Canad. Math. Bull. Vol. **41** (1), 1998 pp. 109–117

ON GENERALIZED THIRD DIMENSION SUBGROUPS

KEN-ICHI TAHARA, L.R. VERMANI AND ATUL RAZDAN

ABSTRACT. Let G be any group, and H be a normal subgroup of G. Then M. Hartl identified the subgroup $G \cap (1 + \triangle^3(G) + \triangle(G)\triangle(H))$ of G. In this note we give an independent proof of the result of Hartl, and we identify two subgroups $G \cap (1 + \triangle(H)\triangle(G)\triangle(H) + \triangle([H,G])\triangle(H))$, $G \cap (1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H))$ of G for some subgroup K of G containing $[H,G]$.

1. **Introduction.** Let G be a group, ZG the integral group ring of G and $\triangle(G)$ the augmentation ideal of ZG. Let H be a normal subgroup of G and write

$$D_1(G,H) = G \cap \left(1 + \triangle^3(G) + \triangle(G)\triangle(H)\right)$$
$$D_2(G,H) = G \cap \left(1 + \triangle(H)\triangle(G)\triangle(H) + \triangle([H,G])\triangle(H)\right).$$

Moreover let $w: G/H \longrightarrow G$ be a map satisfying πw= identity on G/H, where $\pi: G \longrightarrow G/H$ is the natural projection, and $w(1) = 1$. Since, for any $\alpha, \beta \in G/H$,

$$\pi\Big(w(\alpha\beta)^{-1} w(\alpha) w(\beta)\Big) = (\alpha\beta)^{-1}\alpha\beta = 1,$$

there exists a unique element $W(\alpha,\beta) \in H$ such that

$$w(\alpha)w(\beta) = w(\alpha\beta)W(\alpha,\beta). \tag{1}$$

Define K to be the subgroup of H generated by $\{[H,G], W(\alpha,\beta) \mid \alpha,\beta \in G/H\}$. Write

$$D_3(G,H) = G \cap \left(1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H)\right).$$

Let $\gamma_n(G)$, $n \geq 1$, denote the n-th term of the lower central series of the group G. We also write $\gamma_2(G) = G'$. The subgroup $D_1(G,H)$ has been computed by Sandling [9], Passi [7], Passi-Sharma [8], Khambadkone [3] and Karan-Vermani [4], when H is a certain special subgroup of G. Recently this subgroup has been computed by Hartl [2] (which is under circulation in preprint form only) for any normal subgroup H of G. Since we make use of this result in our investigations, we give here an independent proof of this result of Hartl and prove

THEOREM A. *Let G be any group, and H be a normal subgroup of G. Then*

$$D_1(G,H) = \gamma_3(G)\langle [x^m, y] \mid x^m, y^m \in HG' \text{ for some } m \geq 1, x, y \in G\rangle.$$

Received by the editors August 16, 1996.
The first author was supported in part by Grant-in-Aid for Science Research, Ministry of Education.
The second author was supported in part by National Board for higher Mathematics.
The third author was supported by Council of Scientific & Industrial Research.
AMS subject classification: Primary 20C07; Secondary 16S34.

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

Our proof of this result is via free group rings while Hartl's proof is based on homological arguments.

The subgroups $G \cap \big(1 + \triangle^2(G)\triangle(H)\big)$ and $G \cap \big(1 + \triangle(G)\triangle(H)\triangle(G)\big)$ have been computed by Ram Karan and Vermani [5] and also by Vermani, Razdan and Karan [10], when H is a special normal subgroup of G. We are now able to compute subgroups $D_2(G, H)$ and $D_3(G, H)$ similar to the above subgroups. We prove

THEOREM B. *Let G be any group, and H be any normal subgroup of G. Then*

$$D_2(G, H) = \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in [H, G] \text{ for some } m \geq 1, x, y \in H\rangle.$$

THEOREM C. *Let G be any group, and H be any normal subgroup of G. Then*

$$D_3(G, H) = \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in K \text{ for some } m \geq 1, x, y \in H\rangle.$$

2. **Proof of Theorem A.** We record the following simple observation

LEMMA 2.1. *Let J be an ideal of ZG containing $\triangle^2(K)$, K being a subgroup of G. Then*

$$G \cap \big(1 + J + \triangle(K)\big) = \big(G \cap (1 + J)\big) \bullet K$$

Let $F = \langle x_1, x_2, \ldots, x_r\rangle$ be a free group of rank r and R be the normal closure

$$R = \langle x_1^{e_1}\xi_1, x_2^{e_2}\xi_2, \ldots, x_r^{e_r}\xi_r, \xi_{r+1}, \ldots\rangle^F$$

where $e_1|e_2|\cdots|e_r, e_r \geq 0$, $\xi_i \in F'(i \geq 1)$. We prove

THEOREM 2.2. *With notation as in the previous paragraph,*

$$F \cap \big(1 + \triangle^3(F) + \triangle(F)\triangle(R)\big) = \gamma_3(F)U,$$

where U is the subgroup $\langle [x^m, y] \mid x^m, y^m \in RF' \text{ for some } m \geq 1, x, y \in F\rangle$ of F.

PROOF. Note that the left hand side in the formula of Theorem 2.2 is $D_1(F, R)$ as defined previously. By [9], $D_1(F, R) \subseteq [F, R]\gamma_3(F)$.

For $u, v \in F$, using the identities, modulo $\triangle^3(F)$,

$$(u^n - 1)(v - 1) \equiv n(u - 1)(v - 1) \equiv (u - 1)(v^n - 1) \tag{2}$$

$$([u, v] - 1) \equiv \{(u - 1)(v - 1) - (v - 1)(u - 1)\} \tag{3}$$

in ZF, it follows that

$$U \subseteq F \cap \big(1 + \triangle^3(F) + \triangle(F)\triangle(R)\big) = D_1(F, R).$$

Since $\gamma_3(F) \subseteq D_1(F, R)$ is clear,

$$\gamma_3(F)U \subseteq F \cap \big(1 + \triangle^3(F) + \triangle(F)\triangle(R)\big) = D_1(F, R). \tag{4}$$

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

To show reverse inclusion we proceed as follows. Suppose $w-1 \in \triangle^3(F)+\triangle(F)\triangle(R)$. Then by [9], $w \in [F,R]\gamma_3(F)$. Since

$$[F,R] = \langle [x_i^{e_i}\xi_i, x_j], [\xi_q, x_k] \mid 1 \le i \le r, 1 \le j,k \le r, q \ge r+1\rangle^F,$$

we can write any $w \in [F,R]$, modulo $\gamma_3(F)$, as

$$w \equiv \prod_{1\le i<j\le r} [x_i, x_j]^{e_j a_{ij}} \prod_{1\le q<k\le r} [x_k, x_q]^{e_q b_{kq}} \tag{5}$$

where $a_{ij}, b_{kq} \in Z$.

For $w \in [F,R]$ as in (5), define

$$s_k(w) = \prod_{i>k} x_i^{a_{ki}(e_i/e_k)-b_{ik}}, \quad 1 \le k \le r-1. \tag{6}$$

Now it follows from (5) that, modulo $\gamma_3(F)$,

$$w \equiv \prod_{k=1}^{r-1}\Big(\prod_{i>k}[x_k^{e_k}, x_i]^{(e_i/e_k)a_{ki}-b_{ik}}\Big)$$

$$w \equiv \prod_{k=1}^{r-1}[x_k^{e_k}, s_k(w)],\ s_k(w) \quad \text{as in (6)} \tag{7}$$

We now claim that if $w-1 \in \triangle^3(F)+\triangle(F)\triangle(R)$ with $w \in [F,R]\gamma_3(F)$, then

$$\big(s_k(w)\big)^{e_k} \in RF' \quad \text{for } k = 1,2,\ldots,r-1,$$

and so, in view of (7),

$$w \in \langle [x^m, y] \mid x^m, y^m \in RF' \text{ for some } m \ge 1, x, y \in F\rangle \gamma_3(F).$$

This will complete the proof of Theorem 2.2. We proceed to prove our claim.

Let w, as in (7), be such that

$$w-1 \in \triangle^3(F)+\triangle(F)\triangle(R).$$

Expansion of $w-1$, modulo $\triangle^3(F)$, yields (in view of (7))

$$w-1 \equiv \sum_{k=1}^{r-1}\Big\{(x_k^{e_k}-1)\big(s_k(w)-1\big) - \big(s_k(w)-1\big)(x_k^{e_k}-1)\Big\},$$

Since $x_k^{e_k}\xi_k \in R$, where $\xi_k \in F'$, so $x_k^{e_k} \in RF'$ and thus

$$x_k^{e_k}-1 \in \triangle^2(F)+\triangle(R).$$

Whence

$$\begin{aligned} w-1 &\equiv \sum_{k=1}^{r-1}(x_k^{e_k}-1)\big(s_k(w)-1\big)\big(\mathrm{mod}\ \triangle^3(F)+\triangle(F)\triangle(R)\big)\\ &\equiv \sum_{k=1}^{r-1}(x_k-1)\big(s_k(w)^{e_k}-1\big)\big(\mathrm{mod}\ \triangle^3(F)+\triangle(F)\triangle(R)\big) \quad \text{(by (2)).}\end{aligned}$$

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

Already $w-1 \in \triangle^3(F)+\triangle(F)\triangle(R)$. It, therefore, follows that

$$\sum_{k=1}^{r-1}(x_k-1)\big(s_k(w)^{e_k}-1\big) \in \triangle^3(F)+\triangle(F)\triangle(R),$$

and consequently, $\triangle(F)$ being free right ZF-module with $\{x_k-1 \mid 1 \le k \le r\}$ as a basis, we get

$$\big(s_k(w)^{e_k}-1\big) \in \triangle^2(F)+ZF\triangle(R)=\triangle^2(F)+\triangle(R),$$

and so,

$$\big(s_k(w)\big)^{e_k} \in F\cap\big(1+\triangle^2(F)+\triangle(R)\big)=RF', \quad 1\le k\le r-1, \quad \text{(by Lemma 2.1)}$$

as asserted in our claim. ∎

PROOF OF THEOREM A. Let G be a group and H be a normal subgroup of G. Define U_1 to be the subgroup of $[H,G]\gamma_3(G)$ generated by $\{[x^m,y] \mid x^m, y^m \in HG'$ for some $m\ge 1$, $x,y\in G\}$. Then U_1 is a normal subgroup of G containing H'. That $U_1\gamma_3(G)$ is contained in $D_1(G,H)$ follows as in (4) above.

For proving the reverse inclusion, using standard arguments, we can assume G to be a finitely generated group. Let $G=F/T, H=R/T$, where F is a free group and T, R are normal subgroups of F with $T\subseteq R$. Then

$$\begin{aligned} G\cap\big(1+\triangle^3(G)+\triangle(G)\triangle(H)\big) &= F\cap\big(1+\triangle^3(F)+\triangle(F)\triangle(R)+ZF\triangle(T)\big)/T \\ &= F\cap\big(1+\triangle^3(F)+\triangle(F)\triangle(R)\big)T/T. \end{aligned} \tag{8}$$

The last equality holds by Lemma 2.1. On the other hand, $\gamma_3(G)U_1=UT/T, U$ as in Theorem 2.2. Hence, in view of (8), the proof of Theorem A follows from Theorem 2.2. ∎

3. Proof of Theorem B.

First we have the following:

LEMMA 3.1. *If H and K are any subgroups of a group G each contained in the normalizer of the other, then*

$$\triangle(H)\triangle(K)+\triangle([H,K])=\triangle(K)\triangle(H)+\triangle([H,K]).$$

PROOF. Let x be any element of H, and y be any element of K. Then

$$(x-1)(y-1)=(xyx^{-1}-1)(x-1)+(y-1)([y,x^{-1}]-1)+([y,x^{-1}]-1)$$

and so $(x-1)(y-1)\in\triangle(K)\triangle(H)+\triangle([H,K])$, which implies

$$\triangle(H)\triangle(K)\subseteq\triangle(K)\triangle(H)+\triangle([H,K]).$$

Similarly $\triangle(K)\triangle(H)\subseteq\triangle(H)\triangle(K)+\triangle([H,K])$, and the result follows. ∎

COROLLARY 3.2. *Let G be a group, and H be a normal subgroup of G. Then*

$$\triangle(H)\triangle(G)\triangle(H) + \triangle([H,G])\triangle(H) = \triangle(G)\triangle^2(H) + \triangle([H,G])\triangle(H).$$

Let G be a group, H be a normal subgroup of G, and write N for $[H,G]$. Let S denote a left transversal of H in G. Then each $g \in G$ can be uniquely written as $g = sh$ for some $s \in S$ and $h \in H$. Let τ_H denote the extension to ZG by linearity of the map defined on G by

$$g = sh \longrightarrow h.$$

Then $\tau_H\colon ZG \longrightarrow ZH$ is easily seen to be a homomorphism of right ZH-modules, the action of H on ZG being through multiplication in G. We know that each element $u \in ZG$ can be written uniquely as a finite sum of the form

$$u = \sum_{s\in S} su_s, \quad \text{where } u_s \in ZH,$$

namely, ZG is a free right ZH-module with the set S as a free basis. Then τ_H is,in fact, the map which maps $u = \sum_{s\in S} su_s$ to $\sum_{s\in S} u_s$. Consequently we have

(9) $$\tau_H\big(\triangle(G)\triangle^n(H)\big) = \triangle^{n+1}(H) \quad \text{for all } n \geq 1.$$

PROOF OF THEOREM B. Since $D_2(G,H) \subseteq H' \subseteq H$,

$$\begin{aligned} G\cap\big(1+\triangle(H)\triangle(G)\triangle(H)+\triangle(N)\triangle(H)\big) &= H\cap\big(1+\triangle(H)\triangle(G)\triangle(H)+\triangle(N)\triangle(H)\big)\\ &= H\cap\big(1+\triangle(G)\triangle^2(H)+\triangle(N)\triangle(H)\big)\end{aligned}$$

by Corollary 3.2 of Lemma 3.1, and hence

$$\begin{aligned} G\cap\big(1+\triangle(H)\triangle(G)\triangle(H)+\triangle(N)\triangle(H)\big) &\subseteq H\cap\Big(1+\tau_H\big(\triangle(G)\triangle^2(H)+\triangle(N)\triangle(H)\big)\Big)\\ &= H\cap\big(1+\triangle^3(H)+\triangle(N)\triangle(H)\big), \quad \text{by (9)}\\ &\subseteq H\cap\big(1+\triangle(G)\triangle^2(H)+\triangle(N)\triangle(H)\big).\end{aligned}$$

Therefore,

$$D_2(G,H) = H\cap\big(1+\triangle^3(H)+\triangle(N)\triangle(H)\big).$$

The result then follows from Theorem A. ■

4. **The proof of Theorem C.** Let G a group, and H be a normal subgroup of G. Then we can take $w(G/H)$ as representatives of H in G, and any element of G can be uniquely written in the form

$$g = w(\alpha)x \quad \text{with } \alpha \in G/H,\ x \in H.$$

Let $W(\alpha,\beta)$ and K be as in the introduction. For any elements $w(\alpha)x, w(\beta)y$ ($\alpha,\beta \in G/H, x, y \in H$),

$$\begin{aligned}\tau_H\Big(\big(w(\alpha)x-1\big)\big(w(\beta)y-1\big)\Big) &= W(\alpha,\beta)x[x,w(\beta)]y - x - y + 1\\ &\equiv \big(W(\alpha,\beta)-1\big) + \big([x,w(\beta)]-1\big) \bmod \triangle^2(H),\end{aligned} \tag{10}$$

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

and hence

$$\tau_H\big(\triangle^2(G)\big) \subseteq \triangle^2(H) + \triangle(K).$$

For $h \in H, g \in G$, it follows that

$$[h,g] - 1 = \tau_H([h,g] - 1) \in \tau_H\big(\triangle^2(G)\big),$$

and hence $\triangle(N) \subseteq \tau_H\big(\triangle^2(G)\big)$. Moreover, by (10),

$$W(\alpha,\beta) - 1 \equiv \tau_H\Big(\big(w(\alpha)x - 1\big)\big(w(\beta)y - 1\big)\Big) - \big([x, w(\beta)] - 1\big) \bmod \triangle^2(H),$$

and so

$$\triangle(K) \subseteq \tau_H\big(\triangle^2(G)\big),$$

which implies $\triangle^2(H) + \triangle(K) \subseteq \tau_H\big(\triangle^2(G)\big)$. Thus it follows that

$$\triangle^2(H) + \triangle(K) = \tau_H\big(\triangle^2(G)\big).$$

Since τ_H is right ZH-module homomorphism, we have

PROPOSITION 4.1.

$$\tau_H\big(\triangle^2(G)\triangle(H)\big) = \triangle^3(H) + \triangle(K)\triangle(H).$$

PROOF OF THEOREM C.

$$\begin{aligned} D_3(G,H) &= G \cap \big(1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H)\big) \\ &\subseteq G \cap \big(1 + \triangle(G)\triangle(H)\big) = H' \subseteq H, \end{aligned}$$

and hence

$$\begin{aligned} D_3(G,H) &= H \cap \big(1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H)\big) \\ &= \tau_H\Big(H \cap \big(1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H)\big)\Big) \\ &\subseteq H \cap \big(1 + \triangle^3(H) + \triangle(K)\triangle(H)\big) \\ &\subseteq G \cap \big(1 + \triangle^2(G)\triangle(H) + \triangle(K)\triangle(H)\big). \end{aligned}$$

Thus

$$D_3(G,H) = H \cap \big(1 + \triangle^3(H) + \triangle(K)\triangle(H)\big).$$

Thus the result follows from Theorem A. ∎

COROLLARY 4.2. *Let H be a normal subgroup of a group G such that $W(\alpha,\beta) \in H \cap G'$ for any $\alpha, \beta \in G/H$. Then*

$$G \cap \big(1 + \triangle^2(G)\triangle(H)\big) = \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \textit{ for some } m \geq 1, x, y \in H\rangle.$$

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

PROOF. Assume that $W(\alpha,\beta) \in H \cap G'$ for any $\alpha,\beta \in G/H$. Then $K = \langle [H,G], W(\alpha,\beta) \mid \alpha,\beta \in G/H\rangle \subseteq H \cap G'$, and hence

$$\begin{aligned} G \cap \big(1 + \triangle^2(G)\triangle(H)\big) &= H \cap (1 + \tau_H(\triangle^2(G)\triangle(H) \\ &= H \cap \big(1 + \triangle^3(H) + \triangle(K)\triangle(H)\big) \quad \text{by Proposition 4.1} \\ &\subseteq H \cap \big(1 + \triangle^3(H) + \triangle(H \cap G')\triangle(H)\big) \\ &\subseteq G \cap \big(1 + \triangle^2(G)\triangle(H) + \triangle(H \cap G')\triangle(H)\big) \\ &= G \cap \big(1 + \triangle^2(G)\triangle(H)\big). \end{aligned}$$

Thus it follows

$$\begin{aligned} G \cap \big(1 + \triangle^2(G)\triangle(H)\big) &= H \cap \big(1 + \triangle^3(H) + \triangle(H \cap G')\triangle(H)\big) \\ &= \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \text{ for some } m \geq 1, x, y \in H\rangle. \end{aligned}$$

■

We again obtain an identification of the normal subgroup $G \cap \big(1 + \triangle^2(G)\triangle(H)\big)$ similar to the one in Corollary 4.2 for a suitable normal subgroup H of a group G.

We show easily

LEMMA 4.3 ([5, LEMMA 2.1]). *Let $G = H \bullet K$ where H and K are subgroups of G with H normal in G. Then*

$$\triangle^2(G)\triangle(H) = \triangle^3(H) + \triangle([H,K])\triangle(H) + \triangle(K)\triangle^2(H) + \triangle^2(K)\triangle(H).$$

THEOREM 4.4. *Let $G = H \mid K$, H and K be subgroups of G such that H is normal in G and $H \cap K \subseteq H \cap G'$. Then*

$$G \cap \big(1 + \triangle^2(G)\triangle(H)\big) = \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \textit{ for some } m \geq 1, x, y \in H\rangle.$$

PROOF. Let x, y be any elements of H with $x^m, y^m \in H \cap G'$ for some $m \geq 1$. Then

$$\begin{aligned} [x^m, y] - 1 &\equiv -(y-1)(x^m - 1) \bmod \triangle^2(G)\triangle(H) \\ &\equiv -m(y-1)(x-1) \bmod \triangle^2(G)\triangle(H) \\ &\equiv -(y^m - 1)(x-1) \bmod \triangle^2(G)\triangle(H) \\ &\equiv 0 \bmod \triangle^2(G)\triangle(H) \end{aligned}$$

and hence $[x^m, y] \in G \cap \big(1 + \triangle^2(G)\triangle(H)\big)$. Thus

$$\gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \text{ for some } m \geq 1, x, y \in H\rangle$$

is contained in $G \cap \big(1 + \triangle^2(G)\triangle(H)\big)$. To see the reverse inclusion, let $h \in G \cap \big(1 + \triangle^2(G)\triangle(H)\big) \subseteq G \cap \big(1 + \triangle(G)\triangle(H)\big) = H' \subseteq H$. Then

$$\begin{aligned} h - 1 &\in \triangle^2(G)\triangle(H) \cap \triangle^2(H) \\ &= \big(\triangle^3(H) + \triangle([H,K])\triangle(H) + \triangle(K)\triangle^2(H) + \triangle^2(K)\triangle(H)\big) \cap \triangle^2(H) \\ &= \triangle^3(H) + \triangle([H,K])\triangle(H) + \Big(\big(\triangle(K)\triangle^2(H) + \triangle^2(K)\triangle(H)\big) \cap \triangle^2(H)\Big) \end{aligned}$$

https://doi.org/10.4153/CMB-1998-017-1 Published online by Cambridge University Press

by Lemma 4.3, and hence

$$h-1 \in \triangle^3(H) + \triangle([H,K])\triangle(H) + \big(\triangle(K)\triangle(H) \cap \triangle^2(H)\big)$$

It follows from [5, Lemma 2.2] that $\triangle(K)\triangle(H) \cap \triangle^2(H) = \triangle(H \cap K)\triangle(H)$. We thus have

$$h-1 \in \triangle^2(H) + \triangle([H,K])\triangle(H) + \triangle(H \cap K)\triangle(H)$$

which is contained in $\triangle^3(H) + \triangle(H \cap G')\triangle(H)$ by our hypothesis. Therefore, by Theorem A

$$h \in \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \text{ for some } m \geq 1, x, y \in H\rangle,$$

as desired. ■

REMARK 4.5. Observe that the hypothesis of Theorem 4.4, for example, is satisfied if the exact sequence

$$HG'/G' \longrightarrow G/G' \longrightarrow G/HG'$$

splits. This is true in the following cases, *e.g.*

(a) if G/H is free abelian,
(b) if H is a divisible subgroup of G,
(c) if H splits over G.

M. Curzio and C. K. Gupta [1] have obtained an identification of the subgroup $G \cap \big(1 + \triangle^2(G)\triangle(H)\big)$ when G is a finitely generated group and H a normal subgroup of G. We can conjecture as follows

CONJECTURE 4.6. *Let G be a group, and H be a normal subgroup of G. Then*

$$G \cap \big(1 + \triangle^2(G)\triangle(H)\big) = \gamma_3(H)\langle [x^m, y] \mid x^m, y^m \in H \cap G' \textit{ for some } m \geq 1, x, y \in H\rangle.$$

REFERENCES

1. M. Curzio and C. K. Gupta, *Second Fox subgroup of arbitrary groups*. Canad. Math. Bull. **38**(1995), 177–181.
2. M. Hartl, *On relative polynomial construction of degree* 2 *and application*. (preprint).
3. M. Khambadkone, *On the structure of augmentation ideals in group rings*. J. Pure Appl. Algebra **35**(1985), 35–45.
4. R. Karan and L. R. Vermani, *A note on polynomial maps*, J. Pure Appl. Algebra **51**(l988),169–173.
5. ———, *Augmentation quotients of integral group rings*. J. Indian Math. Soc. **54**(1989), 107–120.
6. W. Magnus, A. Karass and A. Solitar, *Combinatorial Group Theory*. 2nd Ed. Dover, 1974.
7. I. B. S. Passi, *Polynomial maps on groups-II*. Math. Z. **135**(1974), 137–141.
8. I. B. S. Passi and S. Sharma, *The third dimension subgroup* mod n. J. London Math. Soc. **9**(1974), 176–182.

9. R. Sandling, *The dimension subgroup problem*. J. Algebra **21**(1972), 216–231.
10. L. R. Vermani, A. Razdan and R. Karan, *Some remarks on subgroups determined by certain ideals in integral group rings*. Proc. Indian Acad. Sci. Math. Sci. **103**(1993), 249–256.

Department of Mathematical Science
Aichi University of Education
Kariya-shi
Japan 448
e-mail: tahara@auems.aichi-edu.ac.jp

Department of Mathematics
Kurukshetra University
Kurukshetra 136 119 (Haryana)
India
e-mail: kuru@doe.ernet.in

School of Sciences
IGNOU, Maidan Garhi
New Dehli 110 068
India
e-mail: ignou@giasdl01.vsnl.net.in