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Stepping-Stone Model with Circular Brownian Migration

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Abstract. In this paper we consider the stepping-stone model on a circle with circular Brownian migration. We first point out a connection between Arratia flow on the circle and the marginal distribution of this model. We then give a new representation for the stepping-stone model using Arratia flow and circular coalescing Brownian motion. Such a representation enables us to carry out some explicit computations. In particular, we find the distribution for the first time when there is only one type left across the circle.

1 Introduction

The stepping-stone model is a mathematical model for population genetics. A discrete site stepping-stone model describes the simultaneous evolution of interacting populations over a collection of finite or countable colonies. There are mutation, selection and resampling within each colony, and there is migration among different colonies. See [Kim53, Shi88] for some early work.

The continuous site stepping-stone model introduced in [Eva97] is a process taking values from the space $\Xi := \{\mu : E \to M_1(\mathbb{K})\}$, where E denotes the continuous *site space*, $\mathbb{K}$ denotes the *type space*, and $M_1(\mathbb{K})$ denotes the space of all probability measures on $\mathbb{K}$. Intuitively, such a map μ simultaneously represents the relative frequencies of different types in populations at various sites. More precisely, for $e \in E$ and $B \subset \mathbb{K}$, $\mu(e)(B)$ represents the "proportion of the population at the site e possessing types from the set B". The "moments" of the continuous site stepping-stone model are specified using the so-called *migration processes* taking values in E. The paper [DEFKZ00] considered models of this type where E is a general *Lusin* space and the migration process belongs to a very general class of *Borel right processes*.

In this paper we only consider a stepping-stone model with site space $\mathbb{T}$, a circle of circumference 1, with type space $\mathbb{K} = [0, 1]$, and with Brownian migration on $\mathbb{T}$. We call it a stepping-stone model with circular Brownian migration (in short, a SSCBM) and write it as X throughout the paper.

The distribution of SSCBM is uniquely determined by a family of coalescing Brownian motions on $\mathbb{T}$. To present an explicit formula, we will require the following notation.

Given a positive integer n, let $\mathcal{P}_n$ denote the set of *partitions* of $\mathbb{N}_n := \{1, \dots, n\}$. That is, an element π of $\mathcal{P}_n$ is a collection $\pi = \{A_1, \dots, A_h\}$ of disjoint subsets of

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$\mathbb{N}_n$ such that $\bigcup_i A_i = \mathbb{N}_n$. The sets $A_1, \dots, A_h$ are the *blocks* of the partition π. The integer h is called the *length* of π and is denoted by $|\pi|$. Equivalently, we can think of $\mathcal{P}_n$ as the set of equivalence relations on $\mathbb{N}_n$ and write $i \sim_\pi j$ if and only if i and j belong to the same block of $\pi \in \mathcal{P}_n$.

Given $\pi \in \mathcal{P}_n$, let $\alpha_i := \min A_i$, $1 \le i \le |\pi|$. $\{\alpha_i\}$ is the collection of *minimal elements* for π.

By a circular (instantaneously) coalescing Brownian motion we mean a collection of Brownian motions on $\mathbb{T}$ such that any two of them will move together as soon as they first meet. Given a circular coalescing Brownian motion $(Z_1, \dots, Z_n)$ starting at $\mathbf{e} = (e_1, \dots, e_n)$. For $t > 0$, let $\pi^{\mathbf{e}}(t)$ be a $\mathcal{P}_n$-valued random partition such that $i \sim_{\pi^{\mathbf{e}}(t)} j$ if and only if $Z_i(t) = Z_j(t)$. Then $\pi^{\mathbf{e}}(t)$ is the random partition *induced* by $(Z_i(t))$. Write

$$\Gamma^{\mathbf{e}}(t) := \{\alpha_i(t) : 1 \le i \le |\pi^{\mathbf{e}}(t)|\}$$

for the collection of minimal elements for $\pi^{\mathbf{e}}(t)$.

SSCBM is then a Ξ-valued, continuous Markov process X with distribution specified as follows. Given $\mu \in \Xi$ and $n > 0$, for any $f_i \in C(\mathbb{T})$, $K_i \subset \mathbb{K}$, $i = 1, \dots, n$,

$$\begin{aligned}(1.1)\quad \mathbb{Q}^\mu\Big[\prod_{i=1}^n \int_{\mathbb{T}} de f_i(e) X_t(e)(K_i)\Big] \\ = \int_{\mathbb{T}^n} d\mathbf{e}\Big(\prod_{i=1}^n f_i(e_i)\Big)\,\mathbb{P}\Big[\prod_{i\in\Gamma^{\mathbf{e}}(t)} \mu(Z_i(t))\Big(\bigcap_{j\sim_{\pi^{\mathbf{e}}(t)} i} K_j\Big)\Big],\end{aligned}$$

where $\mathbb{Q}^\mu$ denotes the probability law of X when its initial value is μ. See [DEFKZ00, Theorem 4.1] for a result on a general continuous site stepping-stone model and for a rigorous treatment of the measurability requirements on $\mu(\,\cdot\,)$ and the sense in which X is continuous and uniquely characterized by (1.1).

In [DEFKZ00] a particle representation for X was given using the Poisson random measure on $D_{\mathbb{T}}[0,\infty[\, \times \mathbb{K}$ and a “look down” scheme similar to that in [DK96]. It leads to better insight into the model. In the same spirit we are going to propose another representation for X in this paper.

Evans [Eva97] showed that the analogue of X on the site space $E = \mathbb{R}$ degenerates, *i.e.*, for any $t > 0$, for almost all $e \in \mathbb{R}$, $X_t(e)$ becomes a point mass on some $k \in \mathbb{K}$. A stronger version of this clustering behavior was later shown in [DEFKZ00] on the site space $E = \mathbb{T}$ for general initial measure $\mu \in \Xi$ and in [Zho03] on the site space $E = \mathbb{R}$ for the special case where $\mu(e) = \theta \in M_1(\mathbb{K})$ for all $e \in \mathbb{R}$. In fact, when the site space is $E = \mathbb{T}$, there exists a random partition of $\mathbb{T}$ such that $\mathbb{T}$ is divided into a finite number of disjoint intervals and $X_t(e)$ is a point mass on the same $k \in \mathbb{K}$ for almost all e in each interval (see [DEFKZ00, Theorem 10.2]). In this way we can identify X with a step-function-valued process.

Let Ξ' be the space of $\mathbb{K}$-valued right continuous step functions on $\mathbb{T}$ equipped with the topology inherited from the Skorohod J_1 topology on $D_{\mathbb{K}}(\mathbb{T})$. For each $\mu \in \Xi$, we are going to construct a Ξ'-valued process $(X'_t, t > 0)$ which can be regarded as SSCBM with initial value μ under the above-mentioned identification.

To this end, we first point out an interesting connection between Arratia flow and SSCBM in Section 2. This connection allows us to specify the *entrance law* of X' using the pre-image of Arratia flow. Then we give an explicit construction of X' using Arratia flow and circular coalescing Brownian motion, and we will show that X' so defined does have the right distribution under the above-mentioned identification. In this sense X' provides a nice version for X. Such a representation enables us to compute the distribution of the time when there is only a single type of individuals left across $\mathbb{T}$ in Section 3. It also allows us to obtain a result on the type that survives eventually.

2 A Representation of Stepping-Stone Model with Circular Brownian Migration

We adopt some conventions for the rest of this paper. We identify $\mathbb{T}$ with interval $[0, 1)$ oriented anti-clockwise. Whenever we write $(e_1, \dots, e_m) \in \mathbb{T}^m$, unless otherwise specified, it implies that $e_1, \dots, e_m$ have been already arranged in anti-clockwise order on $\mathbb{T}$. Given $u, v \in \mathbb{T}$, write $[u, v[$ for an interval starting at u and ending at v in anti-clockwise order. Write $v - u$ for the length of the interval $[u, v[$. For $(\kappa_1, \dots, \kappa_m) \in \mathbb{K}^m$ and $(e_1, \dots, e_m) \in \mathbb{T}^m$, write $\sum_{i=1}^m \kappa_i 1\{[e_i, e_{i+1}[\}, e_{m+1} := e_1$, for a right continuous step function on $\mathbb{T}$.

Arratia flow on $\mathbb{T}$ describes the evolution of a stochastic system in which there is one Brownian motion starting at each point on $\mathbb{T}$. Two Brownian motions coalesce once they meet. Formally, a simple version of the Arratia flow on $\mathbb{T}$ can be defined as a collection $\{\phi(t, x) : t \geq 0, x \in \mathbb{T}\}$ of $\mathbb{T}$-valued random variables such that

- the random map $(t, x) \mapsto \phi(t, x)$ is jointly measurable,
- for each x, the map $t \mapsto \phi(t, x)$, $t \geq 0$, is continuous,
- for each t, the map $x \mapsto \phi(t, x)$ is non-decreasing and right continuous (in the sense that it lifts to a map $\mathbb{R} \mapsto \mathbb{R}$ which is non-decreasing and right continuous),
- for $(x_1, \dots, x_m) \in \mathbb{T}^m$ the process $(\phi(t, x_1), \dots, \phi(t, x_m))_{t \geq 0}$ has the same distribution as a circular coalescing Brownian motion starting at $(x_1, \dots, x_m)$.

Arratia flow (with $\mathbb{T}$ replaced by $\mathbb{R}$) was first introduced in [Arr79]. We refer to [Har84, Mat89] for definitions and constructions of related coalescing flows of (dependent) diffusions. Also see [JR05] for work on more general coalescing flows.

The existence of such a coalescing Brownian flow on $\mathbb{R}$ was established in [Arr79, Ch. 2, Theorem 1; Ch.4, Lemma5]. Similarly, we can construct $\{\phi(t, x) : t \geq 0, x \in \mathbb{T}\}$ as follows. Given a countable and dense subset Q of $\mathbb{T}$, we first construct $\{\phi(t, x') : t \geq 0, x' \in Q\}$ using independent circular Brownian motions starting at Q and the "collision precedence rule" introduced in [Arr79, Ch. 1]. Then set

$$\phi(t, x) := \lim_{\substack{x' \to x+ \\ x' \in Q}} \phi(t, x') \quad \text{for } t \geq 0 \text{ and } x \notin Q.$$

As in [Arr79], we can show that $\{\phi(t, x) : t \geq 0, x \in \mathbb{T}\}$ defined as above has the desired properties.

For the Arratia flow on $\mathbb{R}$, using a Borel–Cantelli type argument it was shown in [Arr79, Ch. 3, Theorem 12] that almost surely, for every $t > 0$,

$$\big|\{\phi(t,x), x \in (-a,a)\}\big| < \infty, \quad a > 0,$$

where $|A|$ denotes the cardinality for a set A. So, the image $\{\phi(t,x), x \in \mathbb{R}\}$ of map $\phi(t,\cdot)$ is a discrete set for every $t > 0$.

For the flow on $\mathbb{T}$, given $t > 0$ we can directly show that

$$\mathbb{P}\big[\big|\{\phi(t,x), x \in Q\}\big|\big] = 1 + 2\sum_{n=1}^{\infty} \exp\{-n^2\pi^2 t\};$$

see the proof for [DEFKZ00, Corollary 9.3]. Then by the right continuity of $\phi(t,\cdot)$ we have

$$N(t) := \big|\{\phi(t,x), x \in \mathbb{T}\}\big| = \big|\{\phi(t,x), x \in Q\}\big| < \infty. \tag{2.1}$$

Furthermore, there exists a sequence of distinct $\mathbb{T}$-valued random variables $(V_i(t)) \in \mathbb{T}^{N(t)}$ such that

$$\{\phi(t,x), x \in \mathbb{T}\} = \{V_i(t), i = 1,\dots,N(t)\}. \tag{2.2}$$

Since the map $\phi(t,\cdot)$ is non-decreasing, given $(u_1,u_2) \in \mathbb{T}^2$ such that $\phi(t,u_1) = \phi(t,u_2) = v$, we have either $\phi(t,u) = v$ for all $u \in]u_1,u_2[$, or $\phi(t,u) = v$ for all $u \in]u_2,u_1[$. Then the pre-image of $V_i(t)$ is a connected set. Moreover, $\phi(t,\cdot)$ is right continuous. So, there exists another sequence of distinct $\mathbb{T}$-valued random variables $(U_i(t)) \in \mathbb{T}^{N(t)}$ such that

$$\phi(t,x) = V_i(t) \text{ for } x \in [U_i(t), U_{i+1}(t)[, \tag{2.3}$$

where $i = 1,\dots,N(t)$ and $U_{N(t)+1} := U_1$. Note that we have suppressed the dependence of $(U_i(t))$ and $(V_i(t))$ on $N(t)$.

Similar results on the discreteness of the images for other coalescing flows can be found in [Har84, Theorem 7.4], in [Mat89, Theorem 3.4] and in [JR05, Proposition 4.1].

Put $\Delta(\epsilon) := \max_{1\le i\le N(\epsilon)}(U_{i+1}(\epsilon) - U_i(\epsilon))$, $\epsilon > 0$, and

$$\tau := \inf\{t \ge 0 : \phi(t,x) = \phi(t,y), \forall x,y \in \mathbb{T}\}. \tag{2.4}$$

Then τ is the time when the image of Arratia flow on $\mathbb{T}$ first becomes a set of a single element.

Lemma 2.1 *Given $\epsilon > 0$, $\phi(\epsilon,\cdot)$ is a random step function specified by* (2.1), (2.2) *and* (2.3) *using $N(\epsilon)$, $(U_i(\epsilon))$ and $(V_i(\epsilon))$. Moreover, we have $\Delta(\epsilon) \to 0$ in probability as $\epsilon \to 0+$, and $\mathbb{P}\{\tau > 0\} = 1$.*

Proof Given $\epsilon > 0$ and $n \in \mathbb{N}_+$, since

$$\{\phi(\epsilon, i/n), i = 0, \dots, n-1\} \subset \{V_i(\epsilon), i = 1, \dots, N(\epsilon)\},$$

then

$$\bigcap_{i=0}^{n-1} \{|\phi(\epsilon, i/n) - i/n| < 1/2n\} \subset \{\Delta(\epsilon) < 2/n\}.$$

Observe that

$$\mathbb{P}\Big\{\bigcap_{i=0}^{n-1} \{|\phi(\epsilon, i/n) - i/n| < 1/2n\}\Big\} \to 1 \text{ as } \epsilon \to 0+,$$

so, $\Delta(\epsilon) \to 0$ in probability.

Let $\epsilon \to 0+$ in $\mathbb{P}\{\tau \geq \epsilon\} \geq \mathbb{P}\{\phi(\epsilon, 0) \neq \phi(\epsilon, 1/2)\}$ Then $\mathbb{P}\{\tau > 0\} = 1$ follows readily. ■

For any $\mu \in \Xi$, given $N(t)$, $(U_i(t), i = 1, \dots, N(t))$ and $(V_i(t), i = 1, \dots, N(t))$, let $(\kappa_i, i = 1, \dots, N(t))$ be a collection of independent $\mathbb{K}$-valued random variables such that κ_i follows the distribution $\mu(V_i(t))$. Define

$$X'_t(e) = \sum_{i=1}^{N(t)} \kappa_i 1\big\{\, [U_i(t), U_{i+1}(t)[\,\big\}(e), \quad e \in \mathbb{T}, U_{N(t)+1} := U_1. \tag{2.5}$$

We first point out that $X'(t)$, when identified as

$$e \mapsto \sum_{i=1}^{N(t)} \delta_{\kappa_i} 1\big\{\, [U_i(t), U_{i+1}(t)[\,\big\}(e), \quad e \in \mathbb{T}, \tag{2.6}$$

is indeed a version of X_t.

Proposition 2.2 *For any $t > 0$, with the identification* (2.6) *X'_t has the same distribution as X_t under $\mathbb{Q}^\mu$.*

Proof To determine the distribution of X'_t we only need to specify joint distributions such as

$$\mathbb{P}\{X'_t(e_1) \in K_1, \dots, X'_t(e_n) \in K_n\}, \quad K_i \subset \mathbb{K}, i = 1, \dots, n.$$

By definition $(\phi(t, e_1), \dots, \phi(t, e_n))$ is a circular coalescing Brownian motion starting at $(e_1, \dots, e_n)$. Let $\pi^{\mathbf{e}}(t)$ be the induced partition on $\mathbb{N}_n$. Given $N(t)$, $(U_i(t))$ and $(V_i(t))$ as before, observe that $i \sim_{\pi^{\mathbf{e}}(t)} j$ implies $\phi(t, e_i) = \phi(t, e_j) = V_r(t)$ for some r, and so e_i and e_j belong to the same interval $[U_r(t), U_{r+1}(t)[$ where $X'_t(e_i) = X'_t(e_j) = \kappa_r$ follows distribution $\mu(V_r(t))$. Therefore,

$$\begin{aligned}(2.7)\qquad &\mathbb{P}\Big\{\bigcap_{i=1}^{n}\{X'_t(e_i) \in K_i\}\Big\}\\ &= \mathbb{P}\Big[\prod_{i\in\Gamma^{\mathbf{e}}(t)}\sum_{r=1}^{N(t)} 1\Big\{\kappa_r \in \bigcap_{j\sim_{\pi^{\mathbf{e}}(t)} i} K_j\Big\} 1\{e_i \in [U_r(t), U_{r+1}(t)[\}\Big]\\ &= \mathbb{P}\Big\{\prod_{i\in\Gamma^{\mathbf{e}}(t)} \mu(\phi(t, e_i))\Big(\bigcap_{j\sim_{\pi^{\mathbf{e}}(t)} i} K_j\Big)\Big\}.\end{aligned}$$

An inspection of (2.7) reveals that (1.1) holds for $X'(t)$ when it is regarded as Ξ-valued. So X_t and X'_t have the same distribution. ■

By Proposition 2.2, there exists a Ξ-valued SSCBM $(X_t, t \geq 0)$ and a collection of Ξ'-valued random variables $(X'_t, t > 0)$ such that X_t may be identified with X'_t via (2.6) for all $t > 0$. In the rest of the paper, we will assume that $(X_t, t \geq 0)$ is such an SSCBM, and we will consider it to have values in Ξ' at all positive times whenever this is convenient.

We can read off some properties for X_t, $t > 0$, immediately from Proposition 2.2. First, with probability one X_t (as a function of e) can only take finitely many different values from $\mathbb{K}$. Moreover, if $\mu(e)$ is a diffuse measure for almost all $e \in \mathbb{T}$, then with probability one X_t takes different values over different intervals on $\mathbb{T}$, *i.e.*, $X(e) = \kappa$ for $e \in [e_1, e_2[$, whenever $X(e_1) = \kappa = X(e_2)$. Such properties are also discussed in [DEFKZ00, §10].

Conditioning on $X_s = \sum_{i=1}^{m} \kappa_i 1\{[u_i, u_{i+1}[\}$, (1.1) shows that given $t > s$, X_t can only take values from $\{\kappa_i\}$. Moreover, for any $\{\kappa'_j, j = 1, \dots, n\} \subset \{\kappa_i\}$ and any $(z_j) \in \mathbb{T}^n$, by (1.1) we can further show that

$$\begin{aligned}(2.8)\qquad &\mathbb{Q}\Big\{\bigcap_{j=1}^{n}\{X_t(z_j) = \kappa'_j\} \;\Big|\; X_s = \sum_{i=1}^{m}\kappa_i 1\{[u_i, u_{i+1}[\}\Big\}\\ &= \mathbb{Q}\Big\{\bigcap_{j=1}^{n}\{X_s(Z_j(t-s)) = \kappa'_j\}\Big\},\end{aligned}$$

where (Z_j) is a circular coalescing Brownian motion starting at (z_j).

To describe the evolution of X over time we need a lemma on duality between two circular coalescing Brownian motions.

Fix $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{T}^m$ and $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{T}^n$. Let $(Y_1, \dots, Y_m)$ be an m-dimensional circular coalescing Brownian motion starting at $\mathbf{y}$. Let $(Z_1, \dots, Z_n)$ be an n-dimensional circular coalescing Brownian motion starting at $\mathbf{z}$. Put

$$I_{i\vec{j}}(t, \mathbf{z}) := 1\{Y_i(t) \in [z_j, z_{j+1}[\}$$

and

$$I_{ij}^{\leftarrow}(t,\mathbf{y}) := 1\{y_i \in [Z_j(t), Z_{j+1}(t)[\}$$

for $1 \le i \le m$ and $1 \le j \le n$. Recall that $z_{n+1} := z_1$ and $Z_{n+1} := Z_1$.

Lemma 2.3 *The two $(m \times n)$-dimensional arrays $(I_{ij}^{\leftarrow}(t,\mathbf{y}))$ and $(I_{ij}^{\rightarrow}(t,\mathbf{z}))$ have the same distribution.*

Proof We can prove Lemma 2.3 in the same way as [Zho07, Theorem 2.1], *i.e.*, we first show that the corresponding duality holds for circular coalescing random walks, and then apply time space scaling to obtain the desired result for circular coalescing Brownian motions. ■

By Lemma 2.3 we can easily derive the following side result concerning a dual relationship for Arratia flow on $\mathbb{T}$. Such a result was pointed out in [Arr79] for coalescing Brownian flow on the real line.

Proposition 2.4 *When identified as point processes, $(U_i(t))$ and $(V_i(t))$ have the same distribution for any fixed $t > 0$.*

Proof For any $(z_i) \in \mathbb{T}^{2n}$, let (Z_i) be a coalescing Brownian motion starting at (z_i). Consider a sequence of circular coalescing Brownian motions $\{(\phi(t, x_i^m))_{i=1}^m, m = 1, 2, \dots\}$ such that the set $\{x_i^m, i = 1, \dots, m\}$ of starting locations approaches to a dense set in $\mathbb{T}$ as $m \to \infty$. Such a sequence provides an "approximation" for the Arratia flow on $\mathbb{T}$. Then by Lemma 2.3

$$\begin{aligned}(2.9)\qquad \mathbb{P}\Big\{\bigcap_{i=1}^m \Big\{\phi(t, x_i^m) \notin \bigcup_{j=1}^n (z_{2j-1}, z_{2j})\Big\}\Big\}\\ = \mathbb{P}\Big\{\bigcap_{i=1}^m \Big\{x_i^m \notin \bigcup_{j=1}^n (Z_{2j-1}(t), Z_{2j}(t))\Big\}\Big\}.\end{aligned}$$

On one hand, taking limits on both sides of (2.9) as $m \to \infty$, we can show that

$$\mathbb{P}\Big\{\{V_i(t)\} \cap \bigcup_{j=1}^n (z_{2j-1}, z_{2j}) = \varnothing\Big\} = \mathbb{P}\Big\{\bigcap_{j=1}^n \{Z_{2j-1}(t) = Z_{2j}(t)\}\Big\}.$$

On the other hand, $\{U_i(t)\} \cap (z_{2j-1}, z_{2j}) = \varnothing$ if and only if

$$(z_{2j-1}, z_{2j}) \subset [U_i(t), U_{i+1}(t)[$$

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for some i if and only if $\phi(t, z_{2j-1}) = V_i(t) = \phi(t, z_{2j})$ for some i. Consequently, we also have

$$\mathbb{P}\Big\{ \{U_i(t)\} \cap \bigcup_{j=1}^{n}(z_{2j-1}, z_{2j}) = \varnothing \Big\} = \mathbb{P}\Big\{ \bigcap_{j=1}^{n}\{\phi(t, z_{2j-1}) = \phi(t, z_{2j})\} \Big\}.$$

Therefore, $(U_i(t))$ and $(V_i(t))$ have the same *avoidance function*. So, the assertion of this proposition holds (see [Kal76, Theorem 3.3]). ■

Let us go back to the stepping-stone model. We first consider a special initial value ν. Given $\nu = \sum_{i=1}^{m} \delta_{\kappa_i} 1\{[u_i, u_{i+1}[\} \in \Xi$, write $\mathbf{Y} = (Y_i)$ for an m-dimensional circular coalescing Brownian motion starting at $\mathbf{u} := (u_i)$ and define

$$X_t'' = \sum_{i=1}^{m} \kappa_i 1\{[Y_i(t), Y_{i+1}(t)[\}, \quad t \geq 0,$$

with the convention that $1\{[y, y[\} := 0$. We also write $\mathbb{Q}^\nu$ for the probability law of X''.

Lemma 2.5 *X'' has the same distribution as X under $\mathbb{Q}^\nu$.*

Proof (X_t'') is clearly a Markov process from its definition.

Given $\{\kappa_1', \dots, \kappa_n'\} \subset \{\kappa_i, i = 1, \dots, m\}$ and $(v_j) \in \mathbb{T}^n$, let $\mathbf{Z} = (Z_j)$ be a circular coalescing Brownian motion starting at $\mathbf{v} := (v_j)$. Set

$$g(\mathbf{y}; \mathbf{z}) := \prod_{j=1}^{n} \sum_{i:\kappa_i = \kappa_j'} 1\big\{ [y_i, y_{i+1}[\big\} (z_j), \quad \mathbf{y} := (y_i), \mathbf{z} := (z_j).$$

Lemma 2.3 yields that

$$\begin{aligned}
\mathbb{Q}^\nu\Big\{ \bigcap_{j=1}^{n}\{X_t''(v_j) = \kappa_j'\} \Big\} &= \mathbb{P}\Big\{ \bigcap_{j=1}^{n}\Big\{ \sum_{i=1}^{m} \kappa_i 1\{[Y_i(t), Y_{i+1}(t)[\}(v_j) = \kappa_j' \Big\} \Big\} \\
&= \mathbb{P}[g(\mathbf{Y}(t); \mathbf{v})] \\
&= \mathbb{P}[g(\mathbf{u}; \mathbf{Z}(t))] \\
&= \mathbb{P}\Big\{ \bigcap_{j=1}^{n}\Big\{ \sum_{i=1}^{m} \kappa_i 1\{[u_i, u_{i+1}[\}(Z_j(t)) = \kappa_j' \Big\} \Big\} \\
&= \mathbb{Q}^\nu\Big\{ \bigcap_{j=1}^{n}\Big\{ X_0''(Z_j(t)) = \kappa_j' \Big\} \Big\}.
\end{aligned}$$

Thus X'' and X have the same initial distribution under $\mathbb{Q}^\nu$, and it follows from (2.8) that they have the same distribution. ■

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Now we are ready to construct a representation for X with a general initial value $\mu \in \Xi$. Given $\epsilon > 0$, as in (2.5) put

$$X''_\epsilon = \sum_{i=1}^{N(\epsilon)} \kappa_i 1\big\{ [U_i(\epsilon), U_{i+1}(\epsilon)[\big\}. \tag{2.10}$$

Given $N(\epsilon)$, $(U_1(\epsilon), \dots, U_{N(\epsilon)}(\epsilon))$ and $(\kappa_1, \dots, \kappa_{N(\epsilon)})$, write (Y_i) for an $N(\epsilon)$-dimensional circular coalescing Brownian motion starting at $(U_i(\epsilon))$. We further define

$$X''_t = \sum_{i=1}^{N(\epsilon)} \kappa_i 1\big\{ [Y_i(t-\epsilon), Y_{i+1}(t-\epsilon)[\big\}, \quad t \geq \epsilon, \tag{2.11}$$

again, with the convention that $1\{[y, y[\} := 0$. Write $\mathbb{Q}^\mu$ for the probability law of $(X''_t, t \geq \epsilon)$.

It follows from Proposition 2.2 and (2.10) that X''_ϵ and X_ϵ have the same distribution

$$\nu = \sum_{i=1}^{N(\epsilon)} \delta_{\kappa_i} 1\big\{ [U_i(\epsilon), U_{i+1}(\epsilon)[\big\},$$

and we may condition on ν and apply Lemma 2.5 to obtain the following result.

Theorem 2.6 *Given $\mu \in \Xi$ and $\epsilon > 0$, $(X''_t, t \geq \epsilon)$ has the same distribution as $(X_t, t \geq \epsilon)$ under $\mathbb{Q}^\mu$.*

Remark 2.7 The representation (2.11) suggests that SSCBM can also be thought of as a multi-type, nearest-neighbor voter model on $\mathbb{T}$. See [Lig85, CH. 5] for discussions on voter models.

3 The First Time When There Is a Single Type Left

In this section we are going to study properties of X using the representation given in Section 2.

Treating $(X_t, t > 0)$ as Ξ'-valued, put

$$T := \inf\{t > 0 : \exists \kappa \in \mathbb{K}, X_t(e) = \kappa, \forall e \in \mathbb{T}\}.$$

Then T is the first time when a single type of individuals prevails all over $\mathbb{T}$. It is easy to see from the representation (2.11) that $\mathbb{Q}^\mu\{T < \infty\} = 1$, for all $\mu \in \Xi$. Now we are going to find the exact distribution for T.

We start with a preliminary result which is interesting in its own right. Let (Y_i) be an m-dimensional circular coalescing Brownian motion starting at $(y_i) \in \mathbb{T}^m$, $m \geq 2$. Let $T_m := \inf\{t > 0 : Y_1(t) = \dots = Y_m(t)\}$.

Proposition 3.1 *Given any positive integer $m \geq 2$, we have*

$$\mathbb{P}[e^{-\lambda T_m}] = \sum_{i=1}^{m} \frac{\sinh((y_{i+1} - y_i)\sqrt{\lambda})}{\sinh(\sqrt{\lambda})}, \quad \lambda > 0. \tag{3.1}$$

Proof For $i = 1, \ldots, m$, write S_i for the time when Y_{i+1} first overtakes Y_i from Y_i's "clockwise side" *i.e.*, when the interval length $Y_{i+1} - Y_i$ first reaches 1. As usual, we define $Y_{m+1} := Y_1$. Since $(Y_{i+1} - Y_i)/\sqrt{2}$ is again a Brownian motion which starts at $(y_{i+1} - y_i)/\sqrt{2}$ and stops whenever it reaches 0 or $1/\sqrt{2}$, S_i is then the first time that the Brownian motion $(Y_{i+1} - Y_i)/\sqrt{2}$ reaches $1/\sqrt{2}$ before it reaches 0 or $S_i = \infty$ if it reaches 0 before $1/\sqrt{2}$, *i.e.*,

$$S_i := \inf\{t > 0 : Y_{i+1}(t) - Y_i(t) = 1, Y_{i+1}(s) - Y_i(s) > 0, \forall\, 0 < s < t\}.$$

We thus have

$$\mathbb{P}[e^{-\lambda S_i}] = \frac{\sinh((y_{i+1} - y_i)\sqrt{\lambda})}{\sinh(\sqrt{\lambda})}. \tag{3.2}$$

See [RY91, Exercise II.3.10].

Our key observation is that $\{T_m < t\} = \bigcup_{i=1}^{m}\{S_i < t\}$, and the events on the right-hand side of this equation are disjoint. So, (3.1) follows. ■

Remark 3.2 By a standard argument we can show that for any positive integer $m \geq 2$,

$$\mathbb{P}[T_m] = \frac{1}{6} - \frac{1}{6}\sum_{i=1}^{m}(y_{i+1} - y_i)^3,$$

and moreover, $\mathbb{P}[T_m]$ attains its maximum $1/6 - 1/6m^2$ if and only if all the initial values $y_1, \ldots, y_m$ are equally spaced on $\mathbb{T}$.

Remark 3.3 An explicit expression for the distribution of T_m can also be found. By [Kni81, Theorem 4.1.1], we have

$$\mathbb{P}\{S_i \leq t\} = (y_{i+1} - y_i) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi(y_{i+1} - y_i))\exp\{-n^2\pi^2 t\}. \tag{3.3}$$

Therefore,

$$\mathbb{P}\{T_m \leq t\} = 1 + \frac{2}{\pi}\sum_{i=1}^{m}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi(y_{i+1} - y_i))\exp\{-n^2\pi^2 t\}.$$

Proposition 3.4 *For any $t > 0$, probability $\mathbb{P}\{T_m \leq t\}$ reaches its minimum if $y_1, \ldots, y_m$ are equally spaced on $\mathbb{T}$.*

Proof Put

$$\begin{aligned} &h(u_1, \ldots, u_{m-1}, t) \\ &\quad := 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\exp\{-n^2\pi^2 t\}\Big(\sin\Big(n\pi(1 - \sum_{i=1}^{m-1}u_i)\Big) + \sum_{i=1}^{m-1}\sin(n\pi u_i)\Big). \end{aligned}$$

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Then $\mathbb{P}\{T_m \le t\} = h(y_2 - y_1, \dots, y_m - y_{m-1}, t)$. In addition, for $0 \le u < 1$ and $t > 0$, put

$$g(u,t) := u + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi u)\exp\{-n^2\pi^2 t\}.$$

Then by (3.3) $g(u,t)$ is the (defective) distribution function for the time when a Brownian motion starting at $u/\sqrt{2}$ first reaches $1/\sqrt{2}$ before it reaches 0. As a result,

$$\frac{\partial g}{\partial t}(u,t) \ge 0. \tag{3.4}$$

We first notice that

$$\frac{\partial h}{\partial u_i}(1/m, \dots, 1/m, t) = 0, \quad i = 1, \dots, m-1.$$

We also find that

$$\frac{\partial^2 h}{\partial u_i^2}(u_1, \dots, u_{m-1}, t) = \frac{\partial g}{\partial t}\Big(1 - \sum_{k=1}^{m-1} u_k, t\Big) + \frac{\partial g}{\partial t}(u_i, t),$$

and

$$\frac{\partial^2 h}{\partial u_i \partial u_j}(u_1, \dots, u_{m-1}, t) = \frac{\partial g}{\partial t}\Big(1 - \sum_{k=1}^{m-1} u_k, t\Big), \quad i \ne j.$$

Consequently, given $(u_1, \dots, u_{m-1}) \in \mathbb{R}^{m-1}$, for $r \in \mathbb{R}$ such that $1/m + ru_i \ge 0$ and $\sum_{i=1}^{m-1} ru_i \le 1/m$,

$$\frac{dh}{dr}(1/m + ru_1, \dots, 1/m + ru_{m-1}, t)\Big|_{r=0} = 0$$

and

$$\begin{aligned}
&\frac{d^2 h}{dr^2}(1/m + ru_1, \dots, 1/m + ru_{m-1}, t)\\
&\quad = \Big(\sum_{i=1}^{m-1} u_i\Big)^2 \frac{\partial g}{\partial t}\Big(1/m - r\sum_{i=1}^{m-1} u_i, t\Big) + \sum_{i=1}^{m-1} u_i^2 \frac{\partial g}{\partial t}(1/m + ru_i, t)\\
&\quad \ge 0,
\end{aligned}$$

where we have used (3.4) for the last inequality. Therefore, the claim in this proposition holds. ∎

Let κ be the type of individuals left after time T. Then $\kappa = \lim_{t\to\infty} X_t(e)$, $\forall e \in \mathbb{T}$.

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Theorem 3.5 *Given $\mu \in \Xi$ such that $\mu(x)$ is a diffuse probability measure for almost all $x \in \mathbb{T}$, then the Laplace transform for T has the expression*

$$\mathbb{Q}^{\mu}[e^{-\lambda T}] = \frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}, \quad \lambda > 0. \tag{3.5}$$

Moreover,

$$\mathbb{Q}^{\mu}\{\kappa \in K\} = \int_{\mathbb{T}} de\mu(e)(K), \quad K \subset \mathbb{K}. \tag{3.6}$$

Proof Given $(y_i^m) \in \mathbb{T}^m$ for each $m \in \mathbb{N}_+$, we first observe that by (3.1),

$$\lim_{m\to\infty} \mathbb{P}[e^{-\lambda T_m}] = \lim_{m\to\infty} \sum_{i=1}^{m} \frac{(y_{i+1}^m - y_i^m)\sqrt{\lambda}}{\sinh(\sqrt{\lambda})} = \frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}, \tag{3.7}$$

provided $\max_{1\le i\le m}(y_{i+1}^m - y_i^m) \to 0+$.

For $\epsilon > 0$,

$$X_t'' = \sum_{i=1}^{N(\epsilon)} \kappa_i 1\big\{ [Z_i(t-\epsilon), Z_{i+1}(t-\epsilon)[\big\}, \quad t \ge \epsilon,$$

where, given $N(\epsilon)$, (Z_i) is a circular coalescing Brownian motion starting at $(U_i(\epsilon)) \in \mathbb{T}^{N(\epsilon)}$. Put

$$T(\epsilon) := \inf\{t \ge 0 : Z_1(t) = \cdots = Z_{N(\epsilon)}(t)\}.$$

Given $N(\epsilon)$, notice that $\kappa_1, \dots, \kappa_{N(\epsilon)}$, are all different, since $\mu(e)$ is diffuse for almost all $e \in \mathbb{T}$. Then $\epsilon + T(\epsilon)$ is also the first time after ϵ when $X''(e)$ assumes a single value in $\mathbb{K}$ for all $e \in \mathbb{T}$.

It follows from Theorem 2.6, Lemma 2.1 and (3.7) that

$$\begin{aligned} \mathbb{Q}^{\mu}[e^{-\lambda T}; T > 0] &= \lim_{\epsilon\to 0+} \mathbb{Q}^{\mu}[e^{-\lambda T}; T > \epsilon] \\ &= \lim_{\epsilon\to 0+} \mathbb{Q}^{\mu}[\mathbb{Q}^{\mu}[e^{-\lambda(\epsilon+T(\epsilon))}; T(\epsilon) > 0 | X_\epsilon'']] \\ &= \frac{\sqrt{\lambda}}{\sinh(\sqrt{\lambda})}. \end{aligned} \tag{3.8}$$

Letting $\lambda \to 0+$ in (3.8), we have $\mathbb{Q}^{\mu}\{T > 0\} = 1$. Consequently, Laplace transform (3.5) follows.

Finally, by (1.1) we have $\lim_{t\to\infty} \mathbb{Q}^{\mu}\{X_t(e) \in K\} = \lim_{t\to\infty} \mathbb{P}[\mu(Z(t))(K)]$, where Z is a circular Brownian motion starting at $e \in \mathbb{T}$. Thus (3.6) follows. ∎

Remark 3.6 Notice that the distribution of T does not depend on μ as long as $\mu(x)$ is diffuse for almost all $x \in \mathbb{T}$.

Remark 3.7 Let Y be a Brownian motion starting at $0 < y < 1/\sqrt{2}$. Put

$$S_y := \inf\{t \geq 0 : Y_t = 0 \text{ or } 1/\sqrt{2}\}.$$

We observe that for the μ in Theorem 3.5,

$$\begin{aligned}
\mathbb{Q}^\mu[e^{-\lambda T}] &= \lim_{y \to 0+} \frac{\sinh(y\sqrt{2\lambda})}{y\sqrt{2}\sinh(\sqrt{\lambda})} \\
&= \lim_{y \to 0+} \frac{\mathbb{P}[e^{-\lambda S_y}; Y_{S_y} = 1/\sqrt{2}]}{\mathbb{P}\left\{Y_{S_y} = 1/\sqrt{2}\right\}} \\
&= \lim_{y \to 0+} \mathbb{P}\left[e^{-\lambda S_y} \middle| Y_{S_y} = 1/\sqrt{2}\right].
\end{aligned} \tag{3.9}$$

Using (3.9) and [Kni81, Theorem 4.1.1] again we can further find an explicit expression for $\mathbb{Q}^\mu\{T \leq t\}$. For $t > 0$,

$$\begin{aligned}
\mathbb{Q}^\mu\{T \leq t\} &= \lim_{y \to 0+} \mathbb{P}\left\{ S_y \leq t \;\middle|\; Y_{S_y} = 1/\sqrt{2}\right\} \\
&= \lim_{y \to 0+} \frac{1}{\sqrt{2}y}\Big(\sqrt{2}y + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(\sqrt{2}n\pi y)\exp\{-n^2\pi^2 t\}\Big) \\
&= 1 + 2\sum_{n=1}^{\infty}(-1)^n \exp\{-n^2\pi^2 t\}.
\end{aligned} \tag{3.10}$$

Remark 3.8 It is not hard to see from the proof for Theorem 3.5 that the distribution (3.10) coincides with the distribution for τ, which has been defined in (2.4).

Again, for the μ given in Theorem 3.5, for any $\epsilon > 0$, let interval $[U'_\epsilon, U''_\epsilon[$ be the unique interval $[U_i(\epsilon), U_{i+1}(\epsilon)[$ in (2.10)) such that $\kappa = \kappa_i$, *i.e.*, $[U'_\epsilon, U''_\epsilon[$ is the collection of sites at time ϵ whose type eventually prevails.

Proposition 3.9 *For the μ given in Theorem 3.5, as $\epsilon \to 0+$ both (U'_ϵ) and (U''_ϵ) converge in distribution to a uniform distribution on $\mathbb{T}$.*

Proof Clearly $U''_\epsilon - U'_\epsilon \to 0$ in probability. Therefore, we just need to show that for any $[a, b[\subset \mathbb{T}$,

$$\lim_{\epsilon \to 0+} \mathbb{Q}^\mu\{U'_\epsilon \in [a, b[\} = b - a. \tag{3.11}$$

To prove (3.11), we first notice that given $N(\epsilon)$, events $\{S_i < \infty\}, 1 \leq i \leq N(\epsilon)$, are all disjoint, where S_i is defined as in the proof for Proposition 3.1, but for a coalescing

Brownian motion starting at $(U_i(\epsilon), U_{i+1}(\epsilon))$. Consequently, by (3.2)

$$\begin{aligned}\mathbb{Q}^\mu\{U'_\epsilon \in [a,b[\} &= \mathbb{Q}^\mu\Big\{\bigcup_{1\le i\le N(\epsilon)}\{U_i(\epsilon)\in[a,b[, S_i<\infty\}\Big\}\\ &= \mathbb{Q}^\mu\Big\{\sum_{1\le i\le N(\epsilon)}(U_{i+1}(\epsilon)-U_i(\epsilon))1\{U_i(\epsilon)\in[a,b[\}\Big\}\\ &= \mathbb{Q}^\mu[U_{n+1}(\epsilon)-U_m(\epsilon],\end{aligned}$$

where $m := \min\{i : U_i(\epsilon) \in [a,b[\}$ and $n := \max\{i : U_i(\epsilon) \in [a,b[\}$. Therefore, (3.11) follows from Lemma 2.1 readily. ■

Remark 3.10 If $\mu \in \Xi$ is arbitrary, we cannot find the explicit distribution for T under $\mathbb{Q}^\mu$. Nevertheless, similar to the proof for Theorem 3.5 we can still show that

$$\mathbb{Q}^\mu\{T \le t\} \ge 1 + 2\sum_{n=1}^{\infty}(-1)^n \exp\{-n^2\pi^2 t\}, \quad t \ge 0.$$

As a result, T under $\mathbb{Q}^\mu$ for an arbitrary $\mu \in \Xi$ is stochastically smaller than the T in Theorem 3.5.

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