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THE MAXIMUM GENUS OF CARTESIAN PRODUCTS OF GRAPHS

JOSEPH ZAKS

The *maximum genus* $\gamma_M(G)$ of a connected graph G has been defined in [**2**] as the maximum g for which there exists an embedding $h : G \rightarrow S(g)$, where $S(g)$ is a compact orientable 2-manifold of genus g, such that each one of the connected components of $S(g) - h(G)$ is homeomorphic to an open disk; such an embedding is called *cellular*. If G is cellularly embedded in $S(g)$, having V vertices, E edges and F faces, then by Euler's formula

$$V - E + F = 2 - 2g.$$

Let $\beta(G) = E - V + 1$ be the 1-dimensional Betti number of G (see [**1**]); since $F \geqq 1$ and g is an integer, the following holds (see [**2**, Theorem 3]).

THEOREM A. *If G is a connected graph, then $\gamma_M(G) \leqq [\beta(G)/2]$, with equality holding if and only if the embedding has one or two faces according to $\beta(G)$ being even or odd, respectively* (*$[x]$ is the largest integer $\leqq x$*).

The following results are known:

THEOREM B. (see [**2**]). *The maximum genus of the complete graph K_n on n vertices is given by*

$$\gamma_M(K_n) = \left[\frac{(n-1)(n-2)}{4}\right].$$

THEOREM C. (see [**4**]). *The maximum genus of the complete bipartite graph $K_{n,m}$ on n and m vertices is given by*

$$\gamma_M(K_{n,m}) = \left[\frac{(n-1)(m-1)}{2}\right].$$

A connected graph G is called *upperembeddable* (see [**5**]) if $\gamma_M(G) = [\beta(G)/2]$. Theorems B and C state that both K_n and $K_{n,m}$ are upperembeddable, for all $n \geqq 1$ and $m \geqq 1$.

In the recent Conference on Graph Theory and Applications, held at Kalamazoo, Michigan, May 1972, E. A. Nordhaus raised the conjecture that the graph Q_n of the n-cube is upperembeddable. It is the purpose of this paper to present an affirmative answer to this conjecture (Corrollary 1, here), together with some more general results.

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Recall [7] that a *1-factor* F of a graph G is a subgraph of G that contains all the vertices of G, each one with valence 1; a *maximum matching* F of a graph G is a subgraph of G that contains all the vertices of G, each one with valence 0 or 1, and has the maximum possible number of edges; a vertex of valence 0 in a maximum matching is called *isolated* (see [**10**]).

The *Cartesian product* $G \times H$ of the two graphs G and H has been defined in [**6**] (see also [**8**] and [**9**]) as follows: Let $V(K)$ and $E(K)$ denote the set of vertices and the set of edges of the graph K; then

$$V(G \times H) = V(G) \times V(H) = \{(g, h) | g \in G, h \in H\};$$
$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) | g_1 = g_2 \text{ and } h_1h_2 \in E(H) \text{ or else } h_1 = h_2 \text{ and } g_1g_2 \in E(G)\}.$$

Observe that $Q_1 = K_2$ and that inductively $Q_{n+1} = Q_n \times K_2$. Let $\bar{A}$ denote the cardinality of the set A.

The following are our main results.

THEOREM 1. *If G and H are nonempty connected graphs and G has a 1-factor, then*

$$\gamma_M(G \times H) \geqq \overline{V(H)}\, \gamma_M(G) + \tfrac{1}{2}\, \overline{E(H)}\, \overline{V(G)} - \overline{V(H)} + 2,$$

provided that either

(1) $\overline{V(H)} \geqq 3$, *or else*

(2) $H = K_2$ *and G has a cellular embedding into $S(\gamma_M(G))$ such that one edge of G that belongs to two different faces is an edge of some 1-factor of G; in this case* $\gamma_M(G \times K_2) \geqq 2\gamma_M(G) + \frac{1}{2}\overline{V(G)}$.

Observe that if $\beta(G)$ is odd and every edge of G belongs to some 1-factor of G, then G satisfies the condition as described in part 2 of Theorem 1; as a particular case of part 2 of Theorem 1, applied to $G = Q_{n-1}$ and H $= K_2$, we get the following.

COROLLARY 1. $\gamma_M(Q_n) = (n - 2)2^{n-2}$, *for all* $n \geqq 2$.

THEOREM 2. *If a nonempty connected graph G has a 1-factor, then*

$$\gamma_M(G \times K_2) \geqq 2\gamma_M(G) + \tfrac{1}{2}\overline{V(G)} - 1.$$

THEOREM 3. *If G and H are nonempty connected graphs and G has a maximum matching that has exactly one isolated vertex, then*

$$\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{E(H)}\, (\overline{V(G)} - 1).$$

For similar results concerning the (minimum) genus of the Cartesian products of graphs, see [**8**; **9**].

Four main lemmas. The following are the main tool for proving the stated theorems.

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LEMMA 1. *If G_1 and G_2 are connected graphs, $E_i = u_iv_i \in E(G_i)$, $i = 1, 2$, and $h_i : G_i \to S_i(n_i)$ are cellular embeddings, $i = 1, 2$, then there exists a $S(n_1 + n_2)$ and a cellular embedding $\bar{h}$: $G_1 \cup G_2 \cup u_1u_2 \cup v_1v_2 \to S(n_1 + n_2)$, such that*

(a) *$S(n_1 + n_2) \cap S_i(n_i) = T_i(n_i)$ is just $S_i(n_i)$ minus an open disk, $i = 1, 2$;*

(b) *$\bar{h}|_{G_i} = h_i$, $i = 1, 2$;*

(c) *if $y_i \in T_i(n_i) - h_i(G_i)$, $i = 1, 2$, then y_1 and y_2 are not in one face of $\bar{h}(\ldots)$ in $S(n_1 + n_2)$.*

LEMMA 2. *If $h : G \to S(n)$ is a cellular embedding, $E_i = u_iv_i \in E(G)$, $i = 1, 2$, with $E_1 \cap E_2 = \emptyset$, and $u_1u_2 \notin E(G)$, $v_1v_2 \notin E(G)$, then there exists a $S(n + 1)$, and a cellular embedding $\bar{h}$: $G \cup u_1u_2 \cup v_1v_2 \to S(n + 1)$, such that*

(a) *$S(n + 1) \cap S(n) = T(n)$ is just $S(n)$ minus two disjoint open disks;*

(b) *$\bar{h}|_G = h$;*

(c) *if y_1 and $y_2 \in T(n) - h(G)$ and they belong to two different faces of $h(G)$, then they belong to two different faces of $\bar{h}(\ldots)$ in $S(n + 1)$.*

LEMMA 3. *If $h : G \to S(n)$ is a cellular embedding, $E_i = u_iv_i \in E(G)$, $i = 1, 2$, $u_1u_2 \notin E(G)$, $v_1v_2 \notin E(G)$ and both $h(E_1)$ and $h(E_2)$ are in the boundary of two different faces of $h(G)$, then there exists a $S(n + 2)$ and a cellular embedding $\bar{h}$: $G \cup u_1u_2 \cup v_1v_2 \to S(n + 2)$, such that*

(a) *$S(n + 2) \cap S(n)$ is just $S(n)$ less four pairwise disjoint open disks;*

(b) *$\bar{h}|_G = h$.*

LEMMA 4. (compare with [**3**, Theorem 2]). *If G_1 and G_2 are connected graphs, $v_i \in V(G_i)$, $i = 1, 2$, and $h_i : G_i \to S_i(n_i)$ are cellular embeddings, $i = 1, 2$, then there exists a $S(n_1 + n_2)$ and a cellular embedding $\bar{h} : G_1 \cup G_2 \cup v_1v_2 \to S(n_1 + n_2)$, such that*

(a) *$S(n_1 + n_2) \cap S_i(n_i)$ is just $S_i(n_i)$ less an open disk, $i = 1, 2$;*

(b) *$\bar{h}|_{G_i} = h_i$, $i = 1, 2$.*

Remark 1. These Lemmas are similar to [**3**, Theorem 2], and [**4**, Theorem, p. 101], quoted from [**2**]; however we need them in these forms so as to be able to continue our constructions in the proofs of our theorems.

Proof of Lemma 1. Let E_1' and E_2' be simple paths in $S_1(n_1)$ and $S_2(n_2)$ such that $E_i' \cup h_i(E_i)$ is a simple closed curve, $i = 1, 2$, meeting $h_i(G_i)$ at $h_i(E_i)$. E_i' has its interior in one of the connected components A_i of $S_i(n_i) - h_i(G_i)$, $i = 1, 2$. A_i is simply connected since h_i is a cellular embedding; let B_i be the disk in A_i, bounded by $E_i' \cup h_i(E_i)$, $i = 1, 2$.

Let S^1 denote a simple closed curve and let I denote the closed unit interval; the topological Cartesian product $S^1 \times I$ is, of course, a cylinder. Let $x, y \in S^1$, with $x \neq y$.

Let

$$\varphi : S^1 \times \{0, 1\} \to (E_1' \cup h_1(E_1)) \cup (E_2' \cup h_2(E_2))$$

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be an orientation preserving homeomorphism (between two pairs of disjoint simple closed curves), such that $\varphi(x, 0) = u_1$, $\varphi(y, 0) = v_1$, $\varphi(x, 1) = u_2$ and $\varphi(y, 1) = v_2$.

We assume, without loss of generality, that $S_1(n_1) \cap S_2(n_2) = \emptyset$. $S(n_1 + n_2)$ is defined as follows: remove the interiors of B_1 and B_2 from $S_1(n_1) \cup S_2(n_2)$ and then attach to it the cylinder $S^1 \times I$ by identifying z of $S^1 \times \{0, 1\}$ with $\varphi(z)$.

Clearly, $S(n_1 + n_2) \cap S_i(n_i) = T_i(n_i) = S_i(n_i) - B_i$, $i = 1, 2$. The embedding $\bar{h}$ of $G_1 \cup G_2 \cup u_1u_2 \cup v_1v_2$ into $S(n_1 + n_2)$ is defined as follows:

$\bar{h}|_{G_1} = h_1$ and $\bar{h}|_{(G_2 - E_2)} = h_2$ as maps, while on E_2, u_1u_2 and v_1v_2 $\bar{h}$ is defined in such a way that (as sets)

$$\bar{h}(E_2) = E_2',\ \bar{h}(u_1u_2) = \{x\} \times I \subset S^1 \times I \text{ and } \bar{h}(v_1v_2) = \{y\} \times I \subset S^1 \times I.$$

To show that $\bar{h}$ is cellular, observe that the 2-cell A_1 of $h_1(G_1)$ in $S_1(n_1)$ is changed into $(A_1 - \mathrm{int}B_1) \cup \alpha \times [0, 1)$, where α is the arc of S^1 from x to y for which $\varphi(\alpha \times \{0\}) = E_1'$. As for the change in A_2, let A_2^* be the other (with the possibility that $A_2^* = A_2$) 2-cell of $h_2(G_2)$ in $S_2(n_2)$ that has E_2 on its boundary; if $A_2^* \neq A_2$, then A_2^* is replaced by $A_2^* \cup ((S^1 - \alpha) \times (0, 1])$ and A_2 becomes $A_2 - B_2$; while if $A_2^* = A_2$, then A_2 is replaced by $(A_2 - B_2) \cup ((S^1 - \alpha) \times (0, 1])$; the rest of the 2-cells of $S(n_1 + n_2) - \bar{h}(G_1 \cup G_2 \cup u_1u_2 \cup v_1v_2)$ are among the 2-cells of $(S_1(n_1) - h_1(G_1)) \cup (S_2(n_2) - h_2(G_2))$. It follows that $\bar{h}$ is cellular. In addition, no face of $T_1(n_1)$ is joined to a face of $T_2(n_2)$ so as to form part of a face of $\bar{h}(G_1 \cup G_2 \cup u_1u_2 \cup v_1v_2)$ in $S(n_1 + n_2)$.

This completes the proof of Lemma 1.

Proof of Lemma 2. The proof is similar to the proof of Lemma 1 (hence the details are omitted), and it amounts to deleting two open disks B_1 and B_2 from $S(n)$, adding a cylinder $S^1 \times I$ with the use of a similar identification, and shifting one edge (E_2) around the cylinder. This shifting assures that the two halves of the cylinder are attached to two faces A_1 and A_2^* (using similar notations; with $A_1 = A_2^*$ possible), such that one of them is attached along $\alpha \times \{0\}$ and the other — along $(S^1 - \alpha) \times \{1\}$; therefore each face is cellular and no two faces of $T(n) = S(n) - (B_1 \cup B_2)$ merge into one face of $S(n + 1)$.

Proof of Lemma 3. Let $h(E_1)$ be on the boundary of the two different faces F_1 and F_2 of $h(G)$ in $S(n)$, and let $h(E_2)$ be on the boundary of the two different faces P_1 and P_2 of $h(G)$ in $S(n)$. ($\{F_1, F_2\} \cap \{P_1, P_2\}$ need not be empty!). Let D_i be a disk in F_i, $i = 1, 2$, and let D_{2+j} be a disk in P_j, $j = 1, 2$, such that $\mathrm{bd}D_1 \cap \mathrm{bd}F_1 = h(u_1)$, $\mathrm{bd}D_2 \cap \mathrm{bd}F_2 = h(v_1)$, $\mathrm{bd}D_3 \cap \mathrm{bd}P_1 = h(u_2)$, $\mathrm{bd}D_4 \cap \mathrm{bd}P_2 = h(v_2)$, and all the disks have pairwise disjoint interiors. Since $F_1 \neq F_2$ and $P_1 \neq P_2$, we may assume without loss of generality that $F_1 \neq P_1$ and $F_2 \neq P_2$.

First operation. Let $\varphi_1 : S^1 \times \{0, 1\} \to \mathrm{bd}D_1 \cup \mathrm{bd}D_3$ be a homeomorphism, such that for some point x of S^1, $\varphi_1(x, 0) = h(u_1)$ and $\varphi_1(x, 1) = h(u_2)$.

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Remove the interiors of D_1 and D_3 from $S(n)$ and attach a handle $S^1 \times I$ by identifying z of $S^1 \times \{0, 1\}$ with $\varphi_1(z)$; an $S(n+1)$ is obtained (adjust $\varphi_1|_{S^1\times\{0\}}$ and $\varphi_2|_{S^1\times\{0\}}$, if necessary, to get an orientable surface), $h_0 : G \cup u_1u_2 \to S(n+1)$ is defined by $h_0|_G = h$ as maps, where $S(n+1) \cap S(n) = S(n) - (\mathrm{int}D_1 \cup \mathrm{int}D_3)$, and $h_0(u_1u_2) = \{x\} \times I$ as sets. The only changes in the faces are to replace F_1 and P_1 by exactly one face that has

$$(F_1 - \mathrm{int}D_1) \cup (P_1 - \mathrm{int}D_3) \cup ((S^1 - \{x\}) \times I)$$

for its interior; therefore h_0 is cellular.

Second operation. Let $\varphi_2 : S^1 \times \{0, 1\} \to \mathrm{bd}D_2 \cup \mathrm{bd}D_4$ be a homeomorphism, such that for some point y of S^1, $\varphi_2(y, 0) = h(v_1)$ and $\varphi_2(y, 1) = h(v_2)$. Remove the interiors of D_2 and D_4 from $S(n+1)$ and attach a handle $S^1 \times I$ by identifying z of $S^1 \times \{0, 1\}$ with $\varphi_2(z)$; an $S(n+2)$ is obtained. A map $\bar{h} : G \cup u_1u_2 \cup v_1v_2 \to S(n+2)$ is defined by $\bar{h}|_{G\cup u_1u_2} = h_0$ as maps and $\bar{h}(v_1v_2) = \{y\} \times I$ as sets (where $\{y\} \times I$ is taken, of course, along the second added handle). $F_2 \neq P_2$ implies, as in the previous case, that $\bar{h}$ is cellular.

This completes the proof of Lemma 3.

Proof of Lemma 4. Add one handle $S^1 \times I$ to the disjoint union of $S_1(n_1)$ and $S_2(n_2)$, with a suitable deleting of two open disks and a corresponding identification, as done in the proof of the previous lemma. The new edge v_1v_2 is embedded as $\{x\} \times I$, where $\{x\} \times \{0\}$ of $S^1 \times I$ is identified with $h_1(v_1)$ and $\{x\} \times \{1\}$ is identified with $h_2(v_2)$.

We are ready for the proofs of the main results.

Proof of Theorem 1. In case 1, let $V(H) = \{v_1, \ldots, v_k\}$, $k \geqq 3$, and $\gamma_M(G) = \lambda$. Let $h_i : G \to S_i(\lambda)$ be cellular embeddings, where $S_i(\lambda)$ is a $S(\lambda)$ for all $1 \leqq i \leqq k$. Let $E = x_1x_2$ be an edge of a 1-factor F of G, and let $T_0 \subset T_1 \subset \ldots \subset T_{k-1}$ be subtrees of a spanning tree T of H, with $\overline{E(T_j)} = j$, for all $0 \leqq j \leqq k-1$.

Use Lemma 1 k − 1 times to get a cellular embedding of $(G \times V(H)) \cup (E \times T)$ into a $S(k\lambda)$, as follows: if $y_1y_2 \in E(T_1)$, then apply Lemma 1 with $G_i = G \times \{y_i\}$, $i = 1, 2$, $E_i = E \times \{y_i\}$ to get a particular cellular embedding of $(G \times V(T_1)) \cup (E \times T_1)$ into $S(2\lambda)$, and continue inductively as follows: if $(G \times V(T_{j-1})) \cup (E \times T_{j-1})$ has been cellularly embedded into $S(j\lambda)$, $j \geqq 2$, apply Lemma 1 once more with $G_1 = (G \times V(T_{j-1})) \cup (E \times T_{j-1})$ and $G_2 = G \times \{z_2\}$, where $z_1z_2 \in E(T_j) - E(T_{j-1})$ and $z_2 \notin V(T_{j-1})$, and with $E_i = E \times \{z_i\}$, $i = 1, 2$; this yields a particular cellular embedding of $(G \times V(T_j)) \cup (E \times T_j)$ into a $S((j+1)\lambda)$.

To the embedding of $(G \times V(H)) \cup (E \times T)$ into $S(k\lambda)$ we apply, again one at a time, Lemma 2 for each one of the possible $\overline{E(F)} \cdot \overline{E(H)} - (k-1)$ choices of an edge X of F and an edge Y of H, except for those $k-1$ combinations of the edge E of F and an edge of T. The two edges E_1 and E_2 of Lemma 2 are, of course, $X \times \{y_1\}$ and $X \times \{y_2\}$, where $y_1y_2 = Y$.

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We have just cellularly embedded $G \times H$ into $S(t)$, where

$$t = k\lambda + \overline{E(F)}\,\overline{E(H)} - (k-1)$$
$$= \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{V(G)} \cdot \overline{E(H)} - \overline{V(H)} + 1;$$

therefore

$$\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{V(G)} \cdot \overline{E(H)} - \overline{V(H)} + 1.$$

This embedding has the additional property that if z is a vertex of H of valence $\geqq 2$ (the existence of which follows from the connectivity of H and the requirement that $\overline{V(H)} \geqq 3$), then $\{v\} \times \{z\}$ (where v is a vertex of G) is a vertex of $G \times H$ that belongs to at least three faces; to see this, consider the two edges of H incident to z and follow the construction of Lemmas 1 and/or 2. By a theorem of Duke [**1**] (see also [**3**, Theorem 3]), it follows that $G \times H$ is cellularly embeddable in a sphere with one more handle; this completes the proof of case 1 of our Theorem.

In case 2, let $V(H) = \{x_1, x_2\}$ and let $h : G \rightarrow S(\gamma_M(G))$ be a cellular embedding with $h(A_1)$ belonging to two different faces of $h(G)$, for some edge A_1 of G; let F be a 1-factor of G that contains A_1.

Let $h_i : G \rightarrow S_i(\gamma_M(G)), i = 1, 2$, be two reproductions of h, where $S_i(\gamma_M(G))$ are disjoint spheres with $\gamma_M(G)$ handles.

In this case $\overline{V(G)} \geqq 4$, and hence $\overline{E(F)} \geqq 2$; let $A_2 \in E(F) - A_1$, and apply Lemma 1 to $G_i = h_i(G)$, $i = 1, 2$, with $E_i = h_i(A_2)$, $i = 1, 2$; successively apply Lemma 2 $\overline{E(F)} - 2$ times for the edges $A_j \times \{x_1\}$ and $A_j \times \{x_2\}$ of

$$(G \times V(K_2)) \cup \left(\bigcup_{i=2}^{j-1} V(A_i) \times E(K_2)\right),$$

for $j = 3, \ldots, \overline{E(F)}$, where $\{A_2, A_3, \ldots, A_{\overline{E(F)}}\} = E(F) - A_1$. The last step is to apply Lemma 3 to $(G \times V(K_2)) \cup [(V(G) - V(A_1)) \times E(K_2)]$, where the two edges E_1 and E_2 are of course $A_1 \times \{x_1\}$ and $A_1 \times \{x_2\}$; both of $A_1 \times \{x_1\}$ and $A_1 \times \{x_2\}$ belong each to two different faces, since this property is preserved under each one of the applications of Lemmas 1 and 2.

The edge A_2 of F was used to connect $S_1(\gamma_M(G))$ to $S_2(\gamma_M(G))$; the remaining $\overline{E(F)} - 2$ edges of F were adding one handle each, while A_1 was used last to add two more handles; therefore

$$\gamma_M(G \times K_2) \geqq 2\gamma_M(G) + \tfrac{1}{2}\overline{V(G)},$$

and the proof of Theorem 1 has been completed.

Proof of Theorem 2. The proof is similar to the proof of case 2 of Theorem 1 (hence the details are omitted), the only difference being that in the last step we apply again Lemma 2 rather than Lemma 3, so as to get one less handle.

Proof of Theorem 3. Let T be a spanning tree of H and let M be a maximum matching of G with the only isolated vertex v. Let $T_1 \subset T_2 \subset \ldots \subset T_{\overline{E(T)}} = T$ be subtrees of T with $\overline{E(T_j)} = j$ for all $1 \leqq j \leqq \overline{E(T)}$. Let $h : G \rightarrow S(\gamma_M(G))$ be a cellular embedding, and take $h_i : G \rightarrow S_i(\gamma_M(G))$, for $i = 1, \ldots, \overline{V(H)}$, $\overline{V(H)}$ copies of h, with $S_i(\gamma_M(G)) \cap S_j(\gamma_M(G)) = \emptyset$, for all $1 \leqq i < j \leqq \overline{V(H)}$ Apply Lemma 4 to get a particular cellular embedding of $(G \times V(T_1)) \cup (\{v\} \times T_1)$ into $S(2\gamma_M(G))$ and continue applying it $\overline{E(T)} - 1$ more times to get a particular cellular embedding of $G \times V(H) \cup \{v\} \times T$ into a $S(\overline{V(H)} \cdot \gamma_M(G))$.

If $A = a_1a_2 \in E(H) - E(T)$, then we add the new edge $\{v\} \times A$ as follows: if $\{v\} \times \{a_1\}$ and $\{v\} \times \{a_2\}$ belong to the same face of the embedding (of $G \times V(H) \cup \{v\} \times T$ into $S(\overline{V(H)} \cdot \gamma_M(G))$), then take a simple arc α in that face, connecting these two end points, to be the image of $\{v\} \times A$; if they belong to different faces, then delete two open disks, one in each of the faces, and attach a handle so as to merge the two faces into one, while embedding the extra arc $\{v\} \times A$ along that handle. Do it $\overline{E(H)} - \overline{E(T)} = \overline{E(H)} - \overline{V(H)} + 1$ times to get a particular cellular embedding of $G \times V(H) \cup \{v\} \times H$ into a $S(t)$, for some integer $t \geqq \overline{V(H)} \cdot \gamma_M(G)$.

Apply Lemma 2 $\overline{E(M)} \cdot \overline{E(H)}$ successive times, one for each possible choice of an edge of M and an edge of H, as in the proof of Theorem 1, and a cellular embedding is obtained, taking $G \times H$ into $S(r)$, where $r \geqq \overline{V(H)} \cdot \gamma_M(G) + \frac{1}{2}(\overline{V(G)} - 1)\, \overline{E(H)}$. It follows that

$$\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}(\overline{V(G)} - 1)\, \overline{E(H)},$$

and Theorem 3 has been proven.

Proof of Corollary 1. The proof is by induction on n, starting with $n = 2$: by Theorem A, $\gamma_M(Q_2) \leqq [(4 - 4 + 1)/2] = 0$, hence $\gamma_M(Q_2) = 0$, as needed. Suppose, inductively, that for some n, $n \geqq 2$, $\gamma_M(Q_n) = (n - 2)2^{n-2}$. As is well-known, $Q_{n+1} = Q_n \times K_2$, $\overline{V(Q_n)} = 2^n$ and $\overline{E(Q_n)} = n2^{n-1}$; both of these two numbers are even; therefore it follows that in every cellular embedding of Q_n in $S(\lambda)$, the number of faces (by Euler's formula) is $\overline{E(Q_n)} - \overline{V(Q_n)} + 2(1 - \lambda)$, which is even and $\geqq 2$. Consider a cellular embedding of Q_n into $S((n - 2)2^{n-2})$; it has at least two faces; hence at least one edge E of Q_n in that embedding belongs to two different faces of the embedding. Every edge of Q_n belongs, quite elementarily, to a 1-factor of Q_n. Therefore case 2 of Theorem 1 is applicable to $Q_n \times K_2$, and it follows that

$$\begin{aligned}\gamma_M(Q_{n+1}) &= \gamma_M(Q_n \times K_2)\\ &\geqq \overline{V(K_2)} \cdot \gamma_M(Q_n) + \tfrac{1}{2}\overline{E(K_2)} \cdot \overline{V(Q_n)} - \overline{V(K_2)} + 2\\ &= 2(n - 2)2^{n-2} + \tfrac{1}{2}2^n\\ &= (n - 2)2^{n-1} + 2^{n-1}\\ &= (n - 1)2^{n-1}.\end{aligned}$$

On the other hand, Theorem A implies that

$$\begin{aligned}\gamma_M(Q_{n+1}) &\leqq \left[\frac{\overline{E(Q_{n+1})} - \overline{V(Q_{n+1})} + 1}{2}\right]\\ &= \left[\frac{(n+1)2^n - 2^{n+1} + 1}{2}\right]\\ &= \left[\frac{(n-1)2^n + 1}{2}\right]\\ &= (n-1)2^{n-1};\end{aligned}$$

as a result $\gamma_M(Q_{n+1}) = (n-1)2^{n-1}$, and the proof of Corollary 1 is complete.

Corollaries.

COROLLARY 2. *If G is a connected graph and every edge of G belongs to a 1-factor of G, then $G \times Q_n$ is upperembeddable for all $n \geqq 1$, provided G is upperembeddable.*

Proof. Suppose, first, that $\overline{E(G)}$ is even, G being an upperembeddable graph with a 1-factor; any cellular embedding of G into a $S(\lambda)$ (hence, in particular, into a $S(\gamma_M(G))$) has an even number of faces, as follows from Euler's Formula. Applying part 2 of Theorem 1 to $G \times K_2$, we get

$$\gamma_M(G \times K_2) \geqq 2\gamma_M(G) + \tfrac{1}{2}\overline{V(G)}.$$

G is upperembeddable, $\overline{E(G)}$ and $\overline{V(G)}$ are even; therefore

$$\gamma_M(G) = \left[\frac{\beta(G)}{2}\right] = \frac{\overline{E(G)} - \overline{V(G)}}{2}$$

and hence

$$\begin{aligned}\gamma_M(G \times K_2) &\geqq 2\,\frac{\overline{E(G)} - \overline{V(G)}}{2} + \tfrac{1}{2}\overline{V(G)}\\ &= \frac{2\overline{E(G)} - \overline{V(G)}}{2}\\ &= \left[\frac{\beta(G \times K_2)}{2}\right].\end{aligned}$$

On the other hand, $\gamma_M(G \times K_2) \leqq [\beta(G \times K_2)/2]$, by Theorem A; therefore $G \times K_2$ is upperembeddable. Clearly, every edge of $G \times K_2$ belongs to some 1-factor of $G \times K_2$, and both $\overline{E(G \times K_2)}$ and $\overline{V(G \times K_2)}$ are even; hence $G \times Q_n$ is, by induction on n, upperembeddable for all $n \geqq 1$.

In case $\overline{E(G)}$ is odd $\gamma_M(G) = \tfrac{1}{2}(\overline{E(G)} - \overline{V(G)} + 1)$ and it follows by

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Theorem 2 that

$$\begin{aligned}\gamma_M(G \times K_2) &\geqq 2\gamma_M(G) + \tfrac{1}{2}\overline{V(G)} - 1 \\ &= \overline{E(G)} - \overline{V(G)} + \tfrac{1}{2}\overline{V(G)} \\ &= \tfrac{1}{2}(2\overline{E(G)} + \overline{V(G)} - 2\overline{V(G)}) \\ &= \tfrac{1}{2}(\overline{E(G \times K_2)} - \overline{V(G \times K_2)}) \\ &= [\beta(G \times K_2)/2],\end{aligned}$$

where the last equality is due to the eveness of both $\overline{E(G \times K_2)}$ and $\overline{V(G \times K_2)}$. Since the other inequality is given by Theorem A, it follows that $\gamma_M(G \times K_2) = [\beta(G \times K_2)/2]$, and $G \times K_2 = G \times Q_1$ is upperembeddable. The rest of the proof is as in the first case; hence Corollary 2 has been proven.

As particular cases, we have

COROLLARY 3. *$K_{2n} \times Q_m$ and $K_{n,n} \times Q_m$ are upperembeddable for all $n \geqq 1$ and $m \geqq 1$.*

COROLLARY 4. *$K_{4n+1} \times Q_m$ is upperembeddable and*

$$2^{m-2}(16n^2 + 12n + 4mn + 2m) \leqq \gamma_M(K_{4n+3} \times Q_m) \leqq \begin{cases} 2^{m-2}(16n^2 + 12n + 4mn + 3m) & \text{if } m \text{ is even} \\ 2^{m-2}(16n^2 + 12n + 4mn + 3m + 1) & \text{if } m \text{ is odd,}\end{cases}$$

for all $n \geqq 1$ and $m \geqq 1$.

Proof of Corollary 4. Using Theorems A and 3, it follows that $K_{4n+1} \times K_2$ is upperembeddable, with $\gamma_M(K_{4n+1} \times K_2) = 8n^2$; Corollary 2 applied to $K_{4n+1} \times K_2$ $(= K_{4n+1} \times Q_1)$ shows that $K_{4n+1} \times Q_m$ is upperembeddable for all $n \geqq 1$ and $m \geqq 1$. The inequalities for $\gamma_M(K_{4n+3} \times Q_m)$ follow from Theorem A and Theorem 3.

COROLLARY 5. *If G is a connected upperembeddable graph with an even number of edges and a maximum matching that has exactly one isolated vertex, then for every connected graph H*

$$\frac{\beta(G \times H)}{2} - \frac{\beta(H)}{2} \leqq \gamma_M(G \times H) \leqq \left[\frac{\beta(G \times H)}{2}\right];$$

in particular, $G \times T$ is upperembeddable for all trees T (and G as stated).

Proof. It follows from Theorem 3 that

$$\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{E(H)} \cdot (\overline{V(G)} - 1).$$

Since $\overline{E(G)}$ is even, $\overline{V(G)}$ is odd and G is upperembeddable, we have $\gamma_M(G) =$

$\frac{1}{2}(\overline{E(G)} - \overline{V(G)} + 1)$; hence

$$\gamma_M(G \times H) \geqq \overline{V(H)}\,\frac{\overline{E(G)} - \overline{V(G)} + 1}{2} + \tfrac{1}{2}\overline{E(H)}(\overline{V(G)} - 1)$$
$$= \tfrac{1}{2}[\overline{V(G)}\,\overline{E(H)} + \overline{E(G)}\,\overline{V(H)} - \overline{V(H)}\,\overline{V(G)} + 1]$$
$$- \tfrac{1}{2}(\overline{E(H)} - \overline{V(H)} + 1)$$
$$= \frac{\beta(G \times H)}{2} - \frac{\beta(H)}{2}.$$

The other inequality is obtained, again, by Theorem A.

If H is a tree, $\beta(H) = 0$; hence

$$\frac{\beta(G \times H)}{2} \leqq \gamma_M(G \times H) \leqq \left[\frac{\beta(G \times H)}{2}\right],$$

and since $[x] \leqq x$ for all x, equality holds and $G \times H$ is upperembeddable (observe that $\beta(G \times H)$ is an even number in the last case). This completes the proof of Corollary 5.

As particular cases, we have

COROLLARY 6. *$K_{2n+3} \times T$ and $K_{n,n+1} \times T$ are upperembeddable for all $n \geqq 1$ and all trees T.*

Remark 2. If G and H are connected graphs and a maximum matching of G has m isolated vertices, $m \geqq 1$, then

$$\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{E(H)}(\overline{V(G)} - m).$$

The proof is similar to the proof of Theorem 3 and is omitted.

Remark 3. The appearance of the term $\overline{V(G)} \cdot \gamma_M(G)$ in our theorems is quite natural, since $G \times H$ contains a connected subgraph G' of the form $G \times V(H) \cup \{v\} \times T$, where $v \in V(G)$ and T is a spanning tree in H; $\gamma_M(G') = \overline{V(H)} \cdot \gamma_M(G)$, by [**3**, Theorem A]; hence $\gamma_M(G \times H) \geqq \overline{V(H)} \cdot \gamma_M(G)$ by [**2**, Theorem 2].

Remark 4. The strong Cartesian product $G \,\overline{\times}\, H$ of G and H has been defined in [**6**] (see also [**9**]), as $G \times H \cup \{(u_1, v_2)(u_2, v_1) | u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)\}$. Treating each pair of edges of $G \,\overline{\times}\, H$ of the form $(u_1, v_2)(u_2, v_1)$ and $(u_1, v_1)(u_2, v_2)$ (which, of course, are not edges of $G \times H$), we get, by a procedure similar to that of Theorem 1, that the following holds:

"If G and H are connected graphs and G has a 1-factor, then

$$\gamma_M(G \,\overline{\times}\, H) \geqq \overline{V(H)} \cdot \gamma_M(G) + \tfrac{1}{2}\overline{E(H)}\,\overline{V(G)} - \overline{V(H)} + 2 + \overline{E(H)}\,\overline{E(G)}\text{''}.$$

Apology. In trying to keep the geometric flavor of the subject, we did not use Edmond's technique (see [**1**; **2**; **3**]), except, of course when using results from [**1**; **2**; **3**; **4**].

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Michigan State University,
East Lansing, Michigan;
University of Haifa,
Haifa, Israel