AN APPLICATION OF SOME SPACES OF LORENTZ

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1. Introduction. The spaces $\Lambda(\alpha)$ and $M(\alpha)$ were defined by Lorentz **(2)** as follows. Let $0 < \alpha < 1$, $0 < l \leqslant \infty$; let ϕ be measurable on $(0, l)$, and, in case $l = \infty$, let the set where $|\phi(x)| > \epsilon$ have finite measure for each positive ϵ. Define

I $$||\phi(\cdot)||_{\Lambda(\alpha)} = \alpha \int_0^l x^{\alpha-1}\phi^*(x)\,dx$$

where $\phi^*(x)$ is the equi-measurable rearrangement of $|\phi|$ in decreasing order, and

II $$||\phi(\cdot)||_{M(\alpha)} = \sup_E (m(E))^{-\alpha} \int_E |\phi(x)|\,dx, \quad E \subseteq (0, l).$$

The spaces $\Lambda(\alpha)$ and $M(\alpha)$ consist of those ϕ for which

$$||\phi(\cdot)||_{\Lambda(\alpha)} < \infty, \quad ||\phi(\cdot)||_{M(\alpha)} < \infty$$

respectively.

Lorentz **(2**; §5**)** found, among other things, necessary and sufficient conditions that a given sequence be the moment sequence of a function in either $\Lambda(\alpha)$ or $M(\alpha)$, for $l = 1$. It is the object of this paper to find necessary and sufficient conditions that a function $f(s)$ on $s > 0$ be the Laplace transform of a function in $\Lambda(\alpha)$ or $M(\alpha)$ for $l = \infty$. To this end we make use of the Widder-Post inversion operator,

III $$L_{k,t}[f(s)] = \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} f^{(k)}\left(\frac{k}{t}\right),$$

whose theory may be found in **(4**; chap. VII**)**.

Section 2 of this paper contains the theory for the spaces $\Lambda(\alpha)$, and §3 the theory for the spaces $M(\alpha)$.

Henceforth when $l < \infty$, we shall denote the spaces $\Lambda(\alpha)$, $M(\alpha)$, L_p, over $(0, l)$ by $\Lambda(\alpha, l)$, and their respective norms by $||\phi(\cdot)||_{\Lambda(\alpha, l)}$. We shall continue to denote the spaces $\Lambda(\alpha)$ on $(0, \infty)$ by $\Lambda(\alpha)$ and the norms by $||\phi(\cdot)||_{\Lambda(\alpha)}$.

2. The space $\Lambda(\alpha)$. The first theorem yields some properties of the Laplace transform of a function in $\Lambda(\alpha)$, while the second theorem is the representation theorem.

THEOREM 1. *If* $\phi \in \Lambda(\alpha)$, *and*

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt,$$

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then

$$\int_0^\infty s^{-\alpha}|f(s)|\,ds < \infty.$$

If ϕ is positive and decreasing, then the above condition is necessary and sufficient that $\phi \in \Lambda(\alpha)$.

Proof. Suppose $\phi \in \Lambda(\alpha)$. Then

$$\begin{aligned}\int_0^\infty s^{-\alpha}|f(s)|\,ds &\leqslant \int_0^\infty s^{-\alpha}ds \int_0^\infty e^{-st}|\phi(t)|\,dt \\ &= \int_0^\infty |\phi(t)|dt \int_0^\infty e^{-st}s^{-\alpha}ds = \Gamma(1-\alpha)\int_0^\infty t^{\alpha-1}|\phi(t)|\,dt \\ &\leqslant \alpha^{-1}\Gamma(1-\alpha)||\phi(\cdot)||_{\Lambda(\alpha)} < \infty.\end{aligned}$$

Conversely, suppose ϕ is positive and decreasing. Then

$$\int_0^\infty s^{-\alpha}|f(s)|\,ds = \alpha^{-1}\Gamma(1-\alpha)\,||\phi(\cdot)||_{\Lambda(\alpha)},$$

and $\phi \in \Lambda(\alpha)$.

THEOREM 2. *Necessary and sufficient conditions that a function $f(s)$, defined for $s > 0$, be the Laplace transform of a function in $\Lambda(\alpha)$ are that*

(1) *f has derivatives of all orders in $(0, \infty)$ and $f^{(k)}(s) \to 0$ as $s \to \infty$ $(k = 0, 1, 2, \ldots)$,*

(2) *$||L_{k,\cdot}[f(s)]||_{\Lambda(\alpha)} \leqslant N$, where N is independent of k $(k = 0, 1, 2, \ldots)$.*

Proof of necessity. Let

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt,\quad \phi \in \Lambda(\alpha).$$

The necessity of (1) is well known; see (**4**; chap. 2, §5).

Now by (**4**; chap. 7, §6),

$$L_{k,t}[f(s)] = \int_0^\infty K(t,u)\,\phi(u)\,du,$$

where $K(t,u) = (k/t)^{k+1}e^{-ku/t}(u^k/k!)$. Thus $K(t,u) \geqslant 0$, and

$$\int_0^\infty K(t,u)\,du = \int_0^\infty K(t,u)\,dt = 1.$$

Hence, by[1] (**3**; Theorem 3.8.1), for each $a > 0$,

$$\int_0^a L_{k,t}[f(s)]^*dt \leqslant \int_0^a \phi^*(t)\,dt,$$

and thus by (**3**; Theorem 3.4.3), for any $a > 0$,

$$\alpha\int_0^a t^{\alpha-1}L_{k,t}[f(s)]^*dt \leqslant \alpha\int_0^a t^{\alpha-1}\phi^*(t)\,dt.$$

[1]This theorem, like all of Lorentz's, is stated for the case $l = 1$. However, all of Lorentz's theorems used here with one exception (to be noted later) are true for l infinite, as a glance at the proof shows.

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Letting $a \to \infty$, we have

$$||L_{k,\cdot}\, [f(s)]||_{\Lambda(\alpha)} \leqslant ||\phi(\cdot)||_{\Lambda(\alpha)},$$

and (2) is necessary.

Proof of sufficiency. By (**2**; 3.5(7)), if $g(t)$ is positive and non-increasing

$$\int_0^\infty t^{p-1}|g(t)|^p dt \leqslant K_p\left\{\int_0^\infty |g(t)|\, dt\right\}^p,\ p \geqslant 1.$$

Let $p = 1/\alpha$, $g(t) = t^{\alpha-1} L_{k,t}[f(s)]^*$. Then, the above result yields

$$\int_0^\infty |L_{k,t}[f(s)]|^{1/\alpha} dt = \int_0^\infty \{L_{k,t}[f(s)]^*\}^{1/\alpha} dt$$
$$\leqslant K_{1/\alpha}\left\{\int_0^\infty t^{\alpha-1} L_{k,t}[f(s)]^* dt\right\}^{1/\alpha} \leqslant K_{1/\alpha} N^{1/\alpha}.$$

Hence,

$$||L_{k,\cdot}[f(s)]||_{L(1/\alpha)} \leqslant N'$$

where $N' = K^\alpha{}_{1/\alpha} N$.

Thus, by (**4**; chap. 1, §17, and chap. 7, §15), $\phi \in L(1/\alpha)$, and an increasing unbounded sequence $\{k_i\}$ exist such that

(i) $||\phi(\cdot)||_{L(1/\alpha)} \leqslant N'$,

(ii) $f(s) = \int_0^\infty e^{-st}\phi(t)\, dt$

(iii) for any $\psi \in L((1-\alpha)^{-1})$,

$$\lim_{i\to\infty} \int_0^\infty \psi(t)\, L_{k_i,t}[f(s)]\, dt = \int_0^\infty \psi(t)\, \phi(t)\, dt.$$

It remains to be shown that $\phi \in \Lambda(\alpha)$.

But by (**3**; Theorem 3.6.1), for any $\psi \in \mathrm{M}(\alpha)$,

$$\left|\int_0^\infty \psi(t)\, L_{k,t}[f(s)]\, dt\right| \leqslant ||\psi(\cdot)||_{\mathrm{M}(\alpha)} ||L_{k,t}[f(s)]||_{\Lambda(\alpha)} \leqslant N\, ||\psi(\cdot)||_{\mathrm{M}(\alpha)}.$$

Hence, by (iii), and since, by (**2**; 1.3(4)), $L((1-\alpha)^{-1}) \subseteq \mathrm{M}(\alpha)$, for any $\psi \in L((1-\alpha)^{-1})$,

$$\left|\int_0^\infty \psi(t)\, \phi(t)\, dt\right| = \lim_{i\to\infty}\left|\int_0^\infty \psi(t)\, L_{k_i,t}[f(s)]\, dt\right| \leqslant N\, ||\psi||_{\mathrm{M}(\alpha)}.$$

Changing ψ to ψ sgn ϕ, we have for any positive $\psi \in L((1-\alpha)^{-1})$

$$\int_0^\infty \psi(t)|\phi(t)|\, dt \leqslant N\, ||\psi(\cdot)||_{\mathrm{M}(\alpha)},$$

and thus, by (**3**; Theorem 3.4.2), for any positive $\psi \in L((1-\alpha)^{-1})$,

$$\int_0^\infty \psi(t)\, \phi^*(t)\, dt \leqslant N\, ||\psi(\cdot)||_{\mathrm{M}(\alpha)}.$$

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Let $\psi(t) = \alpha t^{\alpha-1}$, $0 < \delta \leqslant t \leqslant R$, $\psi(t) = 0$ otherwise. Then $\psi \in L((1-\alpha)^{-1})$, and $||\psi(\cdot)||_{M(\alpha)} \leqslant 1$. Hence

$$\alpha \int_\delta^R t^{\alpha-1}\phi^*(t)\, dt \leqslant N,$$

and so, letting $\delta \to 0$, $R \to \infty$, we have

$$||\phi(\cdot)||_{\Lambda(\alpha)} < \infty,$$

and $\phi \in \Lambda(\alpha)$.

3. The space $M(\alpha)$. The first theorem of this section yields some properties of the Laplace transform of a function in $M(\alpha)$, while the second theorem is the representation theorem.

THEOREM 3. *If* $\phi \in M(\alpha)$, *and*

$$f(s) = \int_0^\infty e^{-st}\phi(t)\, dt,$$

then $s^\alpha f(s)$ *is bounded for* $s > 0$. *If* ϕ *is positive and decreasing, then the condition that* $s^\alpha f(s)$ *be bounded is necessary and sufficient for* $\phi \in M(\alpha)$.

Proof. Let $\phi \in M(\alpha)$. Then if $s > 0$, by (**3**; Theorem 3.6.1),

$$|f(s)| \leqslant \int_0^\infty e^{-st}|\phi(t)|\, dt \leqslant ||e^{-st}||_{\Lambda(\alpha)}||\phi(\cdot)||_{M(\alpha)}$$
$$= \alpha \int_0^\infty t^{\alpha-1}e^{-st}dt||\phi(\cdot)||_{M(\alpha)} = s^{-\alpha}\Gamma(\alpha+1)||\phi(\cdot)||_{M(\alpha)},$$

and $s^\alpha f(s)$ is bounded.

Conversely, suppose ϕ is positive and decreasing, and $s^\alpha f(s)$ is bounded. Let $\delta > 0$, and

$$\frac{1}{2s} < \delta < \frac{1}{s}.$$

Then

$$\int_0^\delta \phi(t)\, dt \leqslant e^{s\delta}\int_0^\delta e^{-st}\phi(t)\, dt \leqslant e\int_0^\infty e^{-st}\phi(t)\, dt \leqslant Ms^{-\alpha} \leqslant M'\delta^{-\alpha},$$

so that $||\phi(\cdot)||_{M(\alpha)} \leqslant M'$ and $\phi \in M(\alpha)$.

THEOREM 4. *Necessary and sufficient conditions that a function* $f(s)$, *defined for* $s > 0$, *be the Laplace transform of a function in* $M(\alpha)$ *are that*

(1) f *has derivatives of all orders in* $(0, \infty)$, $f^{(k)}(s) \to 0$ as $s \to \infty$ $(k = 0, 1, 2, \ldots)$,

(2) $||L_{k,\cdot}[f(s)]||_{M(\alpha)} \leqslant N$ *where* N *is independent of* k $(k = 0, 1, 2, \ldots)$.

Proof of necessity. Let

$$f(s) = \int_0^\infty e^{-st}\phi(t)\, dt, \qquad \phi \in M(\alpha).$$

The necessity of (1) is well known.

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Now as in Theorem 2,

$$L_{k,t}[f(s)] = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1}\int_0^\infty e^{-ku/t}u^k\phi(u)\,du = \frac{k^{k+1}}{k!}\int_0^\infty e^{-ku}u^k\phi(tu)\,du.$$

Hence, if $m(E) = \delta$,

$$\delta^{-\alpha}\int_E |L_{k,t}[f(s)]|\,dt \leqslant \frac{\delta^{-\alpha}k^{k+1}}{k!}\int_0^\infty e^{-ku}u^k du\int_E |\phi(tu)|\,dt$$
$$= \frac{k^{k+1}}{k!}\int_0^\infty e^{-ku}u^{k+\alpha-1}du\,(u\delta)^{-\alpha}\int_{uE}|\phi(t)|\,dt$$

where $uE = \{t|t = uv, v \in E\}$, so that $m(uE) = um(E)$.
Thus

$$\delta^{-\alpha}\int_E |L_{k,t}[f(s)]|\,dt \leqslant \frac{k^{k+1}}{k!}||\phi(\cdot)||_{M(\alpha)}\int_0^\infty e^{-ku}u^{k+\alpha-1}du$$
$$= ||\phi(\cdot)||_{M(\alpha)}\Gamma(k+\alpha)/k^\alpha\Gamma(k).$$

Hence, since $\Gamma(k+\alpha)/k^\alpha\,\Gamma(k)$ is bounded, we have $||L_{k,\cdot}\,[f(s)]||_{M(\alpha)} \leqslant N$.

Proof of sufficiency. It is clear that

$$||L_{k,\cdot}\,[f(s)]||_{M(\alpha,l)} \leqslant ||L_{k,\cdot}\,[f(s)]||_{M(\alpha)}.$$

Further, by[2] (**2**; Theorem 4), $M(\alpha, l) \subseteq L((1-\alpha')^{-1}, l)$ and

$$||L_{k,\cdot}[f(s)]||_{L((1-\alpha')^{-1},l)} \leqslant K_l||L_{k,\cdot}[f(s)]||_{M(\alpha,l)},$$

for every α', $0 < \alpha' < \alpha$. Let α' be fixed $0 < \alpha' < \alpha$ and let $\{l_i\}$ be a positive increasing unbounded sequence. Then by (**4**; chap. 1, Theorem 17a), since

$$||L_{k,\cdot}[f(s)]||_{L((1-\alpha')^{-1},l_1)} \leqslant K_{l_1}N$$

there is a function $\phi_1 \in L((1-\alpha')^{-1}, l)$ and an increasing unbounded sequence $\{k_{i1}\}$ such that

$$||\phi(\cdot)||_{L((1-\alpha')^{-1},l_1)} \leqslant K_{l_1}N$$

and

$$\lim_{i\to\infty}\int_0^{l_1}\psi(t)\,L_{k_{i1},t}[f(s)]\,dt = \int_0^{l_1}\psi(t)\,\phi_1(t)\,dt,$$

for every $\psi \in L(1/\alpha', l_1)$. Further, since

$$||L_{k_{i1},\cdot}[f(s)]||_{L((1-\alpha')^{-1},l_2)} \leqslant K_{l_2}N$$

there is, by (**4**; chap. 1, Theorem 17a), a function $\phi_2 \in L((1-\alpha')^{-1}, l_2)$ and an increasing unbounded sequence $\{k_{i2}\} \subseteq \{k_{i1}\}$ such that

$$||\phi_2(\cdot)||_{L((1-\alpha')^{-1},l_2)} \leqslant K_{l_2}N,$$

and

[2]Lorentz states that this theorem is true for l infinite also. However, this is not the case, as it would imply untrue relations between the L_p spaces.

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$$\lim_{i\to\infty}\int_0^{l_2}\psi(t)\,L_{k_{i2},t}[f(s)]\,dt = \int_0^{l_2}\psi(t)\,\phi_2(t)\,dt$$

for every $\psi \in L(1/\alpha', l_2)$. Inductively, since

$$||L_{k_{i\,j-1},\cdot}[f(s)]||_{L((1-\alpha')^{-1},\,l_j)} \leqslant K_{l_j}N$$

there is a function $\phi_j \in L((1-\alpha')^{-1}, l_j)$, and an increasing unbounded sequence $\{k_{ij}\} \subseteq \{k_{i\,j-1}\}$ such that

$$||\phi_j(\cdot)||_{L((1-\alpha')^{-1},\,l_j)} \leqslant K_{l_j}N$$

and

$$\lim_{i\to\infty}\int_0^{l_j}\psi(t)\,L_{k_{ij},t}[f(s)]\,dt = \int_0^{l_j}\psi(t)\,\phi(t)\,dt,$$

for every $\psi \in L(1/\alpha', l_j)$.

But, if $j < j'$, $\phi_j(t) = \phi_{j'}(t)$ for almost all t in $0 \leqslant t \leqslant l_j$. For $\phi_j - \phi_{j'} \in L((1-\alpha')^{-1}, l_j)$, and hence if $\psi \in L(1/\alpha', l_j)$ and $\bar{\psi} = \psi$, $0 \leqslant t \leqslant l_j$, $\bar{\psi} = 0$, $t \geqslant l_j$, then since $\bar{\psi} \in L(1/\alpha', l_{j'})$, and $\{k_{ij'}\} \subseteq \{k_{ij}\}$,

$$\begin{aligned}\int_0^{l_j}\psi(t)(\phi_j(t) - \phi_{j'}(t))\,dt &= \int_0^{l_j}\psi(t)\,\phi_j(t)\,dt - \int_0^{l_j'}\bar{\psi}(t)\,\phi_{j'}(t)\,dt\\ &= \lim_{i\to\infty}\int_0^{l_j}\psi(t)\,L_{k_{ij},t}[f(s)]\,dt - \lim_{i\to\infty}\int_0^{l_j'}\bar{\psi}(t)\,L_{k_{ij'},t}[f(s)]\,dt\\ &= \lim_{i\to\infty}\left\{\int_0^{l_j}\psi(t)\,L_{k_{ij'},t}[f(s)]\,dt - \int_0^{l_j}\psi(t)\,L_{k_{ij'},t}[f(s)]\,dt\right\} = 0.\end{aligned}$$

Thus by (**1**; chap. IV, §4.2 and Theorem 3), $\phi_j(t) = \phi_{j'}(t)$ almost everywhere in $0 \leqslant t \leqslant l_j$.

For each $t \geqslant 0$ let $\phi(t) = \phi_j(t)$ where j is the least i such that $t \leqslant l_i$. Then clearly $\phi \in L((1-\alpha')^{-1}, l)$ for each $l > 0$, and if $k_i = k_{ii}$, and $\psi \in L(1/\alpha', l)$,

$$\lim_{i\to\infty}\int_0^{l}\psi(t)\,L_{k_i,t}[f(s)]\,dt = \int_0^{l}\psi(t)\,\phi(t)\,dt.$$

Further, ϕ has a Laplace transform. For if $s > 0$, then

$$e^{-st}\text{sgn}(\phi(t)) \in L(1/\alpha', l) \cap \Lambda(\alpha)$$

and thus by (**3**; Theorem 3.6.1)

$$\begin{aligned}\int_0^{l}e^{-st}|\phi(t)|\,dt &= \left|\int_0^{l}e^{-st}\text{sgn}(\phi(t))\,\phi(t)\,dt\right|\\ &= \lim_{i\to\infty}\left|\int_0^{l}e^{-st}\text{sgn}(\phi(t))\,L_{k_i,t}[f(s)]\,dt\right|\\ &\leqslant ||e^{-st}||_{(\Lambda\alpha)}\limsup_{i\to\infty}||L_{k_i,\cdot}[f(s)]||_{M(\alpha)} \leqslant s^{-\alpha}\Gamma(\alpha+1)\,N.\end{aligned}$$

Thus

$$\int_0^{\infty}e^{-st}\phi(t)\,dt$$

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exists for $s > 0$. Also,

$$\lim_{i\to\infty} \int_0^\infty e^{-st} L_{k_i,t}[f(s)]\, dt = \int_0^\infty e^{-st}\phi(t)\, dt,$$

for each $s > 0$. For, by (**3**; Theorem 3.6.1.),

$$\left| \int_l^\infty e^{-st} L_{k_i,t}[f(s)]\, dt \right| \leqslant ||L_{k_i,t}[f(s)]||_{M(\alpha)} \cdot \alpha \int_l^\infty t^{\alpha-1} e^{-st} dt$$
$$\leqslant N\alpha \int_l^\infty e^{-st} t^{\alpha-1} dt < \epsilon$$

and we may also choose l so large that

$$\int_l^\infty e^{-st} |\phi(t)|\, dt < \epsilon.$$

Then,

$$\limsup_{i\to\infty} \left| \int_0^\infty e^{-st}(\phi(t) - L_{k_i,t}[f(s)])\, dt \right|$$
$$\leqslant \limsup_{i\to\infty} \left| \int_0^l e^{-st}(\phi(t) - L_{k_i,t}[f(s)])\, dt \right| + 2\epsilon = 2\epsilon,$$

and thus since ϵ is arbitrary,

$$\lim_{i\to\infty} \int_0^\infty e^{-st} L_{k_i,t}[f(s)]\, dt = \int_0^\infty e^{-st}\phi(t)\, dt.$$

But by (**4**; chap. 7, Theorem 11b), this last limit is equal to $f(s)$. Thus $f(s)$ is the Laplace transform of ϕ, and all that remains to be shown is that $\phi \in M(\alpha)$.

But by (**4**; chap. 7, Theorem 6a)

$$\lim_{k\to\infty} L_{k,t}[f(s)] = \phi(t) \text{ a.e.}$$

Hence if E is any subset, of measure δ, then from Fatou's lemma

$$\int_E |\phi(t)|\, dt \leqslant \liminf_k \int_E |L_{k,t}[f(s)]|\, dt \leqslant N\delta^\alpha$$

Hence

$$||\phi(t)||_{M(\alpha)} = \sup_E \delta^{-\alpha} \int_E |\phi(t)|\, dt \leqslant N$$

and $\phi \in M(\alpha)$.

In conclusion it may be mentioned that results of the type obtained in theorems 2 and 4 hold for considerably more general spaces than $\Lambda(\alpha)$ and $M(\alpha)$. For example, analogues of these theorems hold true if the values of $f(s)$ be in a reflexive Banach space; the proof of this fact is much like the proofs given here.

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