

CHARACTERISATIONS OF BLOCH FUNCTIONS IN THE UNIT BALL OF \mathbb{C}^n , I

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Weighted Hadamard product characterisations of Bloch functions in the unit ball of \mathbb{C}^n are studied. In particular, we prove that f belongs to the Bloch space $\mathcal{B}(B_n)$ if and only if the non-weighted Hadamard products of f and g belong to $BMOA(U^n)$ for all f in $G_0^1(B_n)$, a subspace of the Hardy space $H^1(B_n)$.

1. INTRODUCTION

Hadamard products and related coefficient multiplier problems for spaces of holomorphic functions in the unit disk have been studied extensively in many papers; see, for example, [9]. In the higher dimensional extension of such a study, Burbea and Li [3] obtained recently some new descriptions of multi-index coefficient multiplier problems and the properties of weighted Hadamard products of holomorphic functions of several complex variables. In this note, for the higher dimensional setting of these questions, we prove new characterisations of spaces of multipliers of holomorphic functions in the unit ball, which characterise Bloch functions by weighted Hadamard products.

We first set up some basic notations which shall be used in the note, most of them can be found in [3]. Let \mathbb{C}^n denote the n -dimensional vector space. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, set

$$\partial_j = \frac{\partial}{\partial z_j}, \quad 1 \leq j \leq n,$$
$$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \quad \nabla = (\partial_1, \dots, \partial_n),$$

$z \cdot \xi = (z_1 \xi_1, \dots, z_n \xi_n)$ and $\langle z, \xi \rangle = z_1 \bar{\xi}_1 + \dots + z_n \bar{\xi}_n$ if $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$.

Let B_n and U^n denote respectively the unit ball and the unit polydisk in \mathbb{C}^n . For a complete Reinhardt domain Ω of \mathbb{C}^n , by $H(\Omega)$ we denote the space of all holomorphic functions in Ω . Then any $f \in H(\Omega)$ has a unique power series

$$(1) \quad f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}, \quad z \in \Omega$$

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with $a_\alpha = \frac{\partial^\alpha f(0)}{\alpha!}$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $\alpha \in \mathbb{Z}_+^n$. So $H(\Omega)$ may be regarded as a space of multi-index sequences $\{a_\alpha\}$.

For $f(z) = \sum_\alpha a_\alpha z^\alpha$, $g(z) = \sum_\alpha b_\alpha z^\alpha \in H(B_n)$ and $q \geq 0$, we define the q -Hadamard product of f and g by

$$(f * g)_q(z) = \sum_\alpha \omega_\alpha(q) a_\alpha b_\alpha z^\alpha,$$

where

$$\omega_\alpha(q) = \frac{\alpha! \Gamma(n+q)}{\Gamma(n+q+|\alpha|)}, \quad \alpha \in \mathbb{Z}_+^n.$$

The interesting feature of this product is that it is lying not only in $H(B_n)$ but also in $H(U^n)$ and

$$(2) \quad (f * g)_q(z) = \langle f, g_{\bar{z}}^* \rangle_q := \int f(\tau\xi)g(z \cdot \bar{\xi}) dv_q(\xi)$$

for any $z \in U^n$, where

$$\begin{aligned} f_\xi(z) &= f(\xi \cdot z), \\ g^*(z) &= \overline{g(\bar{z})}, \\ dv_q(z) &= \frac{1}{\pi^n} \frac{\Gamma(n+q)}{\Gamma(q)} (1 - |z|^2)^{q-1} dv(z) \end{aligned}$$

and dv denotes the usual volume of Lebesgue measure on \mathbb{C}^n (see [3, Proposition 3.2]).

Let X and Y be two vector spaces of multi-index sequences. A multi-index sequence $\{\lambda_\alpha\}$ is said to be a multiplier from X to Y , if $\{\lambda_\alpha x_\alpha\} \in Y$ whenever $\{x_\alpha\} \in X$. The set of all multipliers from X to Y is denoted by (X, Y) . A multi-index sequence $\{\lambda_\alpha\}$ is said to be a q -multiplier, $q \geq 0$, from X to Y if $\{\lambda_\alpha \omega_\alpha(q) x_\alpha\} \in Y$ whenever $\{x_\alpha\} \in X$. The set of all q -multipliers from X to Y is denoted by $(X, Y)_q$. Recently Burbea and Li have shown in [3] that

$$(3) \quad \mathcal{B}(B_n) = (A_q^1(B_n), BMOA(U^n))_q,$$

where $\mathcal{B}(B_n)$ is the Bloch space and $BMOA(U^n)$ the space of functions in $H(U^n)$ with bounded mean oscillation on the torus $T^n := \partial U^n$ and $A_q^1(B_n)$ is the Bergman-type space ($A_0^1(B_n) = H^1(B_n)$, the Hardy space). In this note we prove that the space $A_q^1(B_n)$ in (3) can be replaced by its restricted subspaces (Theorem 1).

We next introduce some further notations and recall some function theoretic concepts. For $s > 0$, let

$$(\mathcal{D}^s f)(z) = \sum_\alpha (|\alpha| + 1)^s a_\alpha z^\alpha$$

be the fractional derivative of $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ of order s .

The Bloch space on the unit ball, denoted by $\mathcal{B}(B_n)$, is defined as the space of $f \in H(B_n)$ for which

$$\|f\|_{\mathcal{B}(B_n)} = |f(0)| + \sup_{z \in B_n} (1 - |z|^2) |\nabla f(z)| < \infty.$$

The small Bloch space $\mathcal{B}_0(B_n)$ is the $\mathcal{B}(B_n)$ -closure of the holomorphic polynomials in $z \in \mathbb{C}^n$ (see [11] for more information). It is easy to see that $f \in \mathcal{B}(B_n)$ if and only if

$$\sup_{z \in B_n} (1 - |z|^2)^s |(D^s f)(z)| < \infty, \quad s > 0$$

(refer [7, Theorem 2.1]). Similarly, the Bloch space on the unit polydisk U^n , denoted by $\mathcal{B}(U^n)$, is the space of $f \in H(U^n)$ satisfying

$$\|f\|_{\mathcal{B}(U^n)} = \sup_{z \in U^n} \sum_{j=1}^n (1 - |z_j|^2) |f(z) + n z_j \partial_j f(z)| < \infty.$$

The small Bloch space $\mathcal{B}_0(U^n)$ is the $\mathcal{B}(U^n)$ -closure of the holomorphic polynomials in $z \in \mathbb{C}^n$.

In order to give the definition of *BMOA* space on the unit polydisk U^n , we first define differential operator D^n of order n with $D^1 = d/dz$. Let $\delta : \mathbb{Z} \rightarrow \mathbb{Z}_+$ be defined by $\delta(m) = \max(m, 0)$, $m \in \mathbb{Z}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, let $\delta(\alpha) = (\delta(\alpha_1), \dots, \delta(\alpha_n)) \in \mathbb{Z}_+^n$ and $\pi(\alpha) = \prod_{i=1}^n \{\alpha_i : \alpha_i \neq 0\}$. We define

$$(D^n f)(z) = \sum_{\alpha} \pi(\alpha) a_{\alpha} z^{\delta(\alpha-1)}.$$

BMOA(U^n) is the space of functions f in $H(U^n)$ with bounded mean oscillation on the torus T^n :

$$\|f\|_{BMOA(U^n)}^2 = |f(0)|^2 + \sup_{\xi \in U^n} \int_{U^n} |(D^n f)(z)|^2 \prod_{j=1}^n \frac{(1 - |z_j|^2)(1 - |\xi_j|^2)}{|1 - z_j \bar{\xi}_j|^2} \frac{dv(z)}{\pi^n} < \infty$$

(see [4, p. 238] for $n = 1$). Equipped with the norm $\|\cdot\|_{BMOA(U^n)}$, *BMOA*(U^n) is a Banach space. The closure in *BMOA*(U^n) of the holomorphic polynomials in $z \in \mathbb{C}^n$ gives rise to the space of functions in $H(U^n)$ with vanishing mean oscillation on T^n and is denoted by *VMOA*(U^n).

We now introduce a new quasi-Banach space,

$$G_q^1(B_n) = \{f \in H(B_n) : \|f\|_{G_q^1(B_n)} < \infty\}, \quad 0 \leq q < \infty,$$

where

$$\|f\|_{G_q^1(B_n)} = \int_{B_n} |D^1 f(z)| (1 - |z|^2) dv_q(z) < \infty.$$

2. THEOREM AND ITS PROOF

Now we state our main theorem which gives a new characterisation of the Bloch space $\mathcal{B}(B_n)$ with q -Hadamard products.

THEOREM 1. For any $q \geq 0$, we have

$$\mathcal{B}(B_n) = (G_q^1(B_n), BMOA(U^n))_q.$$

PROOF: Let $A(U^n)$ denote the space of functions which holomorphic in U^n and continuous on $\overline{U^n}$. It is well-known that $A(U^n) \subset BMOA(U^n)$. To prove

$$\mathcal{B}(B_n) \subset (G_q^1(B_n), BMOA(U^n))_q,$$

it suffice to show that

$$\mathcal{B}(B_n) \subset (G_q^1(B_n), A(U^n))_q.$$

For $q > 0$. Let $A_q^1(B_n)$ be the space of $f \in H(B_n)$ for which

$$\|f\|_{A_q^1(B_n)} = \int_{B_n} |f(z)| dv_q(z) < \infty$$

[3]. By [10, Theorem 1(b)], $G_q^1(B_n) \subset A_q^1(B_n)$. Then it is sufficient to prove that

$$\mathcal{B}(B_n) \subset (A_q^1(B_n), A(U^n))_q,$$

where we used [1, Lemma 1].

Suppose that $\Gamma_q(B_n)$ is the dual of $A_q^1(B_n)$ with respect to the q -pairing. Then $\Gamma_q(B_n) = \mathcal{B}(B_n)$ [2]. Applying [3, Propositions 3.1 and 3.3], (2) gives

$$(4) \quad |(f * g)_q(z)| \leq \|g\|_{\Gamma_q(B_n)} \|f_{\overline{z}}^*\|_{A_q^1(B_n)} \leq C \|g\|_{\mathcal{B}(B_n)} \|f\|_{A_q^1(B_n)}.$$

For $f \in A_q^1(B_n)$, $z \in \overline{U^n}$, the correspondence $z \rightarrow f_z \in A_q^1(B_n)$ is continuous. That is $\|f_{z_1} - f_{z_2}\|_{A_q^1(B_n)} \rightarrow 0$ as $|z_1 - z_2| \rightarrow 0$ (refer [8, Lemma 4 (iii)]). Combining this with (4) yields

$$\begin{aligned} |(f * g)_q(z_1) - (f * g)_q(z_2)| &= |((f_{z_1} - f_{z_2}) * g)_q(\mathbf{1})| \\ &\leq \|f_{z_1} - f_{z_2}\|_{A_q^1(B_n)} \|g\|_{\mathcal{B}(B_n)} \\ &\rightarrow 0, \end{aligned}$$

as $|z_1 - z_2| \rightarrow 0$. Hence $(f * g)_q \in A(U^n)$ for all $f \in A_q^1(B_n)$.

For $q = 0$. Let $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha} \in G_0^1(B_n)$, $g(z) = \sum_{\alpha} b_{\alpha} z^{\alpha} \in \mathcal{B}(B_n)$, we have

$$(5) \quad \sum_{\alpha} (|\alpha| + 1)^2 r^{2|\alpha|} a_{\alpha} b_{\alpha} \omega_{\alpha}(0) z^{\alpha} = \int_{\partial B_n} \mathcal{D}^1 f(r\xi) \mathcal{D}^1 g(rz \cdot \bar{\xi}) \, dv(\xi).$$

Noting that $\int_0^1 r^{2n+1} \log 1/r \, dr = 1/(4(n+1)^2)$. We obtain

$$\begin{aligned} |(f * g)_0(z)| &= \left| 4 \int_0^1 r \log \frac{1}{r} \int_{\partial B_n} \mathcal{D}^1 f(r\xi) \mathcal{D}^1 g(rz \cdot \bar{\xi}) \, dv(\xi) \, dr \right| \\ &\leq 4 \int_0^1 \int_{\partial B_n} (1-r) |\mathcal{D}^1 f(r\xi)| |\mathcal{D}^1 g(rz \cdot \bar{\xi})| \, dv(\xi) \, dr \\ &\leq 4 \|g\|_{\mathcal{B}(B_n)} \|f\|_{G_0^1(B_n)}. \end{aligned}$$

It is easy to see that

$$\|(f * g)_0(z_1) - (f * g)_0(z_2)\|_{G_0^1(B_n)} \rightarrow 0$$

as $|z_1 - z_2| \rightarrow 0$. Thus $(f * g)_0 \in A(U^n)$ for all $f \in G_0^1(B_n)$.

We now prove $(G_q^1(B_n), BMOA(U^n))_q \subset \mathcal{B}(B_n)$ by showing that $(G_q^1(B_n), \mathcal{B}(U^n))_q \subset \mathcal{B}(B_n)$. For $g(z) = \sum_{\alpha} b_{\alpha} z^{\alpha}$, we defined a linear operator

$$T_g : G_q^1(B_n) \rightarrow \mathcal{B}(U^n)$$

by $T_g(f) = (f * g)_q$. So T_g is a bounded linear operator from $G_q^1(B_n)$ to $\mathcal{B}(U^n)$ and

$$(6) \quad \|(f * g)_q\|_{\mathcal{B}(U^n)} \leq C_g \|f\|_{G_q^1(B_n)}$$

for all $f \in G_q^1(B_n)$, where C_g is a constant depending on g ([3, Lemma 3.4]).

Let $a \in \mathbb{C}$ and $m \in \mathbb{Z}_+$, $(a)_m$ stands for 1 if $m = 0$ and $a(a+1) \dots (a+m-1)$ if $m > 0$, let $\lambda \in U$ and

$$G_{a,b}(\lambda) := \sum_{m=0}^{\infty} (m+1)^{-b} \frac{(a)_m}{m!} \lambda^m, \quad a, b \in \mathbb{C}.$$

Setting $f := G_{n+q,-b+1}(\langle \cdot, r\bar{\xi} \rangle)$, $\xi \in \partial B_n$, $0 \leq r < 1$, $1 < b$, then $\mathcal{D}^1 f = G_{n+q,-b}(\langle \cdot, r\bar{\xi} \rangle)$. For $\eta \in U^n$, we have

$$\begin{aligned} h(\eta) &:= (f * g)_q(\eta) \\ &= \int g(\eta \cdot \bar{z}) G_{n+q,-b+1}(\langle z, r\bar{\xi} \rangle) \, dv_q(z) \\ &= \sum_{\alpha} (|\alpha| + 1)^{b-1} b_{\alpha} (r\xi\eta)^{\alpha} \\ &= \mathcal{D}^{b-1} g_{r\xi}(\eta). \end{aligned}$$

This implies

$$(7) \quad \mathcal{D}^1 h(r\mathbf{1}) = \mathcal{D}^b g(r^2\xi).$$

Using [3, Proposition 3.5(iii)] we have $G_{n+q,-b}(\lambda) = (1 - \lambda)^{-(n+q+b)} F(\lambda)$, $\lambda \in U$, where F satisfies

$$\sup_{z \in B_1} (1 - |z|^2)^{m-n-q-b} |\mathcal{D}^{n+q+b} F(z)| < \infty$$

for some $m > 0$ (see [3, p. 246]). Proposition 3.5(ii) gives

$$(8) \quad \begin{aligned} \|f\|_{G_q^1(B_n)} &\leq \|F\|_\infty \int |1 - \langle z, r\bar{\xi} \rangle|^{-(n+q+b)} (1 - |z|^2) dv_q(z) \\ &\leq C \|F\|_\infty (1 - r^2)^{1-b}, \end{aligned}$$

where C is a constant depending only on q, n and b . Combining (6) with (7) and (8) we get

$$|\mathcal{D}^b g(r^2\xi)| \leq CC_g \|F\|_\infty (1 - r^2)^{-b}, \quad \xi \in \partial B_n, \quad 0 < r < 1.$$

It follows that $g \in \mathcal{B}(B_n)$. The proof of Theorem 1 is finished. □

REMARKS.

- (1) Let $q = 0$ in Theorem 1, we have

$$\mathcal{B}(B_n) = (G_0^1(B_n), BMOA(U^n))_0.$$

That is f belongs to the Bloch space $\mathcal{B}(B_n)$ if and only if the non-weighted Hadamard products of f and g belong to $BMOA(U^n)$ for all f in $G_0^1(B_n)$, a subspace of the Hardy space $H^1(B_n)$.

- (2) When $n = 1$ and $q = 0$, Theorem 1 was proved in [5].

We next introduce the generalised mean Lipschitz space $\mathcal{L}(U^n)$. For $z = (z_1, \dots, z_n)$, let $z_{(j)} = (z_1, \dots, z_{j-1}) \in \mathbb{C}^{j-1}$ if $j > 1$ and $z^{(j)} = (z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-j}$ for $j < n$. So $(z_{(j)}, z^{(j)}) \in \mathbb{C}^{n-1}$ if $n > 1$. If f is a function defined in a \mathbb{C}^n -neighbourhood $N^n(z)$ of z , then $f(z_{(j)}, \cdot, z^{(j)})$ is a function defined in $N_j^n(z) = \{\ell \in \mathbb{C} : (z_{(j)}, \ell, z^{(j)}) \in N^n(z)\}$. Define $\mathcal{L}(U^n)$ the space of $f \in H(U^n)$ with

$$\|f\|_{\mathcal{L}(U^n)}^2 = |f(0)|^2 + \sum_{j=1}^n \sup_{(z_{(j)}, z^{(j)}) \in U^{n-1}} \int_0^1 M_\infty^2(r, (z_{(j)}, z^{(j)}) : \partial_j f) (1 - r^2) 2r dr < \infty,$$

where $M_\infty(r, (z_{(j)}, z^{(j)}) : f) = \sup\{|f(z_{(j)}, r\ell, z^{(j)})| : \ell \in \partial B_1\}$. Equipped with the norm $\|\cdot\|_{\mathcal{L}(U^n)}$, $\mathcal{L}(U^n)$ is a Banach space. Let $H^\infty(U^n)$ denote the space of all bounded holomorphic functions in U^n . The proof of Theorem 1 implies the following.

COROLLARY 2. For $q \geq 0$

$$\begin{aligned} \mathcal{B}(B_n) &= (G_q^1(B_n), A(U^n))_q = (G_q^1(B_n), H^\infty(U^n))_q \\ &= (G_q^1(B_n), \mathcal{L}(U^n))_q = (G_q^1(B_n), VMOA(U^n))_q \\ &= (G_q^1(B_n), \mathcal{B}_0(U^n))_q = (G_q^1(B_n), \mathcal{B}(U^n))_q. \end{aligned}$$

PROOF: In fact, the proof of Theorem 1 implies that

$$\mathcal{B}(B_n) = (G_q^1(B_n), A(U^n))_q = (G_q^1(B_n), \mathcal{B}(U^n))_q.$$

Noting that $A(U^n) \subset H^\infty(U^n) \subset \mathcal{B}(U^n)$. We have

$$\mathcal{B}(B_n) = (G_q^1(B_n), H^\infty(U^n))_q.$$

From [3, Theorem 5.6], $\mathcal{B}(B_n) \subset (A_q^1(B_n), \mathcal{L}(U^n))_q$. Combining this with the inclusion $G_q^1(B_n) \subset A_q^1(B_n)$, we obtain

$$\mathcal{B}(B_n) \subset (G_q^1(B_n), \mathcal{L}(U^n))_q.$$

Since $\mathcal{L}(U^n) \subset VMOA(U^n)$ ([3, Proposition 2.4]) and $VMOA(U^n) \subset \mathcal{B}_0(U^n) \subset \mathcal{B}(U^n)$, we have

$$\mathcal{B}(B_n) = (G_q^1(B_n), \mathcal{L}(U^n))_q = (G_q^1(B_n), VMOA(U^n))_q = (G_q^1(B_n), \mathcal{B}_0(U^n))_q.$$

Corollary 2 is proved. □

In particular, Corollary 2 gives

$$\mathcal{B}(B_n) = (G_0^1(B_n), A(U^n))_0 = (G_0^1(B_n), \mathcal{B}(U^n))_0,$$

which is a result in [6]. When $G_q^1(B_n)$ is replaced by $A_q^1(B_n)$, Corollary 2 was proved by Burbea and Li in [3].

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