all polars in $G$ is a Boolean algebra under inclusion. The meet operation is set-theoretic intersection, and the complement of $A \in \mathscr{P}(G)$ is $A'$. We write $S''$ for $(S')'$, and if $g \in G$, we write $g''$ for $\{g\}''$. $A \in \mathscr{P}(G)$ if and only if $A = A''$. We denote the join operation in $\mathscr{P}(G)$ by $\blacktriangledown$.

An $l$-group $G$ is the *cardinal sum* $A \oplus B$ of $l$-ideals $A$ and $B$ of $G$ if $A \cap B = 0$ and $A + B = G$. If this is the case, then $B = A'$, and $A$ and $B$ are called (*cardinal*) *summands* of $G$. The collection of all summands of $G$ is a Boolean subalgebra of $\mathscr{P}(G)$.

If $G$ is an $l$-subgroup of an $l$-group $H$ such that $G \cap C \neq 0$ for each non-zero

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convex $l$-subgroup $C$ of $H$, then we say $H$ is an *essential extension* of $G$. If for each $h \in H$ with $h > 0$ there exists $g \in G$ such that $0 < g \leqq h$, then we say $G$ is *dense* in $H$. If $G$ is a dense $l$-subgroup of $H$, then $H$ is an essential extension of $G$.

Let $H$ be an essential extension of $G$, and let * denote the polar operation in $H$. Then $A \rightarrow A'^*$ is a Boolean isomorphism of $\mathscr{P}(G)$ onto $\mathscr{P}(H)$. [**7**, Theorem 3.4]. If $S$ is a subset of $G$, then $S''^{**} = S^{**}$ and $S'^{**} = S^*$. [**6**, Section 2.] Thus $S'^* = S^{**}$.

If each element of $\mathscr{P}(G)$ is a summand of $G$, then $G$ is an *SP-group* (strongly projectable $l$-group). If $H$ is an essential extension of $G$, and $H$ is an $SP$-group, and no proper $l$-subgroup of $H$ that contains $G$ is an $SP$-group, then we say $H$ is an *SP-hull* of $G$.

If $g''$ is a summand of $G$ for each $g \in G$, then $G$ is a *P-group* (projectable $l$-group). If $H$ is an essential extension of $G$, and $H$ is a $P$-group, and no proper $l$-subgroup of $H$ containing $G$ is a $P$-group, then we say $H$ is a *P-hull* of $G$.

If $G$ is an $l$-subgroup of a cardinal product of totally-ordered groups, we say $G$ is *representable*. $G$ is representable if and only if $g''$ is a normal subgroup of $G$ for each $g \in G$. Thus if $G$ has a $P$-hull or an $SP$-hull, then $G$ must be representable. Conversely, if $G$ is representable, then $G$ has a $P$-hull and an $SP$-hull. Moreover, these hulls are unique. Versions of these results have been obtained by all the authors previously mentioned.

Let $f$ be an element of the cardinal product $\prod_{x \in X} T_x$, where each $T_x$ is a totally-ordered group. We denote the $x$-component of $f$ by $f(x)$, and we define $S(f) = \{x \in X | f(x) \neq 0\}$. If $K$ is a subset of $\prod_{x \in X} T_x$, we define

$$S(K) = \{x \in X | f(x) \neq 0 \quad \text{for some } f \in K\}.$$

Throughout this paper $G$ denotes an $l$-group and $'$ is the polar operation in $G$. Where a statement involves another $l$-group, it is often necessary or convenient to use a different symbol for the polar operation in this second $l$-group. We often use $^\perp$ for this purpose. We also use * for this purpose, but never without express designation, since we sometimes use * in other ways. *We assume throughout that all l-groups are representable.*

**1. Fields of sets and extensions of $l$-groups.** A *field of subsets* of a set $X$ is a collection $\mathscr{F}$ of subsets of $X$ such that (i) $\emptyset \in \mathscr{F}$, (ii) $A \cap B \in \mathscr{F}$ if $A, B \in \mathscr{F}$, and (iii) $X \backslash A \in \mathscr{F}$ if $A \in \mathscr{F}$.

Each field of subsets of $X$ is a Boolean algebra under the partial-ordering of inclusion. On the other hand, suppose $\mathscr{B}$ is a collection of subsets of $X$ satisfying (i) $\emptyset \in \mathscr{B}$, and (ii) $A \cap B \in \mathscr{B}$ for all $A, B \in \mathscr{B}$. Then, as is well-known in the theory of Boolean algebras, it is possible that $\mathscr{B}$ is a Boolean algebra under the partial-ordering of inclusion but not a field of subsets of $X$. This is possible even if we assume $X \in \mathscr{B}$. (The collection of all regular open subsets of the real line is an example of this phenomenon.)

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The following technical lemma is crucial to the development in succeeding sections. I owe its proof to an anonymous referee.

LEMMA 1.1. *Suppose $\mathscr{B}$ is a collection of subsets of a set $X$ such that* (i) $\emptyset \in \mathscr{B}$, *and* (ii) $A \cap B \in \mathscr{B}$ *if* $A, B \in \mathscr{B}$. *If $\mathscr{B}$ is a Boolean algebra under the partial-ordering of inclusion, then there exists a Boolean isomorphism $\eta$ of $\mathscr{B}$ onto a field $\mathscr{F}$ of subsets of $X$ with $\eta B \supseteq B$ for all $B \in \mathscr{B}$.*

*Proof.* Let $\mathscr{E}_x$ be the collection of all $B \in \mathscr{B}$ such that $x \in B$. Then $\mathscr{E}_x$ is empty or $\mathscr{E}_x$ is a filter in $\mathscr{B}$. If $\mathscr{E}_x$ is empty, let $\mathscr{U}_x$ be any ultrafilter in $\mathscr{B}$; otherwise let $\mathscr{U}_x$ be an ultrafilter in $\mathscr{B}$ such that $\mathscr{U}_x \supseteq \mathscr{E}_x$.

Define, for all $B \in \mathscr{B}$, $\eta B = \{x \in X | B \in \mathscr{U}_x\}$. It is clear that $\eta$ preserves inclusion, that $\eta(\emptyset) = \emptyset$, and that $\eta$ maps the largest element of $\mathscr{B}$ to $X$. If $A, B \in \mathscr{B}$ we have immediately that $\eta(A \cap B) \subseteq \eta A \cap \eta B$. On the other hand, if $x \in \eta A \cap \eta B$, then $A, B \in \mathscr{U}_x$ whence $A \cap B \in \mathscr{U}_x$, and thus $x \in \eta(A \cap B)$. Thus $\eta(A \cap B) = \eta A \cap \eta B$.

Denote the complement of $B$ in $\mathscr{B}$ by $B'$. We show $\eta(B') = X\backslash \eta B$. $B \cap B' = \emptyset$ so $\eta(B) \cap \eta(B') = \eta(B \cap B') = \emptyset$. Suppose $x \notin \eta B$. Then $B \notin \mathscr{U}_x$ and so there exists $K \in \mathscr{U}_x$ such that $B \cap K = \emptyset$. $K \subseteq B'$ and hence $B' \in \mathscr{U}_x$. Thus $x \in \eta(B')$. We have shown $\eta B \cap \eta(B') = \emptyset$ and $\eta B \cup \eta(B') = X$. Thus $\eta(B') = X\backslash \eta B$.

Suppose $A, B \in \mathscr{B}$ and there exists $z \in B$ such that $z \notin A$. If $A' \cap B = \emptyset$, then $B \subseteq A'' = A$, a contradiction. Thus there exists $x \in A' \cap B$. $B \in \mathscr{U}_x$ and $A \notin \mathscr{U}_x$ (since $A' \in \mathscr{U}_x$). Thus $x \in \eta B$ and $x \notin \eta A$. Hence, if $A \neq B$, then $\eta A \neq \eta B$.

We have shown that $\eta$ is a one-to-one Boolean homomorphism of $\mathscr{B}$ into the field of all subsets of $X$. If $x \in B$, then $B \in \mathscr{E}_x$, whence $B \in \mathscr{U}_x$, and $x \in \eta B$. Thus $\eta B \supseteq B$ for all $B \in \mathscr{B}$, and the proof is complete.

Now, let $G$ be an $l$-subgroup of the cardinal product $\prod_{x \in X} T_x$, where each $T_x$ is a totally-ordered group, and let $\mathscr{F}$ be a field of subsets of $X$. Suppose $h \in \prod T_x$ is such that for some finite partition of $X$ in $\mathscr{F}$, say $F_1, \ldots, F_n$, there exist $g_i \in G$ such that $h(x) = g_i(x)$ for all $x \in F_i$ $(i = 1, \ldots, n)$. We then write $h = [g_i|F_i]$, and we denote the set of all such $h$ by $G[\mathscr{F}]$.

LEMMA 1.2. *$G[\mathscr{F}]$ is an $l$-*subgroup of $\prod T_x$ *that contains $G$.*

*Proof.* Suppose $h = [g_i|F_i]$ and $f = [g_j|F_j]$ are elements of $G[\mathscr{F}]$. Then $h - f = [g_i - g_j|F_i \cap F_j] \in G[\mathscr{F}]$ and $h \vee 0 = [g_i \vee 0|F_i] \in G[\mathscr{F}]$. Thus $G[\mathscr{F}]$ is an $l$-subgroup of $\prod T_x$. Also, if $g \in G$, then $g = [g|X] \in G[\mathscr{F}]$.

LEMMA 1.3. *For $F \in \mathscr{F}$ define $\varphi(F) = \{h \in G[\mathscr{F}] |\ S(h) \subseteq F\}$. Then $\varphi(F)$ is a cardinal summand of $G[\mathscr{F}]$; in fact,*

$$G[\mathscr{F}] = \varphi(F) \oplus \varphi(X\backslash F).$$

*Proof.* Clearly $\varphi(F)$ and $\varphi(X\backslash F)$ are $l$-ideals of $G[\mathscr{F}]$, and $\varphi(F) \cap \varphi(X\backslash F) = 0$. Suppose $h = [g_i|F_i] \in G[\mathscr{F}]$. Let $h_1 \in \prod T_x$ be such that $h_1(x) = g_i(x)$ for all

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$x \in F \cap F_i$ and $h_1(x) = 0$ for all $x \in (X \backslash F) \cap F_i$. Let $h_2 \in \prod T_x$ be such that $h_2(x) = g_i(x)$ for all $x \in (X \backslash F) \cap F_i$ and $h_2(x) = 0$ for all $x \in F \cap F_i$. Then $h_1, h_2 \in G[\mathscr{F}]$, $h_1 \in \varphi(F)$, $h_2 \in \varphi(X \backslash F)$, and $h = h_1 + h_2$. Thus

$$G[\mathscr{F}] = \varphi(F) \oplus \varphi(X \backslash F).$$

LEMMA 1.4. *Let $H$ be an $l$-subgroup of $\prod_{y \in Y} T_y$, where each $T_y$ is a totally-ordered group, and let $G$ be an $l$-subgroup of $H$. Suppose $\mathscr{A}$ is a subalgebra of $\mathscr{P}(H)$ such that each $A \in \mathscr{A}$ is a summand of $H$. Then $\mathscr{F}^* = \{S(A) | A \in \mathscr{A}\}$ is a field of subsets of $S(H)$, and $G[\mathscr{F}^*]$ is an $l$-subgroup of $H$.*

*Proof.* We have $\emptyset = S(0) \in \mathscr{F}^*$ and $S(H) \in \mathscr{F}^*$. If $A$ and $B$ are convex $l$-subgroups of $H$, then $S(A \cap B) = S(A) \cap S(B)$. Also, since $A$ is a summand of $H$, we have $S(A) \cup S(A^\perp) = S(H)$ and $S(A) \cap S(A^\perp) = \emptyset$. (Here $^\perp$ denotes the polar operation in $H$.) Thus $\mathscr{F}^*$ is a field of subsets of $S(H)$.

Let $g \in G$ and $F = S(A) \in \mathscr{F}^*$. Let $f \in G[\mathscr{F}^*]$ be such that $f(x) = g(x)$ for all $x \in F$ and $f(x) = 0$ for all $x \in X \backslash F = S(A^\perp)$. We can write $g = r + s$ where $r \in A$ and $s \in A^\perp$. Now $S(r) \subseteq S(A)$, $S(s) \subseteq S(A^\perp)$, and $S(A) \cap S(A^\perp) = \emptyset$. Thus $r(x) = g(x)$ for all $x \in S(A)$ and $r(x) = 0$ for all $x \in S(A^\perp)$. Thus $f = r \in H$, and since each element of $G[\mathscr{F}^*]$ is the sum of finitely many elements like $f$, we conclude $G[\mathscr{F}^*] \subseteq H$.

**2. The $SP$-hull of an $l$-group.** Let $G$ be an $l$-subgroup of $\prod_{x \in X} T_x$, where each $T_x$ is a totally-ordered group. The map $J \to S(J)$ is a one-to-one inclusion preserving function of $\mathscr{P}(G)$ onto a collection $\mathscr{B}$ of subsets of $X$; moreover, the inverse map is also inclusion-preserving. Thus $\mathscr{B}$ is a Boolean algebra of subsets of $X$ with respect to the partial-ordering of inclusion. If $I, J \in \mathscr{P}(G)$, then $S(I) \cap S(J) = S(I \cap J) \in \mathscr{B}$; also, $\emptyset = S(0) \in \mathscr{B}$. Thus by Lemma 1.1 there exists a Boolean isomorphism $\eta$ of $\mathscr{B}$ onto a field $\mathscr{F}$ of subsets of $X$. We will prove $G[\mathscr{F}]$ is the $SP$-hull of $G$.

LEMMA 2.2. *If $g \in G$, $J \in \mathscr{P}(G)$, and $S(g) \cap S(J) = \emptyset$, then $S(g) \cap \eta S(J) = \emptyset$.*

*Proof.* $S(g) \cap S(J) = \emptyset$ implies $g'' \cap J = 0$, and hence $S(g'') \cap S(J) = \emptyset$. Thus $\eta S(g'') \cap \eta S(J) = \emptyset$. Since $S(g) \subseteq S(g'') \subseteq \eta S(g'')$ we conclude

$$S(g) \cap \eta S(J) = \emptyset.$$

LEMMA 2.2. (i) *If $h \in G[\mathscr{F}]$, $J \in \mathscr{P}(G)$, and $S(h) \cap S(J) = \emptyset$, then*

$$S(h) \cap \eta S(J) = \emptyset.$$

(ii) *If $0 < h \in G[\mathscr{F}]$, there exists $g \in G$ with $0 < g \leqq h$.*

*Proof.* Suppose $z \in S(h) \cap \eta S(J)$. Let $F_1 \in \mathscr{F}$ and $g_1 \in G$ be such that $z \in F_1 \subseteq \eta S(J)$ and $h(x) = g_1(x)$ for all $x \in F_1$. $F_1 = \eta S(J_1)$ for some $J_1 \in \mathscr{P}(G)$. We have $z \in S(g_1) \cap F_1$, so by Lemma 2.1 $S(g_1) \cap S(J_1) \neq \emptyset$. Since $g_1(x) = h(x)$ for all $x \in S(J_1)$, we conclude $S(h) \cap S(J_1) \neq \emptyset$. Thus since $S(J_1) \subseteq S(J)$ we have $S(h) \cap S(J) \neq \emptyset$, and (i) is proved.

Now let $h > 0$. Since $S(g_1) \cap S(J_1) \neq \emptyset$, there exists $0 < k \in J_1$ with $S(g_1) \cap S(k) \neq \emptyset$. Let $g = |g_1| \wedge k$. Then $0 < g \leqq h$, and (ii) is proved.

LEMMA 2.3. *Let $J \in \mathscr{P}(G)$ and $F = \eta S(J)$. Let $\varphi(F) = \{h \in G[\mathscr{F}] | S(h) \subseteq F\}$. Then $\varphi(F) = J'^{\perp}$* (where $^{\perp}$ denotes the polar operation in $G[\mathscr{F}]$).

*Proof.* Let $h \in \varphi(F)$ and $g \in J'$. Then $S(g) \cap S(J) = \emptyset$, so by Lemma 2.1, $S(g) \cap F = \emptyset$. Thus $S(g) \cap S(h) = \emptyset$, and $|g| \wedge |h| = 0$. That is, $h \in J'^{\perp}$.

Now suppose $h \in G[\mathscr{F}]$ and $h \notin \varphi(F)$. Then $h(x) \neq 0$ for some $x \in X \backslash F = \eta S(J')$. Thus by Lemma 2.2(i), $S(h) \cap S(J') \neq \emptyset$. Thus $h \notin J'^{\perp}$.

THEOREM 2.4. *$G[\mathscr{F}]$ is the SP-hull of $G$.*

*Proof.* By Lemma 1.2, $G[\mathscr{F}]$ is an $l$-group, and $G$ is an $l$-subgroup of $G[\mathscr{F}]$. $G$ is a dense $l$-subgroup of $G[\mathscr{F}]$ by Lemma 2.2(ii). Thus each polar in $G[\mathscr{F}]$ is of the form $J'^{\perp}$ where $J \in \mathscr{P}(G)$. Thus by Lemma 2.3 and Lemma 1.3, $G[\mathscr{F}]$ is $SP$.

Suppose now that $K$ is an $l$-subgroup of $G[\mathscr{F}]$ containing $G$ and that $K$ is an $SP$-group. Let * denote the polar operation in $K$. Then $J^{**}$ is a summand of $K$ for all $J \in \mathscr{P}(G)$. Thus $S(J^{**}) \cup S(J^*) = S(K) = S(G[\mathscr{F}])$. (Note $S(G) \subseteq S(K) \subseteq S(G[\mathscr{F}]) = S(G)$.) Since $G[\mathscr{F}]$ is an essential extension of $G$, it is also an essential extension of $K$, and thus $J^{**} \subseteq J^{**\perp\perp} \subseteq J^{\perp\perp}$ and $J^* \subseteq J^{*\perp\perp} = J^{\perp}$. Thus $S(J^{**}) \subseteq S(J^{\perp\perp})$ and $S(J^*) \subseteq S(J^{\perp})$. But $S(J^{\perp\perp}) \cap S(J^{\perp}) = \emptyset$. Thus $S(J^{**}) = S(J^{\perp\perp})$ and $S(J^*) = S(J^{\perp})$.

Now if $g \in G$ and $J \in \mathscr{P}(G)$, we can write $g = r + s$ where $r \in J^{**}$ and $s \in J^*$. We have then $r(x) = g(x)$ for all $x \in S(J^{\perp\perp}) = S(J^{**})$, and $r(x) = 0$ for all $x \in X \backslash S(J^{\perp\perp})$. Also, $S(J^{\perp\perp}) = \eta S(J) \in \mathscr{F}$. Thus $r \in G[\mathscr{F}]$. But each element of $G[\mathscr{F}]$ is the sum of finitely many elements like $r$. Thus $K = G[\mathscr{F}]$, and $G[\mathscr{F}]$ is an $SP$-hull of $G$.

THEOREM 2.5. *Suppose $G$ is an $l$-subgroup of an $l$-group $M$ such that*

(i) *$M$ is $SP$,*

(ii) *if $N$ is $SP$ and $N$ is an $l$-subgroup of $M$ containing $G$, then $N = M$, and*

(iii) *there exists a Boolean isomorphism $\tau$ of $\mathscr{P}(G)$ onto $\mathscr{P}(M)$ such that $J \subseteq \tau(J)$ for all $J \in \mathscr{P}(G)$.*

*Then there exists an $l$-isomorphism $\beta$ of $G[\mathscr{F}]$ onto $M$ such that $\beta(g) = g$ for all $g \in G$.*

*Proof.* Let $M$ be an $l$-subgroup of $\prod_{y \in Y} T_y$, where each $T_y$ is a totally-ordered group, and $S(M) = Y$. Then $\mathscr{F}^* = \{S(K) | K \in \mathscr{P}(M)\}$ is a field of subsets of $Y$ since $M$ is $SP$. Let $\mathscr{B}^* = \{S(J) | J \in \mathscr{P}(G)\}$, where here we take $S(J)$ as a subset of $Y$. Define $\eta^* : \mathscr{B}^* \to \mathscr{F}^*$ by $\eta^*(S(J)) = S(\tau(J))$. Then $\eta^*$ is a surjective Boolean isomorphism and $S(J) \subseteq \eta^* S(J)$. By Theorem 2.4, $G[\mathscr{F}^*]$ is an $SP$-hull of $G$. By Lemma 1.4 $G[\mathscr{F}^*]$ is an $l$-subgroup of $M$. Thus $G[\mathscr{F}^*] = M$.

Now define $\beta: G[\mathscr{F}] \to G[\mathscr{F}^*]$ by $\beta[g_i|\eta S(J_i)] = [g_i|\eta^* S(J_i)]$. We show that $\beta$ is a well-defined function.

Suppose $[g_i|\eta S(J_i)] = [0|X] = 0$. Then $S(g_i) \cap \eta S(J_i) = \emptyset$, and thus $S(g_i'') \cap S(J_i) = \emptyset$, and $g_i'' \cap J_i = 0$. Hence $S(g_i'') \cap \eta^* S(J_i) = \emptyset$, and thus $S(g_i) \cap \eta^* S(J_i) = \emptyset$. Since this is true for each $i$, we conclude that

$$[g_i|\eta^* S(J_i)] = [0|X] = 0.$$

Now suppose $[g_i|\eta S(J_i)] = [g_j|\eta S(J_j)]$. Then

$$[g_i - g_j|\eta S(J_i \cap J_j)] = [g_i - g_j|\eta S(J_i) \cap \eta S(J_j)] = [0|X].$$

Thus $[g_i - g_j|\eta^* S(J_i \cap J_j)] = [0|Y]$ and hence $[g_i|\eta^* S(J_i)] = [g_j|\eta^* S(J_j)]$.

Thus $\beta$ is well-defined. It is readily verified that $\beta$ is a surjective $l$-homomorphism. We show it is an isomorphism. Suppose $[g_i|\eta S(J_i)] \neq 0$. Then $S(g_i) \cap \eta S(J_i) \neq \emptyset$ for some $i$, and hence $S(g_i) \cap S(J_i) \neq \emptyset$, using Lemma 2.1. Thus $S(g_i) \cap \eta^* S(J_i) \neq \emptyset$ and hence $\beta[g_i|\eta S(J_i)] \neq 0$.

Finally, $\beta(g) = \beta[g|X] = \beta[g|\eta S(G)] = [g|\eta^* S(G)] = [g|Y] = g$ for all $g \in G$. This completes the proof of Theorem 2.5.

If $M$ is an $SP$-hull of $G$, then the hypotheses of Theorem 2.5 are satisfied with $\tau(J) = J'^{\perp}$. It follows that $G$ has a unique $SP$-hull (up to isomorphism over $G$). Following [**6**] we denote the $SP$-hull of $G$ by $G^{SP}$.

Our model of the $SP$-hull makes many of its properties almost self-evident. We list these below as corollaries. Many have appeared in one form or another at various places in the literature.

COROLLARY 2.6. *If $G$ is an $l$-subgroup of a cardinal product $\prod R_x$ of copies of the real numbers $R$, then $G^{SP}$ is an $l$-subgroup of the same cardinal product.* (c.f., [**9**, Theorem 3.3].)

COROLLARY 2.7. *If $0 < h \in G^{SP}$, then there exist $g, \bar{g} \in G$ such that $0 < g \leqq h \leqq \bar{g}$. In particular, if $G$ is archimedean, then so is $G^{SP}$.*

*Proof.* By Lemma 2.2(ii) there exists $g \in G$ with $0 < g \leqq h$. Write $h = [g_i|F_i] \in G[\mathscr{F}]$, in the notation of Section 1, and let $\bar{g} = \vee g_i$. Then $h \leqq \bar{g}$.

COROLLARY 2.8. *If $G$ is divisible* (respectively, a vector lattice, an $f$-ring) *then so is $G^{SP}$. If $G$ belongs to an equationally-closed class $\mathscr{C}$ of $l$-groups, then so does $G^{SP}$.*

*Proof.* Only the case that $G$ is divisible is treated here; the proofs of the remaining assertions are similar.

Suppose $G$ is divisible. We can view $G$ as an $l$-subgroup of $\prod_{M \in [\mathscr{M}]} G/M$ where each $M \in \mathscr{M}$ is a minimal prime subgroup of $G$ (and hence $G/M$ is a totally-ordered group). Suppose $h = [g_i|F_i] \in G[\mathscr{F}]$ and $n$ is a positive integer. There exists $f_i \in G$ such that $nf_i = g_i$. Now $\bar{h} = [f_i|F_i] \in G[\mathscr{F}]$ and $n\bar{h} = [nf_i|F_i] = h$. Thus $G[\mathscr{F}] = G^{SP}$ is divisible.

An $l$-group with the property that each of its $l$-epimorphic images is archimedean will be called *hyperarchimedean*. It is proved in [**8**, p. 2.17] that $G$ is hyperarchimedean if and only if $G$ is (isomorphic to) an $l$-subgroup of a cardinal product $\prod R_x$, where each $R_x$ is a copy of the real numbers, such that if $0 < g, \bar{g} \in G$ there exists a positive integer $n$ such that $n\bar{g}(x) > g(x)$ whenever $\bar{g}(x) \neq 0$.

COROLLARY 2.9. *If $G$ is hyperarchimedean, then so is $G^{SP}$.*

*Proof.* Let $G$ be represented as in the preceding paragraph. Let $0 < h, \bar{h} \in G[\mathscr{F}]$. Write $h = [g_i|F_i]$, $\bar{h} = [g_j|F_j]$. There exists an integer $n_{ij}$ such that $n_{ij}g_j(x) > g_i(x)$ whenever $g_j(x) \neq 0$. Let $n$ be the largest of the $n_{ij}$. Then $n\bar{h}(x) > h(x)$ whenever $\bar{h}(x) \neq 0$. Thus $G^{SP}$ is hyperarchimedean.

*Example.* Let $G$ be the $l$-subgroup of $\prod_{n \in N} R_n$ consisting of all eventually constant real sequences. (Here $N$ denotes the natural numbers.) Then $\mathscr{B} = \{S(J)|J \in \mathscr{P}(G)\}$ consists of all subsets of $N$. Thus the map $\eta$ in Lemma 1.1 can be taken to be the identity, and hence by Theorem 2.4 $G^{SP}$ is the $l$-group of all real sequences which have finite range. $G^{SP}$ is hyperarchimedean. However, the Dedekind completion of $G$ is the $l$-group of all bounded real sequences, and this is not hyperarchimedean.

COROLLARY 2.10. *Suppose $H$ is an essential extension of $G$, and $H$ is an $SP$-group. Suppose $H$ is an $l$-subgroup of a cardinal product $\prod_{y \in Y} T_y$ of a totally-ordered groups $T_y$. Then $\mathscr{F}^* = \{S(J)|J \in \mathscr{P}(H)\}$ is a field of subsets of $S(H)$, and $G[\mathscr{F}^*]$ is the $SP$-hull of $G$.*

*Proof.* This was proved in the first paragraph of the proof of Theorem 2.5.

*Example.* Suppose $H$ is the $l$-group of all continuous almost-finite extended-real-valued functions on an extremally disconnected compact Hausdorff space $Y$, and $H$ is an essential extension of $G$. Then $\mathscr{F}^*$ is the collection of regular open subsets of $Y$, and $G[\mathscr{F}^*]$ is the $SP$-hull of $G$ by Corollary 2.10.

Corollary 2.10 can be generalized somewhat. Suppose $G$ is an $l$-subgroup of $H$. Let us say $H$ is a *weak-essential* extension of $G$ if $(J + J')^\perp = 0$ for all $J \in \mathscr{P}(G)$. (Here $^\perp$ denotes polar in $H$.) $H$ is a weak-essential extension of $G$ if and only if the map $J \rightarrow J'^\perp$ is a Boolean isomorphism of $\mathscr{P}(G)$ into $\mathscr{P}(H)$. [**5**, Theorem 4.1.] It is clear that each essential extension of $G$ is a weak-essential extension of $G$.

COROLLARY 2.11. *Suppose $H$ is a weak-essential extension of $G$, and $H$ is an $SP$-group. Then $G^{SP}$ is an $l$-subgroup of $H$.*

*Proof.* Represent $H$ as an $l$-subgroup of $\prod_{y \in Y} T_y$, where each $T_y$ is a totally-ordered group, and $S(H) = Y$. Let $\mathscr{B}^* = \{S(J)|J \in \mathscr{P}(G)\}$, where here $S(J)$ is taken in $Y$. Let $\eta S(J) = S(J'^\perp)$. Then $\eta$ is a Boolean isomorphism onto a field of subsets of $Y$, and $\eta S(J) \supseteq S(J)$ for all $J \in \mathscr{P}(G)$. Thus $G[\mathscr{F}^*] = G^{SP}$ by Theorem 2.4. By Lemma 1.4, $G[\mathscr{F}^*]$ is an $l$-subgroup of $H$.

*Remark.* Suppose $G \in \mathscr{P}(H)$. Then $J \in \mathscr{P}(G)$ if and only if $J \in \mathscr{P}(H)$ and $J \subseteq G$. It follows by an argument similar to that for Corollary 2.11 that $G^{SP}$ is an $l$-subgroup of $H^{SP}$.

**3. Further properties of the $SP$-hull.** In the first three lemmas in this section, $G$ and $H$ are $l$-groups which need not be representable, and $^\perp$ denotes the polar operation in $H$.

LEMMA 3.1. *Let $\alpha:G \rightarrow H$ be a surjective l-homomorphism. If $S$ is a subset of $G$ such that* $\ker \alpha \subseteq S'$, *then* $\alpha(S') = \alpha(S)^\perp$. *If $J \in \mathscr{P}(G)$ and $J \supseteq$* $\ker \alpha$, *then* $\alpha(J) = (\alpha(J'))^\perp$, *an element of* $\mathscr{P}(H)$.

*Proof.* Suppose $h \in \alpha(S')$. Then $h = \alpha f$ for some $f \in S'$, and $f \wedge s = 0$ for all $s \in S$. Thus $h \wedge \alpha s = 0$ for all $s \in S$, and hence $h \in \alpha(S)^\perp$.

On the other hand, suppose $h \in \alpha(S)^\perp$. Then $h = \alpha g$ for some $g \in G$, and $\alpha g \wedge \alpha s = 0$ for all $s \in S$. Thus $g \wedge s \in \ker \alpha$ and hence by hypothesis $g \wedge s \in S'$. Thus $(g \wedge s) \wedge s = 0$ for all $s \in S$, and thus $g \in S'$. Thus $h \in \alpha(S')$.

The last statement in the lemma follows by taking S $= J'$. Then $S' = J'' = J$, and hence $\alpha(J) = \alpha(J')^\perp$.

LEMMA 3.2. *Let $\alpha:G \rightarrow H$ be a surjective l-homorphism such that* $\ker \alpha \in \mathscr{P}(G)$. *If $K \in \mathscr{P}(H)$, then $\alpha^{-1}(K) = \{g \in G | \alpha g \in K\}$ is an element of $\mathscr{P}(G)$.*

*Proof.* $K = \alpha(S)^\perp$ for some subset $S$ of $G$. Let $A = \ker \alpha$, and let $D = \{s \wedge b | s \in S \text{ and } b \in A'\}$. The sentences that follow are equivalent (using $A = A''$ to get from the fifth to the fourth). $g \in \alpha^{-1}(K)$. $\alpha g \in K$. $\alpha g \wedge \alpha s = 0$ for all $s \in S$. $g \wedge s \in A$ for all $s \in S$. $g \wedge s \wedge b = 0$ for all $s \in S$ and $b \in A'$. $g \in D'$.

Thus $\alpha^{-1}(K) = D'$ is a polar in $G$.

LEMMA 3.3. *Let $\alpha:G \rightarrow H$ be a surjective l-homorphism such that* $\ker \alpha \in \mathscr{P}(G)$. *Define $\bar{\alpha}:\mathscr{P}(G) \rightarrow \mathscr{P}(H)$ by $\bar{\alpha}(J) = \alpha(J \blacktriangledown$* $\ker \alpha)$. *Then $\bar{\alpha}$ is a surjective Boolean homorphism.*

*Proof.* Let $\mathscr{A} = \{I \in \mathscr{P}(G) | I \supseteq \ker \alpha\}$. Then $\mathscr{A}$ is a Boolean algebra with $\ker \alpha$ as least element. The map $J \rightarrow J \blacktriangledown \ker \alpha$ is a Boolean homomorphism of $\mathscr{P}(G)$ onto $\mathscr{A}$. Also, by Lemmas 3.1 and 3.2, the map $I \rightarrow \alpha(I)$ is a Boolean isomorphism of $\mathscr{A}$ onto $\mathscr{P}(H)$. Thus $\bar{\alpha}$ is a surjective Boolean homorphism.

THEOREM 3.4. *Suppose $G$ and $H$ are representable l-groups, and $\alpha:G \rightarrow H$ is a surjective l-homomorphism such that* $\ker \alpha \in \mathscr{P}(G)$. *Then there exists a surjective l-homomorphism $\beta:G^{SP} \rightarrow H^{SP}$ such that $\beta g = \alpha g$ for all $g \in G$.*

*Proof.* Let $G$ be an $l$-subgroup of $\prod_{x \in X} T_x$, where each $T_x$ is a totally-ordered group. Let $\mathscr{B} = \{S(J) | J \in \mathscr{P}(G)\}$, and let $\eta:\mathscr{B} \rightarrow \mathscr{F}$ be as in Lemma **1.1**. Then $G^{SP} = G[\mathscr{F}]$ by Theorem 2.4.

Similarly, let $H$ be an $l$-subgroup of $\prod_{y \in Y} T_y$, where each $T_y$ is a totally-ordered group. Let $\mathscr{B}^* = \{S(K) | K \in \mathscr{P}(H)\}$, and let $\eta^*: \mathscr{B}^* \to \mathscr{F}^*$ be as in Lemma 1.1. Again, $H^{SP} = H[\mathscr{F}^*]$ by Theorem 2.4.

Let $\bar{\alpha}$ be as in Lemma 3.3. Define $\beta: G[\mathscr{F}] \to H[\mathscr{F}^*]$ by $\beta[g_i | \eta S(J_i)] = [\alpha g_i | \eta^* S(\bar{\alpha}(J_i))]$. We show that $\beta$ is well-defined. For this, as in the proof of Theorem 2.5, it is enough to show that if $[g_i | \eta S(J_i)] = 0$, then

$$[\alpha g_i | \eta^* S(\bar{\alpha}(J_i))] = 0.$$

Suppose $[g_i | \eta S(J_i)] = 0$. Then $g_i'' \cap J_i = 0$, and hence

$$(g_i'' \blacktriangledown \ker \alpha) \cap (J_i \blacktriangledown \ker \alpha) = \ker \alpha,$$

and hence $\alpha(g_i'' \blacktriangledown \ker \alpha) \cap \alpha(J_i \blacktriangledown \ker \alpha) = 0$. Now, $\alpha(g_i'' \blacktriangledown \ker \alpha)$ is a polar in $H$ by Lemma 3.1, and $\alpha g_i$ is an element of $\alpha(g_i'' \blacktriangledown \ker \alpha)$. Hence $(\alpha g_i)^{\perp\perp} \subseteq \alpha(g_i'' \blacktriangledown \ker \alpha)$ and thus $(\alpha g_i)^{\perp\perp} \cap \alpha(J_i \blacktriangledown \ker \alpha) = 0$. Therefore $S((\alpha g_i)^{\perp\perp}) \cap S(\alpha(J_i \blacktriangledown \ker \alpha)) = \emptyset$, and hence by Lemma 2.2,

$$S(\alpha g_i) \cap \eta^* S(\alpha(J_i \blacktriangledown \ker \alpha)) = \emptyset.$$

Thus $[\alpha g_i | \eta^* S(\alpha(J_i \blacktriangledown \ker \alpha))] = 0$, and $\beta$ is well-defined.

It is easily verified that $\beta$ is an $l$-homomorphism. To see that $\beta$ is surjective, it is enough to note that $\alpha$ is surjective and that each finite partition of $Y$ in $\mathscr{F}^*$ is the image of a finite partition of $X$ in $\mathscr{F}$. The latter is true because each finite partition of $H$ in $\mathscr{P}(H)$ is the image under $\bar{\alpha}$ of some finite partition of $G$ in $\mathscr{P}(G)$. (This is an elementary fact about Boolean algebras.)

Finally, if $g \in G$, then

$$\beta g = \beta[g|X] = \beta[g|\eta S(G)] = [\alpha g|\eta^* S(\bar{\alpha} G)] = [\alpha g|\eta^* S(H)] = [\alpha g|Y] = \alpha g.$$

THEOREM 3.5. *If* $A \in \mathscr{P}(G)$, *then* $G^{SP} \simeq (G/A)^{SP} \oplus (G/A')^{SP}$.

*Proof.* Let $G \subseteq \prod_{x \in X} T_x$, $\eta$, and $\mathscr{F}$ be as in the construction of $G^{SP}$ in Section 2, and let $F = \eta S(A')$. Denote by $g|_F$ the element of $\prod_{x \in F} T_x$ such that $g|_F(x) = g(x)$ for all $x \in F$. Then $L = \{g|_F \,| g \in G\}$ is an $l$-subgroup of $\prod_{x \in F} T_x$ and $\alpha: G \to L$ by $\alpha g = g|_F$ is a surjective $l$-homomorphism. If $g \in A$, then $S(g) \cap S(A') = \emptyset$, and so by Lemma 2.2, $S(g) \cap F = \emptyset$, and hence $g \in \ker \alpha$. Moreover, if $g \in \ker \alpha$, then $S(g) \cap F = \emptyset$, and thus $S(g) \cap S(A') = \emptyset$ and $g \in A'' = A$. Thus $L \simeq G/\ker \alpha = G/A$.

By Lemmas 3.1 and 3.2 the polars in $L$ are of the form $\alpha J$ where $J \in \mathscr{P}(G)$ and $J \supseteq A$. Let $\mathscr{B}^* = \{S(\alpha(J)) | J \in \mathscr{P}(G) \text{ and } J \supseteq A\}$ and $\mathscr{F}^* = \{E \in \mathscr{F} | E \subseteq F\}$. Define $\eta^*: \mathscr{B}^* \to \mathscr{F}^*$ by $\eta^*(S(\alpha J)) = \eta S(J) \cap \eta S(A')$. If $E \in \mathscr{F}^*$, then $(X \backslash E) \cap F \in \mathscr{F}$, and hence there exists $C \in \mathscr{P}(G)$ such that $\eta S(C) = (X \backslash E) \cap F$. $C \subseteq A'$ since $\eta S(C) \subseteq \eta S(A') = F$. Thus $C' \supseteq A'' = A$, and

$$\eta^* S(\alpha(C')) = \eta S(C') \cap F = (X \backslash \eta S(C)) \cap F = (X \backslash (X \backslash E) \cap F)) \cap F = E,$$

since $E \subseteq F$. Thus $\eta^*$ is a surjective function. Also, it is clear that $\eta^*$ preserves inclusion.

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Suppose $I, J \in \mathscr{P}(G)$, $I \supseteq A$, $J \supseteq A$, and that $\eta^*S(\alpha I) \subseteq \eta^*S(\alpha J)$. Then $\eta S(I \cap A') = \eta(S(I) \cap S(A')) = \eta S(I) \cap \eta S(A') = \eta^*S(\alpha I) \subseteq \eta^*S(\alpha J) = \eta S(J \cap A')$, and thus $I \cap A' \subseteq J \cap A'$. Now since $I \supseteq A$ and $J \supseteq A$, we have $I = A \blacktriangledown I = (A \blacktriangledown I) \cap (A \blacktriangledown A') = A \blacktriangledown (I \cap A') \subseteq A \blacktriangledown (J \cap A') = J$. It follows that $\eta^*$ is one-to-one and that its inverse preserves inclusion.

Thus $\eta^*$ is a Boolean isomorphism of $\mathscr{B}^*$ onto $\mathscr{F}^*$. Hence

$$(G/A)^{SP} \simeq L^{SP} \simeq L[\mathscr{F}^*]$$

by Theorem 2.4. $L[\mathscr{F}^*]$ can be identified with $\varphi(F) = \{h \in G[\mathscr{F}] | S(h) \subseteq F\}$. Similarly, $(G/A')^{SP}$ is isomorphic to $\varphi(X \backslash F)$. By Lemma 1.3 we conclude $G[\mathscr{F}] = \varphi(F) \oplus \varphi(X\backslash F) \simeq (G/A)^{SP} \oplus (G/A')^{SP}$.

**4. The $P$-hull of an $l$-group.** In this section we generalize the results of Section 2, and we consider the $P$-hull of an $l$-group.

We assume $G$ is an $l$-subgroup of $\prod_{x \in X} T_x$, where each $T_x$ is a totally-ordered group, and that $\mathscr{A}$ is a subalgebra of $\mathscr{P}(G)$ such that, for each $g \in G$, $g''$ is an element of $\mathscr{A}$. We let $\mathscr{C} = \{S(A) | A \in \mathscr{A}\}$ and let $\eta : \mathscr{C} \to \mathscr{E}$ be a Boolean isomorphism onto a field $\mathscr{E}$ of subsets of $X$ (as in Lemma 1.1). $G[\mathscr{E}]$ is an $l$-subgroup of $\prod T_x$ by Lemma 1.2.

THEOREM 4.1. *$G$ is a dense $l$-subgroup of $G[\mathscr{E}]$, and if $A \in \mathscr{A}$, then $A'^{\perp}$ is a summand of $G[\mathscr{E}]$. If $H$ is an $l$-subgroup of $G$ containing $G$ which has the property that $A'^*$ is a summand of $H$ for each $A \in \mathscr{A}$, then $H = G[\mathscr{E}]$.* (Here $^{\perp}$ denotes polar in $G[\mathscr{E}]$, and $^*$ denotes polar in $H$.) *Moreover, these properties characterize $G[\mathscr{E}]$ up to isomorphism over $G$.*

The interested reader can without difficulty modify the proofs of Theorems 2.4 and 2.5 to obtain a proof of Theorem 4.1.

THEOREM 4.2. *$G[\mathscr{E}]$ is a $P$-group.*

*Proof.* Let $E \in \mathscr{E}$ and $g \in G$. Let $k \in G[\mathscr{E}]$ be given by $k(x) = g(x)$ if $x \in E$ and $k(x) = 0$ if $x \in X\backslash E$. We show $k^{\perp\perp} = \varphi(E \cap \eta S(g''))$ from which it follows by Lemma 1.3 that $k^{\perp\perp}$ is a summand of $G[\mathscr{E}]$.

Let $0 < f \in k^{\perp\perp}$. Then $f \wedge r = 0$ for all $r \in k^{\perp}$. Suppose that there exists $z \in (X\backslash E) \cap S(f)$. Let $r \in G[\mathscr{E}]$ be given by $r(x) = f(x)$ for all $x \in X\backslash E$ and $r(x) = 0$ for all $x \in E$. Then $r \in k^{\perp}$ but $f \wedge r > 0$. We conclude from this contradiction that $S(f) \subseteq E$. Also, $k^{\perp\perp} \subseteq g^{\perp\perp} = g'^{\perp}$, and hence $S(f) \subseteq S(g'^{\perp}) = \eta S(g'')$. Thus $S(f) \subseteq E \cap \eta S(g'')$, and $f \in \varphi(E \cap \eta S(g''))$.

On the other hand, suppose $0 < f \notin k^{\perp\perp}$. Then there exists $r \in k^{\perp}$ with $k \wedge r > 0$. We have $S(r) \cap E \cap S(g) = \emptyset$. Let $E = \eta S(J)$ where $J \in \mathscr{P}(G)$. Then $S(r) \cap S(J) \cap S(g) = \emptyset$, and hence $S(r) \cap S(J) \cap S(g'') = \emptyset$. Thus $S(r) \cap S(J \cap g'') = \emptyset$, and hence

$$S(r) \cap E \cap \eta S(g'') = S(r) \cap \eta S(J) \cap \eta S(g'') = S(r) \cap \eta(S(J) \cap S(g'')) = S(r) \cap \eta S(J \cap g'') = \emptyset,$$

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where the last equality is by (the appropriate analogue to) Lemma 2.2. Since $k \wedge r > 0$ we conclude there exists $x \in X \backslash (E \cap \eta S(g''))$ such that $k(x) \neq 0$. Thus $k \notin \varphi(E \cap \eta S(g''))$.

Finally, if $0 < h \in G[\mathscr{E}]$, then $h = k_1 \vee \ldots \vee k_n$ where each $k_i$ is like $k$ in the preceding paragraphs. Thus $h^{\perp\perp}$ is the join in $\mathscr{P}(G[\mathscr{E}])$ of $k_1^{\perp\perp}, \ldots, k_n^{\perp\perp}$. Since the cardinal summands of an $l$-group $H$ always form a subalgebra of $\mathscr{P}(H)$, we conclude $h^{\perp\perp}$ is a summand of $G[\mathscr{E}]$.

THEOREM 4.3. *If $\mathscr{A}$ is the subalgebra of $\mathscr{P}(G)$ generated by $\{g'' | g \in G\}$, then $G[\mathscr{E}]$ is the P-hull of G.*

*Remark.* The possibility of using the subalgebra of $\mathscr{P}(G)$ generated by $\{g'' | g \in G\}$ to produce the $P$-hull of $G$ was first utilized by D. Chambless [**4**] in his direct limit construction.

*Proof of Theorem* 4.3. $G$ is a dense $l$-subgroup of $G[\mathscr{E}]$ by Theorem 4.1, and by Theorem 4.2 $G[\mathscr{E}]$ is a $P$-group.

Suppose now that $K$ is an $l$-subgroup of $G[\mathscr{E}]$ containing $G$ and that $K$ is a $P$-group. Let $\mathscr{A}_1$ be the subalgebra of $\mathscr{P}(K)$ generated by $\{g^{**} | g \in G\}$, and $\mathscr{A}_2$ the subalgebra of $\mathscr{P}(G[\mathscr{E}])$ generated by $\{g^{\perp\perp} | g \in G\}$. (* denotes polar in $K$, and $^{\perp}$ denotes polar in $G[\mathscr{E}]$.)

Since $G[\mathscr{E}]$ and $K$ are essential extensions of $G$, the maps $A \mapsto A^{\perp\perp}$ and $A \mapsto A^{**}$ are Boolean isomorphisms of $\mathscr{A}$ onto $\mathscr{A}_2$ and $\mathscr{A}_1$, respectively. Since $G[\mathscr{E}]$ is a $P$-group, we conclude that $A^{\perp\perp}$ is a summand of $G[\mathscr{E}]$ for all $A \in \mathscr{A}$. Similarly, $A^{**}$ is a summand of $K$ for all $A \in \mathscr{A}$.

One can now imitate the argument in the body of the proof of Theorem 2.4 and get $K = G[\mathscr{E}]$. Thus $G[\mathscr{E}]$ is a $P$-hull of $G$.

THEOREM 4.4. *Let $\mathscr{A}$ and $\mathscr{E}$ be as in Theorem* 4.3. *If $M$ is a P-hull of G, then there exists an l-group isomorphism $\beta$ of $G[\mathscr{E}]$ onto $M$ with $\beta g = g$ for all $g \in G$. Thus G has a unique P-hull.*

*Proof.* Let $M$ be an $l$-subgroup of $\prod_{y \in Y} T_y$, where each $T_y$ is a totally-ordered group, and $S(M) = Y$. Let $\mathscr{C}^* = \{S(A) | A \in \mathscr{A}\}$, where $S(A)$i s taken in $Y$. The map $\mu$ by $\mu(A) = A^{\perp\perp}$ is a Boolean isomorphism of $\mathscr{A}$ into $\mathscr{P}(M)$. Also, $\mu(g'') = g''^{\perp\perp} = g^{\perp\perp}$ is a summand of $M$ for all $g \in G$, since $M$ is a $P$-group. Thus $A^{\perp\perp}$ is a summand of $M$ for all $A \in \mathscr{A}$, and hence $\mathscr{E}^* = \{S(A^{\perp\perp}) | A \in \mathscr{A}\}$ is a field of subsets of $Y$. Now $\eta^*: \mathscr{C}^* \rightarrow \mathscr{E}^*$ by $\eta^*(S(A)) = S(A^{\perp\perp})$ is a surjective Boolean isomorphism and $C \subseteq \eta^* C$ for all $C \in \mathscr{C}^*$. By Theorem 4.3, $G[\mathscr{E}^*]$ is a $P$-hull for $G$.

The remainder of the proof is exactly similar to the arguments used in proving Theorem 2.5.

The results in Section 2 extend easily to the general setting of this section. The same does not seem to be true for the results in Section 3.

Let $G$ be an $l$-subgroup of a product of totally-ordered groups. The only non-constructive step in our existence proofs for $G^{SP}$ and $G^P$ is the proof of Lemma 1.1.

In cases where the map $\eta$ of the Lemma can be produced constructively, we get a fairly concrete model of the $P$-hull or $SP$-hull. We give an example illustrating this possibility.

*Example.* The $P$-hull of the free vector lattice $FVL2$ on two generators.

Let $X = R^2\backslash\{0\}$ and let $W = \prod_{x \in X} R_x$. ($R$ denotes the real numbers; thus $X$ is the plane punctured at the origin.) Then $W$ is the vector lattice of all real-valued functions on $X$. By a cone in $X$ we mean a subset $K$ of $X$ such that $rk \in K$ whenever $k \in K$ and $0 < r \in R$. $K$ is an open (closed) cone in $X$ if and only if $K$ is a topologically open (closed) subset of $X$.

Let $H = \{f \in W | f$ is continuous, and there exist a finite number of closed cones $K_1 \ldots K_n$ in $X$ with $K_1 \cup \ldots \cup K_n = X$ and there exist linear functionals $f_1, \ldots, f_n : R^2 \rightarrow R$ such that $f(x) = f_i(x)$ for all $x \in K_i\}$. It was shown in [**1**] that $FVL2 \subseteq H$ and in [**3**] that $FVL2 = H$.

The collection $\{f'' | f \in FLV2\}$ is a Boolean subalgebra of $\mathscr{P}(FVL2)$. $\mathscr{C} = \{S(f'') | f \in FVL2\}$ consists of the regular open cones in $X$ that have only finitely many connected components. (See [**1**] for proofs of these last two sentences.) $\mathscr{C}$ is a Boolean algebra of subsets of $X$. Let $C \in \mathscr{C}$ with $C \neq \emptyset$ and $C \neq X$. Each component of $C$ has a boundary which consists of two rays, one at the clockwise-most extremity of the component, the other on the counter-clockwise side. We let $\eta C$ be the union of $C$ and the clockwise boundary rays of its components, and we let $\eta\emptyset = \emptyset$ and $\eta X = X$. Then $\eta$ is a Boolean homomorphism of $\mathscr{C}$ onto a field $\mathscr{E}$ of subsets of $X$ and $\eta C \supseteq C$ for $C \in \mathscr{C}$.

By Theorem 4.3, $H[\mathscr{E}]$ is the $P$-hull of $FVL2$. $\mathscr{E}$ consists of $\emptyset$, $X$, and all those cones in $X$ with finitely many components, each of which is closed on the clockwise side and open on the counter-clockwise side.

Now we can give the following nice description of the $P$-hull of $FVL2$: A function $f : X \rightarrow R$ is in the $P$-hull of $FVL2$ if and only if $f = 0$ or there exist a finite number of connected cones $E_1, \ldots, E_n$ in $X$ with $E_1 \cup \ldots \cup E_n = X$ and with each $E_i$ closed on the clockwise side and open on the counterclockwise side, and there exist linear functionals $f_1, \ldots, f_n : R^2 \rightarrow R$ such that $f(x) = f_1(x)$ for all $x \in E_i$.

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