TENSOR PRODUCTS OF BANACH ALGEBRAS*

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Introduction. In [3] Gelbaum defined the tensor product $A \otimes_C B$ of three commutative Banach algebras, A, B and C and established some of its properties. Various examples are given and the particular case where A, B and C are group algebras of L.C.A. groups G, H and K respectively, is discussed there. It is shown there that if K is compact $L_1(G) \otimes_{L_1(K)} L_1(H)$ is isomorphic to $L_1(\hat{S})$ where $\hat{S}$ is L.C.A. if and only if $L_1(G) \otimes_{L_1(K)} L_1(H)$ is semisimple.

It is the purpose of this paper to extend these results to the case where K is L.C.A. group and to point out the connection between the tensor product and spectral synthesis.

This paper is divided into three sections: section 1 is a collection of definitions and theorems which appear in [3]; section 2 deals with group algebras as a topological module; and in section 3 we discuss the case of the tensor product of group algebras.

1. Preliminaries. Let A, B and C be the commutative Banach algebras where A and B are C modules: $\|ac\| \leq \|a\| \, \|c\|$, $\|bc\| \leq \|b\| \, \|c\|$ for $a \in A$, $b \in B$ and $c \in C$.

We construct the commutative algebra

$$F_C(A, B) = \{f : f \in C^{A \times B},\ f(a, 0) = f(0, b) = 0,\ \gamma_1(f) = \Sigma \, \| f(a, b) \| \, . \, \| a \| \, . \, \| b \| < \infty \}$$

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where addition and multiplication by scalars are defined as usual and where multiplication of two elements $f_1, f_2 \in F_C(A, B)$ is defined by

$$f_1 * f_2(a, b) = \begin{cases} \{\Sigma f_1(a_1, b_1) f_2(a_2, b_2) : a_1 a_2 = a, \ b_1 b_2 = b\} & \text{if } \|a\| . \|b\| > 0 \\ 0 & \text{otherwise .} \end{cases}$$

In $F_C(A, B)$ we consider the closed ideal I (with respect to the semi-norm γ_1) generated by the functions of the following type:

1. $f(a_1 + a_2, b_1) = -f(a_1, b_1) = -f(a_2, b_1)$

   $f(a, b) = 0$ otherwise

2. $f(a_1, b_1 + b_2) = -f(a_1, b_1) = -f(a_1, b_2)$

   $f(a, b) = 0$ otherwise

3. $f(a_1 \phi_1, b_1) = -f(a_1, b_1 \phi_1)$

   $f(a, b) = 0$ otherwise

4. $f(a_1 \phi_1, b_1)\phi_1 = -f(a_1, b_1)$

   $f(a, b) = 0$ otherwise

where ϕ_1 represents either a scalar or an element of C.

With the above notations the tensor product $D = A \otimes_C B$ is defined to be $F_C(A, B)/I$: D is then a commutative Banach algebra with γ_1 as a (quotient) norm. If C is the complex numbers we obtain the usual tensor product $A \otimes_\gamma B$ endowed with the "greatest cross norm" - the "projective tensor product".

As is customary we denote by $\mathfrak{m}_A$, $\mathfrak{m}_B$, $\mathfrak{m}_C$ and $\mathfrak{m}_D$ the maximal spaces of A, B, C and D respectively.

In order to simplify our next theorems we add the following assumption: For every $(M_A, M_B) \in \mathfrak{m}_A \times \mathfrak{m}_B$ there exist $a \in A$, $b \in B$, $c_1, c_2 \in C$ such that $\widehat{ac_1}(M_A)\widehat{bc_2}(M_B) \neq 0$.

[In the general case the one point compactification of $\mathfrak{M}_A$ (resp. $\mathfrak{M}_B$) or equivalently the adjunction of module identity is needed.]

THEOREM 1. (i) There are continuous mappings $\mu : \mathfrak{M}_A \to \mathfrak{M}_C$, $\nu : \mathfrak{M}_B \to \mathfrak{M}_C$ such that, for $(a, b, c) \in A \times B \times C$ and $(M_A, M_B) \in \mathfrak{M}_A \times \mathfrak{M}_B$, $\hat{ac}(M_A) = \hat{a}(M_A)\hat{c}(\mu(M_A))$, $\hat{bc}(M_B) = \hat{b}(M_B)\hat{c}(\nu(M_B))$.

(ii) Let $\rho = \mu \times \nu : \mathfrak{M}_A \times \mathfrak{M}_B \to \mathfrak{M}_c \times \mathfrak{M}_c$ and let Δ be the diagonal of $\mathfrak{M}_C \times \mathfrak{M}_C$. Then there exists a homomorphism $\tau : \mathfrak{M}_D \to \rho^{-1}(\Delta)$, a locally compact subset of $\mathfrak{M}_A \times \mathfrak{M}_B$. If $\tau(M_D) = (M_A, M_B)$ and if $\mu(M_A) = M_C = \nu(M_B)$ then for every $\bar{f} = f/I \in D$, with $f(a_n, b_n) = c_n$ $n = 1\ldots$ and $f(a, b) = 0$ otherwise,

$$\hat{f}(M_D) = \Sigma\, \hat{c}_n(M_C)\,\hat{a}_n(M_A)\,\hat{b}_n(M_B).$$

THEOREM 2. Let $\{c\}$ be an approximate identity for C. Then $\{c\}$ is also an approximate identity for A if and only if each $a \in A$ is of the form $a_1 c_1$ where $a_1 \in A$ and $c_1 \in C$. Moreover, for $\epsilon > 0$, a_1 and c_1 can be chosen to satisfy $\|c_1\| = 1$ and $\|a_1 - a\| < \epsilon$.

For the proofs of these theorems as well as several other applications we refer the reader to [1], [3] and [5].

In the next sections we shall denote by $\Sigma\, c_n(a_n, b_n)$ and $\Sigma\, c_n(a_n \otimes b_n)$ elements of $F_C(A, B)$ and D respectively.

2. Group Algebras. In this section we shall focus our attention upon group algebras with an additional module property. Although some of the results of this section hold for a larger class of multipliers [5] we shall restrict ourselves to the following particular case [3]:

Let G and K be two L.C.A. groups with dual groups $\hat{G}$ and $\hat{K}$ respectively. Let $\theta : K \to G$ be a (topological) homomorphism of K into G (so that $\theta(K)$ is locally compact and hence $\theta(k)$ is closed [6] and let $\theta^* : \hat{G} \to \hat{K}$ be the (induced) dual mapping defined by [12]: $(\theta(k), \alpha) = (k, \theta^*(\alpha))$ where $\alpha \in \hat{G}$.

With the above notations we define the "module action" as

$$ac(g) = \int_K a(g - \theta(k))c(k)dk \quad \text{for} \quad a \in L_1(G),\ c \in L_1(K).$$

Under this definition $L_1(G)$ is an $L_1(K)$ module with $\|ac\| \leq \|a\| \|c\|$ and $(a_1c)a_2 = (a_1a_2)c$ etc. Indeed, the usual proofs hold here with the obvious modifications.

We now prove several propositions which will be used in the sequel.

LEMMA 3. Let $\alpha \in \hat{G}$. Then $\hat{ac}(\alpha) = \hat{a}(\alpha)\hat{c}(\theta^*(\alpha))$ for $a \in L_1(G)$, $c \in L_1(K)$.

Proof.
$$\hat{ac}(\alpha) = \int_G ac(g)\overline{(g, \alpha)}dg$$
$$= \int_G \int_K a(g - \theta(k))c(k)\overline{(g, \alpha)}\, dk\, dg$$
$$= \int_K \int_G a(g)c(k)\overline{(g, \alpha)}\overline{(\theta(k), \alpha)}\, dg\, dk$$
$$= \hat{a}(\alpha)c(\theta^*(\alpha)) .$$

LEMMA 4. Let $\bar{A}' = \{\Sigma a_ic_i;\ a_i \in L_1(G),\ c_i \in L_1(K)\}$. Then $\bar{A}' = L_1(G)$.

Since $\bar{A}'$ is a closed ideal in $L_1(G)$ it suffices to show that $\bar{A}'$ is not contained in α for every $\alpha \in \hat{G}$. [7, p. 148].

This clearly is the case since $\Phi(a, c) \equiv \hat{ac}(\alpha) \equiv \hat{a}(\alpha)\hat{c}(\alpha) \not\equiv 0$.

PROPOSITION 5. Let $\{u\}$ be an approximate identity for $L_1(K)$. Then $au \to a$ for every $a \in L_1(G)$.

Proof. Let $\epsilon > 0$. Choose $a_i \in L_i(G)$, $c_i \in L_i(K)$, $i = 1, \ldots, n$ and $u \in \{u\}$ such that

$$\| a - \sum_{i=1}^{n} a_ic_i \| < \epsilon/3 \text{ and } \| c_i - c_iu \| \leq \epsilon/3\eta .$$

Then,

$$(\| u \| \leq 1,\ \eta > \max \| a_j \|)\ 1 \leq j \leq n$$

$$\|a - au\| \leq \|a - \Sigma a_ic_i\| + \|\Sigma a_ic_i - \Sigma a_ic_iu\| + \|\Sigma a_ic_iu\| < \epsilon/_3 + \epsilon/_3 + \epsilon/_3 = \epsilon .$$

PROPOSITION 6. (i) Let $a \in L_1(G)$, $c \in L_1(K)$. Suppose $\hat{c} = 1$ on $\mathrm{supp}(\hat{a})$. Then $a = ac$.

(ii) Let $a \in L_1(G)$, and let $\epsilon > 0$. Then there exist $a_1 \in L_1(G)$, $c_1 \in L_1(k)$ such that $\| c_1 \| = 1$, $\| a - a_1 \| < \epsilon$ and $a = a_1 c_1$.

Proof. (i) $\hat{ac}(\alpha) = \hat{a}(\alpha)\hat{c}(\theta^*(\alpha)) = \hat{a}(\alpha)$ by Lemma 3 and our assumption. Hence, since $L_1(G)$ is semisimple $ac = a$.

(ii) Theorem 2.

3. Tensor Products by Group Algebras. We now turn our attention to a particular example by a tensor product over a Banach algebra - the case where the algebras involved are group algebras of a locally compact abelian group and where the module action is in accordance with the previous section.

One purpose in the development will be the realization of D as a group algebra $L_1(\hat{S})$ of a L. C. A., $\hat{S}$ constructible from G, H and K. Another equally important problem will be the semisimplicity of D. It is a well-known open question whether the tensor product of semi-simple Banach algebras is again semisimple. In the case of group algebras this is true since $L_1(G) \otimes L_1(H) = L_1(G \times H)$ [2], [4] and [11]. Yet in the general case this is known to be true if the condition of "monomorphy" ($A \otimes_\gamma B \to A \otimes B$ is 1-1) holds, which is the case if either one of the algebras satisfies the Grothendieck condition of approximation [2], [4], [10], and [11].

To focus our ideas let G, H, and K be three L. C. A. groups with dual groups $\hat{G}$, $\hat{H}$ and $\hat{K}$ respectively. Let $\theta : K \to G$ and $\psi : K \to H$ be homomorphisms of K into G and H respectively. With the previous definitions the tensor product $D = L_1(G) \otimes_{L_1(K)} L_1(H)$ is a well-defined Banach algebra. We first characterize its maximal ideal space - which turns out to be L. C. A. group S - and then define a linear mapping $T : F_{L_1(K)}(L_1(G), L_1(H)) \to L_1(S)$ which turns out to be an isomorphism of D onto $L_1(S)$ provided D is semisimple. Several rather powerful theorems are used in this development. Besides Cohen's factorization theorem (Proposition 6) [1], [5] we need Grothendieck's characterization of the tensor product [4] (these are used in showing that T is surjective) and Calderon's result in spectral synthesis [8].

Some of the ideas involved in this discussion appear in [3]. However, our proof of the semisimplicity of D is entirely different.

To make our discussion complete we indicate the proofs of several propositions which appear already in [3].

THEOREM 7. (i) The maps $\mu : \hat{G} \to \hat{K}$ and $\nu : \hat{H} \to \hat{K}$ are the duals θ^* and ψ^* of the maps θ and ψ.

(ii) $\tau(\mathfrak{m}_D) = \{(\alpha, \beta) : \alpha \in \hat{G},\ \beta = \hat{H},\ \theta^*(\alpha) = \psi^*(\beta)\} = (\theta^* \times \psi^*)^{-1}\Delta$ where Δ = diagonal of $\hat{K} \times \hat{K}$. Hence $\tau(\mathfrak{m}_D)$ is a closed subgroup of $\hat{G} \times \hat{H}$.

(iii) $\widehat{\tau(\mathfrak{m}_D)} = G \times H/\tau(\mathfrak{m}_D)^+$ where $+$ is the annihilator.

(iv) $\tau(\mathfrak{m}_D)^+ = (\theta \times -\psi)\Gamma = Q$ where Γ = diagonal of $K \times K$.

(v) $\tau(\mathfrak{m}_D) = \widehat{G \times H/_Q}$.

(vi) If $\bar{z} \in D$ and $\bar{z} = \Sigma c_m(a_m \otimes b_m)$ then for $M_D \in \mathfrak{m}_D \hat{\bar{z}}(M_D) = \Sigma c_m(\gamma)\hat{a}_m(\alpha)\hat{b}_m(\beta)$, where $\tau(M_D) = (\alpha, \beta)$ and $\theta^*(\alpha) = \psi^*(\beta) = \gamma$.

Proof. (i), (ii) and (vi) follow from Theorem 1, (iii) follows from (ii) by duality and (v) follows from (iv) by duality. To prove (v) we first note that Q is closed since its locally compact group in the relative topology (the mappings are open) [6].

Next we show that $Q \subset \tau(M_D)^+$. Indeed, for $g = \theta(k)$, $h = -\psi(k)$, $k \in K$ and $(\alpha, \beta) \in \tau(M_D)$, we have

$$(g, \alpha)(h, \beta) = (\theta(k), \alpha))(-\psi(k), \beta) = (k, \theta^*(\alpha))\overline{(k, \psi^*(\beta))}$$

$$= (k, \gamma)\overline{(k, \gamma)} = 1$$

by (ii) and (iii).

Finally, let $(\alpha_o, \beta_o) \in Q^+$ then $1 = (\theta(k), \alpha_o)(-\psi(k), \beta_o)$; hence $\theta^*(\alpha_o) = \psi^*(\beta_o)$. Hence $(\alpha_o, \beta_o) \in \tau(\mathfrak{m}_D)$ (by (ii)) and this completes the proof since Q is closed.

Let T be the linear operator defined on the functions in $F_{L_1(K)}(L_1(G), L_1(H))$ with finite support and values in $L_1(G \times H/Q)$; T is defined by

$$Tc(a, b)(\overset{o}{g}, \overset{o}{h}) = \int_Q \int_K a(g - \theta(k_1) - \theta(k_2))c(k_1)b(h + \psi(k_2))dk_1 dq_2$$

$$= \int_Q ac(g - \theta(k_2))b(h + \psi(k_2))dq_2,$$

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where dq represents the Haar measure on $Q = (\epsilon x - \psi)\ \mathrm{diag}(K \times K)$ and $(\overset{o}{g}, \overset{o}{h})$ represents the coset $(g, h) + Q$. By a proper choice of the Haar measures we have that the mapping $F \to \int_Q F((g, h) + q)dq$ is surjective and

$$\int_{G\times H/_Q} \int_Q F((g, h) + q)dq\, d\overset{o}{g}\, d\overset{o}{h} = \int_{G\times H} F(g, h)dgdh,$$

[7], [12].

PROPOSITION 8. (i) T is bounded, $(\|Tf\| \leq \gamma_1(f))$.

(ii) $T[I] = 0$.

(iii) T is multiplicative on D where T denotes the induced mapping by (ii).

(iv) $T: D \to L_1(G \times H/Q)$ is surjective.

(v) T is isomorphic.

Proof. (i), (ii) and (iii) are straight-forward.

(iv) is a consequence of Propositions 6 and the isometric isomorphism between $L_1(G) \otimes_\gamma L_1(H)$ and $L_1(G\times H)$. Indeed, let $\Sigma a'_n b_n \in L_1(G\times H)$; write $a'_n = a_n c_n$ and consider $\Sigma c_n (a_n \otimes b_n) \in D$ by proper choice of a_n and c_n. Then

$$T\Sigma c_n(a_n \otimes b_n) = \int_Q a_n c_n (g-\theta)(k_2)\, b_n(h + \psi(k_2))dq_2$$

which is surjective.

(v) One half is obvious. The second follows directly from the identity $\hat{z}(M_D) = T\hat{z}(\alpha, \beta)$ where $(\alpha, \beta) \longleftrightarrow M_D$ (Theorem 7).

We include a detailed proof of this identity since the involved computations are typical. To this end let $d\sigma = d\overset{o}{g}d\overset{o}{h}$; then

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$$\Sigma\, a_n \hat{c}_n(\alpha)\hat{b}_n(\beta) = \Sigma \int_{G\times H}\int_K a_n(g-\theta(k_1))c_n(k_1)b_n(h)(g,\alpha)(h,\beta)\,dk\,dg\,dh$$

$$= \Sigma \int_{G\times H/Q}\int_Q\int_K a_n(g-\theta(k_1+k_2))c_n(k_1)b_n(h+\psi(k_2))\overline{(g-\theta(k_2),\alpha)(h-\psi(k_2),\beta)}\,dk_1\,dq_2\,d\sigma$$

$$= \Sigma \int_{G\times H/Q}\int_Q\int_K a_n(g-\theta(k_1+k_2))c_n(k_1)b_n(h+\psi(k_2))\overline{(\overset{o}{g},\overset{o}{h}),(\alpha,\beta)}\,)dk_1\,dq_2\,d\sigma$$

$$= \Sigma \int_{G\times H/Q} T(c_n(a_n\otimes b_n)(\overset{o}{g},\overset{o}{h})(\overline{(\overset{o}{g},\overset{o}{h})(\alpha,\beta)})d\sigma = \overline{T\Sigma\, c_n(a_n\otimes b_n)}(\alpha,\beta)\,.$$

In order to simplify some of the statement, we introduce the following definitions:

Let

$$S = \{(\alpha,\beta,\gamma);\ \alpha\in\hat{G},\ \beta\in\hat{H},\ \gamma\in K,\ \theta^*(\alpha) = \psi^*(\beta) = \gamma\}\ .$$

A cube is a set of the form $E = E_{\hat{G}}\times E_{\hat{H}}\times E_{\hat{K}}$ where $E_{\hat{G}}$, $E_{\hat{H}}$ and $E_{\hat{K}}$ are subsets of $\hat{G}$, $\hat{H}$ and $\hat{K}$ respectively and where $E\cap S = \Phi$. An element $X = \sum_{i=1}^{m} a_i b_i c_i$ of $L_1(G\times H\times K)$ where $a_i\in L_1(G)$, $b_i\in L_1(H)$, $\epsilon_i\in L_1(k)$, $i = 1,\dots,n$ will be called a generator. A term abc will be a component of the generator.

LEMMA 9. (i) *S is a closed subgroup of* $\hat{G}\times\hat{H}\times\hat{K}$.

(ii) *If* $E = E_{\hat{G}}\times E_{\hat{H}}\times E_{\hat{K}}$ *is a cube then* $\theta^*(E_{\hat{G}})\cap\psi^*(E_{\hat{H}})\cap E_{\hat{K}} = \phi$.

Proof. (i) If $\lambda = (\alpha,\beta,\gamma)$ does not belong to S then for $\theta^*(\alpha) = \gamma_1 \neq \gamma$ (similarly for $\psi^*(\beta) = \gamma_2 \neq \gamma$) we choose two disjoint neighbourhoods (in $\hat{K}$) V_γ of γ and V_{γ_1} of γ_1. Then $\theta^{*-1}(V_{\gamma_1})\times\hat{H}\times V_\gamma$ is a cube neighbourhood of λ.

(ii) (By contradiction) If $\lambda\in\theta^*(E_{\hat{G}})\cap\psi^*(E_{\hat{H}})\cap E_{\hat{K}}$ then $\gamma = \theta^*(\alpha) = \psi^*(\beta)$ and $(\alpha,\beta,\gamma)\in S\cap E$.

LEMMA 10. *Let* $f\in L_1(G\times H\times K)$ *with* $\hat{f} = 0$ *on* S. *Then for arbitrary* $\epsilon > 0$ *there exists a generator* z *with components* z_i, $i = 1,\dots, L = L(\epsilon, s, t)$ *such that*

(i) $\{\text{supp. } \hat{z}_i\}$ are compact cubes;

(ii) $\|z-f\| < \epsilon$.

Proof. Without loss of generality we may assume that the support of $\hat{f}$ is a compact set disjoint from S. For, by Calderon's Theorem, [1], S is a spectral set and the usual triangle inequality completes the argument.

Let supp. $\hat{f} = \Lambda$ be a compact set disjoint from S. Choose generators x and y such that $\hat{x} = 1$ on Λ and the supp. $\hat{x}_i$ are compact cubes where x_i are the components for x for $i = 1, \ldots, n$ and $\|y-f\| < \epsilon / \|x\|$. Then $z = x * y$ satisfies the lemma.

LEMMA 11. Let $\bar{f} = \Sigma c_n(a_n \otimes b_n) \in L_1(G) \otimes_{L_1(K)} L_1(H)$.

Then $f = \Sigma a_n b_n c_n \in L_1(G \times H \times K)$. If $\hat{\bar{f}} \equiv 0$ then $f = 0$ on S.

Proof. $\|f\| \leq \Sigma \|a_n\| \|b_n\| \|c_n\| = \gamma_1(\Sigma c_n(a_n(a_n, b_n)) < \infty$.

Also, $f(\alpha, \beta, \gamma) = \Sigma \widehat{a_n b_n c_n}(\alpha, \beta, \gamma) = \Sigma \hat{a}_n(\alpha)\hat{b}_n(\beta)\hat{c}_n(\gamma) = f(\alpha, \beta, \gamma) = 0$ by Theorem 7.

LEMMA 12. Let $\bar{f} = c(a \otimes b) \in L_1(G) \otimes_{L_1(K)} L_1(H)$.

Let $\theta^*(\text{supp } \hat{a})$, $\psi^*(\text{supp } \hat{b})$, supp $\hat{c}$ be compact subsets of $\hat{K}$. Then, , if $\theta^*(\text{supp } \hat{a}) \cap \psi^*(\text{supp } \hat{b}) \cap \text{supp } \hat{c} = \phi$, $\bar{f} = 0$.

Proof. Choose V_1, V_2, V_3 neighbourhoods of $\theta^*(\text{supp } \hat{a})$, $\psi^*(\text{supp } \hat{b})$, supp $\hat{c}$ respectively such that $V_1 \cap V_2 \cap V_3 = \phi$. Choose local identities c_1, c_2, $c_3 \in L_1(K)$ such that $\hat{c}_i = 0$ outside of V_i, $i = 1, 2, 3$ and $\hat{c}_1 = 1$ on $\theta^*(\text{supp } \hat{a})$, $\hat{c}_2 = 1$ on $\psi^*(\text{supp } \hat{b})$, $\hat{c}_3 = 1$ on supp $\hat{c}$. Now $V_1 \cap V_2 \cap V_3 = \phi$ implies $c_1 c_2 c_3 = 0$ whence $c(a \otimes b) = cc_1(ac_1 \otimes bc_2) = cc_1c_2c_3(a \otimes b) = 0(a \otimes b) = 0$.

COROLLARY. Let z be a generator with components $z_i = a_i b_i c_i$, $i = 1, \ldots, n$. Let supp $\hat{z}_i$ be a compact cube. Then $z = \sum_1^n c_i(a_i \otimes b_i) = 0$.

Proof. By Lemma 9, $\theta^*(\text{supp } \hat{a}_i) \cap \psi^*(\text{supp } \hat{b}_i) \cap \text{supp } \hat{c}_i = \phi$, $i = 1, \ldots, n$. Hence, by Lemma 11 and the compactness of $\theta^*(\text{supp } a_i)$ etc., we get the required result.

THEOREM 13. D *is semisimple*.

Proof. Let $\overline{y} = \Sigma c_n(a_n \otimes b_n)$ be such that $\hat{\overline{y}} \equiv 0$. Consider, in accordance with Lemma 11, $y = \Sigma a_n b_n c_n \in L_1(G \times H \times K)$. Let $\epsilon > 0$. By Lemma 10 there exists a generator z with components z_i, $i = 1, \ldots, L$ such that $\operatorname{supp} \hat{z}_i$ are compact cubes and $\| z - y \| < \epsilon$.. By the previous corollary $\Sigma c_i(a_i \otimes b_i) = 0$. On the other hand we have that $\gamma_1(\overline{y} - \Sigma c_i(a_i \otimes b_i)) \leq \| y - z \| < \epsilon$. Hence $\gamma_1(\overline{y}) < \epsilon$, and D is semisimple.

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