BULL. AUSTRAL. MATH. SOC. MOS 3504, 3565, 6568
VOL. 1 (1969), 363-374.

# A class of densely invertible parabolic operator equations

**R. S. Anderssen**

Before variational methods can be applied to the solution of an initial boundary value problem for a parabolic differential equation, it is first necessary to derive an appropriate variational formulation for the problem. The required solution is then the function which minimises this variational formulation, and can be constructed using variational methods. Formulations for K-p.d. operators have been given by Petryshyn. Here, we show that a wide class of initial boundary value problems for parabolic differential equations can be related to operators which are densely invertible, and hence, K-p.d. ; and develop a method which can be used to prove dense invertibility for an even wider class. In this way, the result of Adler on the non-existence of a functional for which the Euler-Lagrange equation is the simple parabolic is circumvented.

## 1. Introduction

The use of direct methods such as Ritz and Galerkin for the solution of a (differential) operator equation $Au = f$ , $f$ an element of some real and separable Hilbert space $\underline{H}$ , requires the existence of a functional $F(u)$ such that the solution of the operator equation and the function which minimises $F(u)$ are equivalent. In the classical literature (Courant and Hilbert [2, Chapter IV] and Mikhlin [3, Chapter III]), such a correspondence is defined by the energy functional $F(u) = (Au,u) - (u,f) - (f,u)$ for which the corresponding operator

Received 9 July 1969. The author wishes to thank Professors J.R.M. Radok and R. Výborný, the former for suggesting and stimulating this research, the latter for his valuable criticism throughout its preparation.



https://doi.org/10.1017/S000497270004226X Published online by Cambridge University Press

equation, in the above sense, is the Euler-Lagrange equation. Adler [1] showed that such a correspondence, in the classical sense, did not hold for the first initial boundary value problem for the simple parabolic equation.

A number of authors (Mikhlin [4], Petryshyn [5], [6]) have established results which allow the extension of the correspondence to a wider class of operator equations than was possible with the energy functional. In particular, it follows from Petryshyn [5, Theorem 1.2] that the solution of the operator equation $Au = f$ , $f \in \underline{H}$ , is equivalent to the function which minimises the functional

$$F_K(u) = (Au,Ku) - (Ku,f) - (f,Ku) ,$$

and conversely, if $A$ is K-p.d. (Petryshyn [5, §1]).

In this paper, we shall show that a wide class of non-homogeneous parabolic differential equations, satisfying zero initial and boundary conditions, can be related to densely invertible operator equations (see Petryshyn [6, p. 2]), and hence, are K-p.d. with $K = A$ . This bypasses the difficulty posed by Adler. As is well known, [3, §18], there is no problem regarding the reduction of homogeneous and non-homogeneous parabolic equations satisfying non-zero initial and boundary conditions of the first kind to the above-mentioned form.

An examination of the problem of applying variational methods to parabolic differential equations which can be reduced to the above-mentioned form and can be related to densely invertible operator equations, can be found in Anderssen [8].

## 2. Parabolic operator equations and dense invertibility

We consider the parabolic differential equations

$$(1) \qquad A(t)u = Lu + \frac{\partial u}{\partial t} = f(x,t) , \quad u = u(x,t) = u(x_1,x_2,\ldots,x_m,t) ,$$

where $f(x,t) \in L_2(\Omega)$ , $\Omega = Q \times (0,\infty) = \{(x,t) \; ; \; x \in Q , \; 0 < t < \infty\}$ with $Q$ an open bounded region in $m$-dimensional Euclidean space, and $L$ is the following elliptic differential operator of order $2p$ with real coefficients

(2)
$$Lu = \sum_{1 \leq |s| = |r| \leq p} (-1)^{|r|} D^r \left\{ a_{r_1, r_2, \ldots, r_m; s_1, s_2, \ldots, s_m}(x,t) D^s u \right\}$$
$$= \sum_{|r| \leq p} (-1)^{|r|} D^r a_{r,s}(x,t) D^s u \qquad \text{(for brevity)} ,$$

where

$$D^r u = \frac{\partial^{|r|} u}{\partial x_1^{r_1} \ldots \partial x_m^{r_m}} , \quad |r| = \sum_{j=1}^{m} r_j , \quad 0 \leq r_j \leq p .$$

The coefficients $a_{r,s} = a_{r,s}(x,t)$ are symmetric, $a_{r,s} = a_{s,r}$, and real functions which have $\max(r_j, s_j)$ continuous derivatives with respect to each of the $x_j$, $1 \leq j \leq m$, and one continuous derivative with respect to $t$.

The closure and boundary of $\Omega$ will be denoted by $\bar{\Omega}$ and $\partial\Omega$, and the cylindrical surface of $\Omega$ by $S(\bar{\Omega})$.

Throughout this paper, (1) will be associated with the initial condition

(3) $$u(x,0) = 0 \qquad (x \in \bar{Q})$$

and the boundary conditions

(4) $$\frac{\partial^j u}{\partial \nu^j} = 0 \qquad \left((x,t) \in S(\bar{\Omega}) , \quad j = 0,1,2,\ldots,p-1\right) ,$$

where $\nu$ is the outward normal to $S(\bar{\Omega})$. It is assumed that $S(\bar{\Omega})$ is sufficiently smooth so that there is no difficulty regarding the interpretation of (4). This and the above assumptions will hold throughout the remainder of this paper.

Before associating (1), (3) and (4) with an operator equation

(5) $$Au = f , \qquad f \in \underline{\underline{H}} ,$$

where $A$ maps some appropriate linear manifold of real functions into $\underline{\underline{H}}$, a number of preliminary results will be established.

First, let $C^k(Q)$ denote the set of functions which have all possible $k$'th continuous derivatives with respect to the $x_i$ $(i = 1,2,\ldots,m)$, and $M$

denote the linear manifold of functions which satisfy the initial and boundary conditions (3) and (4) and which are contained in

$$C^{2p}(\overline{Q}) \cap C^*([0,\infty)) \cap L_2(\Omega) ,$$

where $u \in C^*([0,\infty))$ implies that

(i) $\frac{\partial u}{\partial t}$ exists and is continuous,

(ii) $\int_Q u^2 \, dQ$ is uniformly bounded,

(iii) for $T$ sufficiently large,

$$\frac{\partial}{\partial t}\left\{\int_Q u^2 \, dQ\right\} \leqq -K \int_Q u^2 \, dQ \qquad (t > T , \quad K = \text{const.} > 0) ,$$

which implies that $u(x,t) \to 0$ as $t \to \infty$ for $x \in Q$ .

LEMMA 1. *If the coefficients* $a_{r,s}(x,t) = a_{s,r}(x,t)$ *satisfy*

$$(6) \qquad \sum_{|s|=|r|=j} \dot{a}_{r,s}(x,t) \, \xi^r \xi^s \leqq 0 \qquad (j = 1,2,\ldots,p)$$

*for all vectors* $\xi = (\xi_1,\xi_2,\ldots,\xi_m)$ *, then, for* $u \in M$ *,*

$$(7) \qquad \left(Lu , \frac{\partial u}{\partial t}\right) \geqq 0 ,$$

*where* $( \; , \; )$ *denotes the inner product in* $L_2(\Omega)$ *and the dot denotes differentiation with respect to* $t$ *.*

Proof. A simple integration by parts of the left hand side of (7) with respect to one of the $x_i$ , $1 \leqq i \leqq m$ , yields

$$\left(Lu,\frac{\partial u}{\partial t}\right) = \sum_{|r|\leqq p} (-1)^{|r|} \int_0^\infty dt \int_{\partial Q} \frac{\partial u}{\partial t} D^{r-1}\left(a_{r,s} D^s u\right) \cos\{\nu,x_i\} d(\partial Q)$$

$$- \sum_{|r|\leqq p} (-1)^{|r|} \int_0^\infty dt \int_Q D^{r-1}\left(a_{r,s} D^s u\right) D\left(\frac{\partial u}{\partial t}\right) dQ$$

where $\{\nu,x_i\}$ denotes the angle between the outward normal $\nu$ to $\overline{Q}$ and the co-ordinate direction $x_i$ . Since $u$ satisfies (4) and $\Omega$ is a cylindrical region, the first integral on the right hand side of the last expression vanishes; and hence,

$$\left(Lu,\frac{\partial u}{\partial t}\right) = - \sum_{|r|\leq p} (-1)^{|r|} \int_0^\infty dt \int_Q D^{r-1}(a_{r,s}D^s u) D\left(\frac{\partial u}{\partial t}\right) dQ \ .$$

By repeated integration by parts with respect to the $x_i$ , $1 \leqq i \leqq m$ , and the use of the above result, the last expression becomes

$$(8) \qquad \left(Lu,\frac{\partial u}{\partial t}\right) = \sum_{|r|\leqq p} (-1)^{|r|} \int_0^\infty dt \int_Q u \ D^r\left\{a_{r,s}D^s\left(\frac{\partial u}{\partial t}\right)\right\} dQ = I(u) \ .$$

Integration of the l.h.s. of (7) by parts once with respect to $t$ and the use of the fact that $u \in M$ yields

$$(9) \qquad \left(Lu,\frac{\partial u}{\partial t}\right) = - I(u) - \sum_{|r|\leqq p} (-1)^{|r|} \int_0^\infty dt \int_Q u \ D^r\left(\dot{a}_{r,s}D^s u\right) dQ \ .$$

On combining (8) and (9) and then integrating the resulting right hand side $|r|$-times with respect to the $x_i$ , $1 \leqq i \leqq m$ , we obtain

$$(10) \qquad \left(Lu,\frac{\partial u}{\partial t}\right) = - \frac{1}{2} \sum_{|r|\leqq p} \int_0^\infty dt \int_Q \dot{a}_{r,s} \left(D^r u\right)\left(D^s u\right) dQ \ ,$$

which proves the Lemma on using (6).

Note. It is clear from (10) that, if the coefficients $a_{r,s}$ are independent of $t$ , then $\left(Lu,\frac{\partial u}{\partial t}\right) = 0$ .

We now associate (1), (3) and (4) with the *parabolic operator* $\mathsf{A}$ , corresponding to (5), which is defined by $\mathsf{A}u = A(t)u$ with $\underline{\underline{D}}(\mathsf{A}) = M$ , $\underline{\underline{R}}(\mathsf{A}) = \mathsf{A}(M)$ , $\underline{\underline{H}} = L_2(\Omega)$ , where $\underline{\underline{D}}(\mathsf{A})$ and $\underline{\underline{R}}(\mathsf{A})$ denote, respectively, the *domain* and *range* of $\mathsf{A}$ . In making this association, we also define the *elliptic operator* $\mathsf{L}$ such that $\mathsf{L}u = Lu$ with $\underline{\underline{D}}(\mathsf{L}) = M_0$ , $\underline{\underline{R}}(\mathsf{L}) = \mathsf{L}(M_0)$ , $\underline{\underline{H}} = L_2(Q)$ , where $M_0$ is the linear manifold of functions which are contained in $C^{2p}(Q) \cap L_2(\Omega)$ and satisfy the boundary conditions (4).

We are now in a position to establish our main result.

**THEOREM 1.** *The parabolic operator* $\mathsf{A}$ *, defined above, is densely invertible on* $L_2(\Omega)$ *if*

*(a)* $\mathsf{L}$ *is positive definite on* $M_0$ *for all* $t \in [0,\infty)$ ,

https://doi.org/10.1017/S000497270004226X Published online by Cambridge University Press

(b) *the coefficients* $a_{r,s} = a_{s,r}$ *have* $\max(r_i, s_i)$ *continuous derivatives with respect to the* $x_i$ *and one continuous derivative with respect to* $t$ *and satisfy* (6),

(c) $\partial Q$ *is sufficiently smooth, and*

(d) $L$ *is uniformly parabolic.*

Proof. Following the definition of densely invertible, Petryshyn [6, p. 2], it is only necessary to prove that $\underline{\underline{D}}(A) = M$ and $A(\underline{\underline{D}}(A)) = A(M)$ are dense in $L_2(\Omega)$ and $A^{-1}$ is bounded on $\underline{\underline{R}}(A)$ .

The denseness of $M$ in $L_2(\Omega)$ follows on noting that $C_0^{2p}$ {the set of functions with compact support in $\Omega$ which are contained in $C^{2p}(\Omega)$} is contained in $M$ .

On the basis of Lemma 1 and condition *(a)* of the Theorem, we have that

$$\|Au\|^2 = (Au, Au) \geqq \int_0^\infty dt \int_Q \left\{|Lu|^2 + \left|\frac{\partial u}{\partial t}\right|^2\right\} dQ \quad (u \in M)$$
$$\geqq \|Lu\|^2 \geqq k \|u\|^2 \qquad (k > 0) \ ,$$

which proves that $A^{-1}$ is bounded on $\underline{\underline{R}}(A)$ (see Mikhlin [3, §45]).

It only remains to prove that $A(M)$ is dense in $L_2(\Omega)$ . In fact, we shall prove that there exists a set $\overline{M} \subset M$ such that $\overline{M}$ and $A(\overline{M})$ are dense in $L_2(\Omega)$ . The existence of a set $\overline{M} \subset M$ such that $A(\overline{M})$ is dense in $L_2(\Omega)$ can be derived as a consequence of Theorem 19, Chapter 10 of [7], if $L$ is uniformly parabolic and the coefficients $a_{r,s}$ and the boundary $\partial Q$ are sufficiently smooth. In fact, it follows from the mentioned theorem, under the cited conditions, that there exists a suitably large integer $N$ such that if $f \in C_0^N(\Omega) \subset L_2(\Omega)$ there exists a $T$ , $0 < T < \infty$ , such that (1), (3) and (4) has a unique *classical* solution $v(x,t)$ and, further, $v \in C^{2p}\left(\overline{Q} \times [0,T]\right)$ for $p > 0$ . We identify $A(\overline{M})$ with $C_0^N(\Omega)$ and $\overline{M}$ with the corresponding classical solutions $v(x,t)$ . Since $v(x,t) \in C^{2p}(\overline{Q})$ and satisfies (3) and (4) and $\frac{\partial u}{\partial t}$ exists and is continuous, it only remains to prove that $v(x,t) \in L_2(\Omega)$ and conditions (ii) and (iii) on $v \in C^*([0,\infty))$ are satisfied in order to show that $\overline{M} \subset M$ . It follows from

https://doi.org/10.1017/S000497270004226X Published online by Cambridge University Press

the above that $v(x,t)$ is bounded in $\overline{Q} \times [0,T]$ and the continuation of $v(x,t)$ for $t > T$ is defined by the homogeneous parabolic equation $A(t)u = 0$ $\{f \equiv 0$ in (1)$\}$, zero boundary conditions and continuous initial condition $v(x,t) = F(x)$ . Hence, for $t > T$ ,

$$\int_Q v \frac{\partial v}{\partial t} dQ = \frac{1}{2} \frac{\partial}{\partial t} \left\{ \int_Q v^2 \, dQ \right\} = - \int_Q v \, \mathrm{L} v \, dQ \, .$$

Since L is positive definite on $M_0$ , it follows that

$$\frac{1}{2} \frac{\partial}{\partial t} \left\{ \int_Q v^2 \, dQ \right\} \leqq -k \int_Q v^2 \, dQ \, .$$

In this way, we obtain

$$\int_T^\infty \int_Q v^2 \, dQ \, dt \leqq \frac{K_1}{k} < \infty \, , \tag{11}$$

where $K_1 = \int_Q F(x)^2 \, dQ$ . Hence, conditions (ii) and (iii) for $v \in C^*([0,\infty))$ are satisfied and $v(x,t) \in L_2(\Omega)$ , and thus, $\overline{M} \subset M$ . The denseness of $\mathrm{A}(\overline{M})$ in $L_2(\Omega)$ now follows from the denseness of $C_0^N(\Omega)$ for arbitrary large $N$ , while the denseness of $\overline{M}$ follows from the denseness of $\mathrm{A}(\overline{M})$ and the boundedness of $\mathrm{A}^{-1}$ .

EXAMPLE. Consider the non-homogeneous parabolic equation

$$A_1(t)u = L_1 u + \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \left(Ke^{-t}x+H\right) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = f \, , \tag{12}$$

which is defined on $S(0,1) = \{(x,t) \; ; \; 0 \leqq x \leqq 1 \; , \; t \geqq 0\}$ and satisfies zero initial and boundary conditions of the first kind with $H$ and $K$ positive constants and $f \in L_2\left(S(0,1)\right)$ . Clearly, $L_1$ is uniformly parabolic. Since $Ke^{-t}x+H$ is a decreasing function of $t$ and $L_1$ is positive definite on $C_0^2\left((0,1)\right)$ for all $t$ , the dense invertibility of $\mathrm{A}_1$ , the parabolic operator corresponding to (12), on $L_2\left(S(0,1)\right)$ follows from Theorem 1.

Note. The dense invertibility of the parabolic operator corresponding to the first initial boundary value problem for the simple heat conduction equation follows from the example on setting $K = 0$ .

The use of Theorem 1 of [6] yields the equivalence of all classical

https://doi.org/10.1017/S000497270004226X Published online by Cambridge University Press

solutions of (5), corresponding to $f \in L_2(S(0,1))$ , with the functions which minimise $F_K(u)$ , K = A .

## 3. The extension of dense invertibility to other parabolic operators

Theorem 1 only gives sufficient conditions for the dense invertibility of operator equations which can be associated with parabolics of the form (1), (3) and (4).

Consider the parabolic differential equation

$$P(t)u = Bu + \frac{\partial u}{\partial t} = f \, , \qquad f \in L_2(\Omega) \, , \tag{13}$$

where $\Omega$ and $u$ are defined as in §2, and

$$\begin{aligned} Bu &= - \sum_{1 \leq |s| = |r| \leq p} (-1)^{|r|} \, b_{r_1,\ldots,r_m;s_1,\ldots,s_m}(x,t) D^{r+s} u \\ &= - \sum_{|r| \leq p} (-1)^{|r|} \, b_{r,s} D^{r+s} u \qquad \text{(for brevity)} \, . \end{aligned} \tag{14}$$

The coefficients $b_{r,s}$ are assumed to have the same properties as listed above for the $a_{r,s}$ and to be such that $B$ is uniformly parabolic and not reducible to the form (2). We associate with (13), (3) and (4) the parabolic operator P defined by $\mathrm{P}u = P(t)u$ , $\underline{\underline{D}}(\mathrm{P}) = M$ , $\underline{\underline{R}}(\mathrm{P}) = \mathrm{P}(M)$ , $\underline{\underline{H}} = L_2(\Omega)$ , where $M$ now denotes the set of all classical solutions of (13) corresponding to $f \in L_2(\Omega)$ . Let B denote the corresponding elliptic operator.

THEOREM 2. P *is densely invertible on* $L_2(\Omega)$ *if*

*(a)* $\|\mathrm{P}u\| \geq k\|u\|$ , $k = \text{const.} > 0$ , $u \in M$ ,

*(b) the backward parabolic equation*

$$\mathrm{P}^*u = \mathrm{B}^*u - \frac{\partial u}{\partial t} = 0 \, ,$$

*where* P* *and* B* *denote the adjoint operators corresponding to* P *and* B , *respectively, has only one solution* $u = 0$ , $u \in \underline{\underline{D}}(\mathrm{P}^*)$ .

Proof. It is only necessary to show that $\underline{\underline{R}}(\mathrm{P})$ is dense in $L_2(\Omega)$ . It follows from Riesz and Sz. Nagy [9, §115] that there exists a closed linear extension $\overline{P}$ of $P$. Next, using the *closed range theorem* of Banach and the fact that $\underline{D}(P^*) \supset C_0^{2p}(\Omega)$, $\underline{R}(\overline{P}) = \underline{N}(\overline{P}^*)^{\perp}$ - the orthogonal projection of the null space of the adjoint of $\overline{P}$. Finally, since $P^*$ is a closed operator, it is seen that $\underline{N}(\overline{P}^*)^{\perp} = \underline{N}(P^*)^{\perp} = L_2(\Omega)$, and hence, that $\underline{R}(P)$ is dense in $L_2(\Omega)$.

In order to confirm that *(b)* of Theorem 2 is satisfied in any given case, Theorem 1.1 of Lions and Malgrange [*10*] can be used. We first observe that the bilinear form $(-B^*u,w)_Q$, corresponding to $B$, can be rewritten as

$$(-B^*u,w)_Q = a_0(t;u,w) + a_1(t;u,w) ,$$

where

$$(u,w)_Q = \int_Q uwdQ ,$$

$$a_0(t;u,w) = \sum_{|r|\leq p} \int_Q b_{r,s} D^r u D^s w \, dQ ,$$

and

$$a_1(t;u,w) = - \sum_{|r|\leq p} \int_Q w \, D^r\left(b_{r,s}\right) D^s u \, dQ .$$

We let $\underline{V} = H_0^p(Q)$ and $\underline{H} = L_2(Q)$, where $H_0^p(Q)$ is the well known Sobolev space, [*11*, Chapter II]. Using the facts that $a_0(t;u,w) = a_0(t;w,u)$ and $\underline{D}(P^*) \supset C_0^{2p}(\Omega)$, Theorem 1.1 of [*10*] can be restated as

LEMMA 2. *Let* $a_0(t;u,w)$ *and* $a_1(t;u,w)$ *be continuous bilinear forms on* $\underline{V} \times \underline{V}$ *and satisfy the conditions*

*(i)* *for* $u,w \in \underline{V}$, *the* $a_j(t;u,w)$ $(j = 0,1)$ *have one continuous derivative with respect to* $t$ *for all* $t \in [0,\infty)$ ;

*(ii)* *for* $w \in \underline{V}$, *there exists a* $\lambda > 0$ *and* $\alpha > 0$ *such that*

$$a_0(t;w,w) + \lambda(|w|^Q)^2 \geq \alpha(|w|_p)^2$$

*where*

$$(|w|^Q)^2 = (w,w)_Q$$

https://doi.org/10.1017/S000497270004226X Published online by Cambridge University Press

*and* $|w|_p$ *is the norm in* $H_0^p(Q)$ *;*

*(iii) for all* $u,w \in \underline{V}$ *, there exists a positive constant* $C_1$ *such that*

$$|a_1(t;u,w)| \leq C_1|u|_p|w|^Q .$$

*Then the only solution of* $\mathsf{P}^*u = 0$ *is* $u = 0$ *,* $u \in \underline{D}(\mathsf{P}^*)$ *.*

As an example, we consider the non-homogeneous parabolic equation

$$(15) \qquad P_1(t)u = B_1u + \frac{\partial u}{\partial t} = -\left(Ke^{-t}x+H\right)\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = f(x,t) ,$$

which is defined on $S(0,1)$ and satisfies zero initial and boundary conditions of the first kind with $H$ and $K$ positive constants and $f \in L_2\left(S(0,1)\right)$ .

THEOREM 3. *The parabolic operator* $\mathsf{P}_1$ *, defined by* $\mathsf{P}_1u = P_1(t)u$ *with* $\underline{D}(\mathsf{P}_1)$ *equal to all classical solutions of* (15) *corresponding to* $f \in L_2\left(S(0,1)\right)$ *, is densely invertible, if* $H^2 > K^2$ *.*

Proof. We start by establishing conditions *(i)*, *(ii)* and *(iii)* of Lemma 2. *(i)* is obvious; *(ii)* follows from

$$a_0(t;w,w) + \lambda\left(|w|^Q\right)^2 = \int_0^1 \left(Ke^{-t}x+H\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial x}\,dx + \lambda\int_0^1 w^2\,dx$$
$$\geq \min(H,\lambda)\,.\,(|w|_1)^2 ;$$

and *(iii)* follows from

$$|a_1(t;u,w)| = K\left|\int_0^1 e^{-t}\frac{\partial u}{\partial x}\,w\,dx\right|$$
$$\leq K|u|_1\,.\,|w|^Q ,$$

where $Q$ is the interval $(0,1)$ . Hence, it only remains to prove *(a)* of Theorem 2. Writing $H^2 = K^2 + H_0$ , $H_0 > 0$ , we obtain

$$\|\mathsf{B}_1u\|^2 \geq H^2\left\|\frac{\partial u}{\partial x}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 - 2K\left\|\frac{\partial u}{\partial x}\right\|\left\|\frac{\partial u}{\partial t}\right\|$$
$$\geq H_0\left\|\frac{\partial u}{\partial x}\right\| + \left(K\left\|\frac{\partial u}{\partial x}\right\| - \left\|\frac{\partial u}{\partial t}\right\|\right)^2 ,$$
$$\geq H_0\|u\|^2 ,$$

where the last step depends on the fact that, [3, §21, equation (7)],

$\left\|\frac{\partial u}{\partial x}\right\| \geqq \|u\|$ for $u \in L_2(S(0,1))$ .

## References


[1] György Adler, "Sulla caratterizzabilità dell'equazione del calore dal punto di vista del calcalo delle variazioni", *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 2 (1957), 153-157.

[2] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 1, (Interscience, New York, 1953).

[3] S.G. Mikhlin, *Variational methods in mathematical physics* (Pergamon Press, Oxford, 1964). Translated from *Variatsionnyye metody v matematicheskoi fizike* (Gostekhizdat, Moscow, 1957).

[4] S.G. Mikhlin, *The numerical performance of variational methods* (to be published by Noordhoff, Groningen). Translated from *Chislennaya realizatsiya variatsionnykh metodov* (Nauka, Moscow, 1966).

[5] W.V. Petryshyn, "Direct and iterative methods for the solution of linear operator equations in Hilbert space", *Trans. Amer. Math. Soc.* 105 (1962), 136-175.

[6] W.V. Petryshyn, "On a class of $K$-p.d. and non $K$-p.d. operator equations", *J. Math. Anal. Appl.* 10 (1965), 1-24.

[7] Avner Friedman, *Partial differential equations of parabolic type* (Prentice-Hall, Englewood Cliffs, New Jersey, 1964).

[8] R.S. Anderssen, "The numerical solution of parabolic differential equations using variational methods", *Numer. Math.* 13 (1969), 129-145.

[9] Frigyes Riesz and Béla Sz.-Nagy, *Functional analysis* (Blackie & Son, London, 1956).

[10] J.-L. Lions et B. Malgrange, "Sur l'unicité rétrograde dans les problèmes mixtes paraboliques", *Math. Scand.* 8 (1960), 277-286.

[11] J.-L. Lions, *Equations différentielles operationnelles et problèmes aux limites* (Grundlehren der mathematischen Wissenschaften, Band 111, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961).

Computer Centre,
Australian National University,
Canberra, ACT.