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A class of majorant functions for contractors and equations

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Majorant functions for contractors can be defined in a natural way. Such a case is considered here in order to find iterative solutions of general equations in Banach spaces by means of contractors. A class of majorant functions is defined which contains in particular the linear majorant ones. Local and global existence and convergence theorems are proved.

1. Natural majorant functions

Let $P : D \subset X \to Y$ be a non-linear operator with domain D containing a sphere $S = S(x_0, r)$ with radius r and centre x_0, X and Y being Banach spaces. Denote by $L(Y \to X)$ the space of all linear bounded operators from Y into X and let $\Gamma : D \to L(Y \to X)$ be a mapping, that is, for fixed $x \in D$, $\Gamma(x) : Y \to X$ is a bounded linear operator from Y into X. Put

$$(1.1) \quad Q(s) = \max\{\|P(x+\Gamma(x)y)-Px-y\| \mid x \in S,\ y \in Y,\ x+\Gamma(x)y \in D,\ \|y\| \le s\} ,$$

assuming that $Q(s)$ is finite for $0 < s \le \eta$, where $\eta > 0$ is a certain number to be defined below. It follows from (1.1) that $Q(0) = 0$ and $Q(s)$ is non-decreasing. It results also from (1.1) that

$$(1.2) \qquad \|P(x+\Gamma(x)y)-Px-y\| \le Q(\|y\|)$$

for $x \in S$, $y \in Y$ whenever $x+\Gamma(x)y \in D$. If there exists a function Q satisfying (1.2), then we say that Γ is a contractor for P with majorant function Q satisfying the contractor inequality (1.2). Thus, a

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majorant function for Γ, if any, can be defined in a natural way by (1.1).

Consider now the problem of solving the operator equation

$$Px = 0 \ , \quad x \in D \ . \tag{1.3}$$

We assume that x_0 is chosen so as to satisfy

$$\|Px_0\| \le \eta \tag{1.4}$$

and Γ is bounded; that is

$$\|\Gamma(x)y\| \le B \quad \text{for} \quad x \in S \ . \tag{1.5}$$

In addition, we suppose that

$$Q(s) < s \quad \text{for} \quad s > 0 \tag{1.6}$$

and there exists the integral

$$I(\eta) = \int_0^{\eta} s[s-Q(s)]^{-1} ds < \infty \tag{1.7}$$

provided that the function $s/(s-Q(s))$ is non-increasing.

In order to solve equation (1.3) we use the following iterative procedure

$$x_{n+1} = x_n - \Gamma(x_n)Px_n \ , \quad n = 0, 1, \ldots \ . \tag{1.8}$$

Simultaneously we consider the following numerical iterative procedure

$$s_{n+1} = Q(s_n) \ , \quad s_0 = \eta \ , \quad n = 0, 1, \ldots \ , \tag{1.9}$$

where Q is the majorant function for Γ defined by (1.1) or (1.2).

An operator P is said to be closed if $x_n \in D$, $x_n \to x$ and $Px_n \to y$ imply $x \in D$ and $y = Px$.

LEMMA 1.1. *Let $Q(s) > 0$ for $s > 0$ with $Q(0) = 0$ be a function satisfying conditions (1.6) and (1.7), where $s/(s-Q(s))$ is non-increasing. Then the series $\sum_{i=0}^{\infty} s_i$ is convergent, where the sequence $\{s_n\}$ is defined by (1.9) and the following estimate holds:*

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$$(1.10) \qquad \sum_{i=n}^{\infty} s_i \le \sum_{i=n}^{\infty} \int_{s_{i+1}}^{s_i} s[s-Q(s)]^{-1} ds = \int_0^{s_n} s[s-Q(s)]^{-1} ds .$$

Proof. We have for $m > n$, by (1.9),

$$(1.11) \qquad \sum_{i=n}^{m-1} s_i = \sum_{i=n}^{m-1} s_i(s_i - s_{i+1})/(s_i - Q(s_i)) \le \sum_{i=n}^{m-1} \int_{s_{i+1}}^{s_i} s[s-Q(s)]^{-1} ds$$

$$= \int_{s_m}^{s_n} s[s-Q(s)]^{-1} ds ,$$

since the function $s/(s-Q(s))$ is non-increasing by assumption. Inequality (1.10) follows from (1.11), and the convergence of the series $\sum_{i=0}^{\infty} s_i$ results from (1.7), and we have

$$(1.12) \qquad \sum_{i=0}^{\infty} s_i \le I(\eta) .$$

Lemma 1.1 will be applied to prove the following.

THEOREM 1.1. *Let $P : D \to Y$ be a closed non-linear operator with domain D containing the sphere S. Suppose that Γ is a contractor for P and satisfies condition* (1.5) *and the contractor inequality* (1.2). *Furthermore, assume that the majorant function Q satisfies conditions* (1.6) *and* (1.7), *where η is given by* (1.4). *Finally let*

$$(1.13) \qquad BI(\eta) = r .$$

Then all x_n lie in S and the sequence $\{x_n\}$ defined by (1.8) *converges to a solution x of equation* (1.3) *and the error estimate is as follows:*

$$(1.14) \qquad \|x-x_n\| \le B \int_0^{s_n} s[s-Q(s)]^{-1} ds .$$

Proof. It follows from the contractor inequality (1.2) with $x = x_n$ and $y = -Px_n$ that

$$(1.15) \qquad \|Px_{n+1}\| \le Q(\|Px_n\|) .$$

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Since Q is non-decreasing we prove by induction, in virtue of (1.15), that

$$\|Px_{n+1}\| \le Q(s_n) = s_{n+1} , \quad n = 0, 1, \ldots . \tag{1.16}$$

It follows from (1.8), (1.5) and (1.16), by using induction, that

$$\|x_{n+1}-x_n\| \le Bs_n , \quad n = 0, 1, \ldots .$$

Hence it follows, in virtue of Lemma 1.1, in virtue of (1.11), that

$$\|x_m-x_n\| \le B \sum_{i=n}^{m-1} s_i \le B \int_{s_m}^{s_n} s[s-Q(s)]^{-1} ds , \tag{1.17}$$

and

$$\|x_n-x_0\| \le B \sum_{i=0}^{n-1} s_i \le B \int_{s_n}^{\eta} s[s-Q(s)]^{-1} ds < BI(\eta) = r , \tag{1.18}$$

in virtue of (1.13). Thus, it results from (1.18) that $x_n \in S$ for $n = 0, 1, \ldots$ and (1.17) shows that the sequence $\{x_n\}$ converges to some element x. On the other hand, by (1.12), the series $\sum_{n=0}^{\infty} s_n$ is convergent and obviously $s_n \to 0$ as $n \to \infty$. Therefore, (1.16) implies that $Px_n \to 0$ as $n \to \infty$. Since P is closed, it follows that $Px = 0$. The error estimate (1.14) follows from (1.17) by letting $m \to \infty$ so that $s_m \to 0$, and the proof is complete.

It is easily seen that the case of a contractor with linear majorant function, investigated in [1], is a particular one of Theorem 1.1. In other words, we have the following.

COROLLARY 1.1. *Under the corresponding hypotheses of Theorem* 1.1 *suppose that the majorant function* Q *is defined by* $Q(s) = qs$ *with* $0 < q < 1$. *Then all assertions of Theorem* 1.1 *hold true and the error estimate* (1.14) *yields*

$$\|x-x_n\| \le B\eta q^n/(1-q) . \tag{1.19}$$

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Proof. Condition (1.6) is obviously fulfilled and $Q(0) = 0$. The function Q is evidently increasing. Since the function $s/\{s-Q(s)\} = 1/(1-q)$ is constant, $I(\eta) = \eta/(1-q) < \infty$, yielding condition (1.7). In virtue of (1.9), we have

$$s_n = \eta q^n , \quad n = 0, 1, \ldots .$$

Hence, it follows that in this particular case the error estimate (1.14) coincides with (1.19) and condition (1.13) is replaced by $B\eta/(1-q) = r$.

EXAMPLE. Let $F : S \to S \subset X$ be a contraction with Lipschitz constant $q < 1$, that is

$$\|Fx-F\bar{x}\| \le q\|x-\bar{x}\| \quad \text{for all} \quad x, \bar{x} \in S .$$

Put $Px = x - Fx$. Then it is easy to verify that $\Gamma(x) \equiv I$ (the identity mapping of X) is a contractor for P with majorant function Q defined by $Q(s) = qs$.

Thus, Theorem 1.1 as well as Corollary 1.1 generalize the well known local Banach contraction principle.

REMARK 1.1. Suppose that in addition to the hypotheses of Theorem 1.1 the contractor $\Gamma(x)$ maps Y onto X . Then the solution of equation (1.3) is unique.

Proof. If x and $\bar{x}$ are two solutions of equation (1.3), then there exists an element $y \in Y$ such that

$$\bar{x} = x + \Gamma(x)y , \text{ since } \Gamma(x) \text{ is onto.}$$

It follows from the contractor inequality (1.2) that $\|y\| < Q(\|y\|) < \|y\|$, if $\|y\| > 0$, in virtue of (1.6). Hence, we obtain a contradiction which proves that $\|y\| = 0$, that is $\bar{x} = x$.

2. Global existence and convergence theorem

Theorem 1.1 yields a local existence and convergence theorem. However, using the same argument one can obtain a global existence and convergence theorem. Let $P : D \subset X \to Y$ be a non-linear operator and let $\Gamma : D \to L(Y \to X)$. We assume that $\Gamma(x)(Y) \subset D$ for all $x \in D$ and that the domain D of P is a linear subset of X . Then we can replace (1.1) by

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$$(2.1) \qquad Q(s) = \max\{\|P(x+\Gamma(x)y)\| \mid x \in D,\ y \in Y,\ \|y\| \le s\}$$

assuming that $Q(s)$ is finite for arbitrary $s > 0$. Then the contractor inequality (1.2) is replaced by the following one:

$$(2.2) \qquad \|P(x+\Gamma(x)y)-Px-y\| \le Q(\|y\|)$$

for $x \in D$ and arbitrary $y \in Y$. Condition (1.5) should be replaced by

$$(2.3) \qquad \|\Gamma(x)\| \le B \quad \text{for} \quad x \in D .$$

As in Paragraph 1 we assume that the majorant function Q for Γ satisfies condition (1.6) and that the function $s/(s-Q(s))$ is non-increasing for $Q \le s < \infty$. Finally, we assume that there exists the integral

$$(2.4) \qquad I(a) = \int_0^a s[s-Q(s)]^{-1}ds < \infty$$

for arbitrary positive a.

Now we can prove the following global existence and convergence.

THEOREM 2.1. *Let* $P : D \to Y$ *be a closed non-linear operator and let* Γ *be a bounded contractor for* P *satisfying the contractor inequality* (2.2), *condition* (2.3) *and* $\Gamma(x)(Y) \subset D$ *for arbitrary* $x \in D$. *Let the majorant function* Q *be non-decreasing and satisfy condition* (1.6) *and let the function* $s/(s-Q(s))$ *be non-increasing for* $0 \le s < \infty$. *If the integral* (2.4) *exists for arbitrary* $a > 0$, *then* P *maps* D *onto the whole of* Y *and the sequence* $\{x_n\}$ *defined by*

$$(2.5) \qquad x_{n+1} = x_n - \Gamma(x_n)[Px_n-l] , \quad n = 1, 2, \dots ,$$

where x_0 *is an arbitrary initial approximation, converges to a solution* x *of* $Px - l = 0$, *where* l *is an arbitrary element of* Y. *The error estimate* (1.14) *holds true, where the sequence* $\{s_n\}$ *is defined by* (1.9) *with* $\|Px_0-l\| \le \eta = s_0$.

Proof. Since for arbitrary $l \in Y$ the operators defined by Px and $Px - l$ have the same contractor Γ, it is sufficient to show that $Px = 0$ has a solution x and that the sequence $\{x_n\}$ defined by (2.5) with $l = 0$ converges to x. Using the same argument as in the proof of

Theorem 1.1 we prove (1.16) and (1.17) by induction. In virtue of Lemma 1.1 the series $\sum_{n=0}^{\infty} s_n$ is convergent. Hence, it follows from (1.17) that the sequence $\{x_n\}$ converges to some element x. Then all assertions follow in the same way as in the proof of Theorem 1.1.

REMARK 2.1. Under the hypotheses of Theorem 2.1 if, in addition $\Gamma(x)$ is onto for every $x \in D$, then P is a one-to-one mapping onto the whole of Y.

Proof. The proof is the same as that of Remark 1.1.

COROLLARY 2.1. *If the majorant function* Q *in Theorem* 2.1 *is defined by* $Q(s) = qs$ *with* $0 < q < 1$, *then all assertions of Theorem* 2.1 *hold true and the error estimate is given by* (1.19) *where* $\|Px_0 - l\| \le \eta$.

Proof. The proof follows from that of Corollary 1.1.

EXAMPLE. Let $F : X \to X$ be a contraction with Lipschitz constant $q < 1$. Then $\Gamma(x) \equiv I$ is a contractor for P defined by $Px = x - Fx$ with majorant function $Q(s) = qs$. Thus, both Theorem 2.1 and Corollary 2.1 generalize the well known global Banach contraction principle. The case of contractors with linear majorant functions is investigated in [1]. More facts about contractors with non-linear majorant functions are presented in [2].

REMARK 2.2. The class of majorant functions satisfying the hypotheses of Lemma 1.1 contains all linear majorant functions Q defined by $Q(s) = qs$ with $0 < q < 1$. It is easily seen that majorant functions Q which cannot be majorised by linear ones necessarily possess the following property:

$$(\alpha) \qquad s/\big(s-Q(s)\big) \to \infty \quad \text{as} \quad s \to 0 .$$

In fact, since the function $s/\big(s-Q(s)\big)$ is non-increasing by assumption it must be bounded by some positive constant M, if condition (α) is not satisfied. But $s/\big(s-Q(s)\big) \le M$ implies $Q(s) \le qs$ with $q = (M-1)/M < 1$ and $M > 1$ so that Q can be majorized by a linear function with $q < 1$.

Let us observe that the error estimate (1.14) is more accurate than the error estimate (1.19) obtained by replacing if possible the majorant function Q by a linear one with $q = (M-1)/M$.

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