Can. J. Math., Vol. XXXVIII, No. 2, 1986, pp. 376-386

# A TRACE CONDITION EQUIVALENT TO SIMULTANEOUS TRIANGULARIZABILITY

HEYDAR RADJAVI

**0. Introduction.** A collection $\mathcal{S}$ of matrices over a field $F$ is said to be triangularizable if there is an invertible matrix $T$ over $F$ such that the matrices $T^{-1}ST$, $S \in \mathcal{S}$, are all upper triangular. It is a well-known and easy fact that any commutative set $\mathcal{S}$ is triangularizable if $F$ is algebraically closed, or if $F$ contains the spectrum of every member of $\mathcal{S}$. Many sufficient conditions are known for triangularizability of matrix collections. Levitzki [**7**] proved that a (multiplicative) semigroup of nilpotent matrices is triangularizable. (His result is valid even over a division ring.) Kolchin [**5**] showed the triangularizability of a semigroup of unipotent matrices, i.e., matrices of the form $I + N$ with $N$ nilpotent. Kaplansky [**3**, **4**] unified and generalized these results.

One of Kaplansky's theorems [**4**] is that if $F$ has characteristic zero, then a semigroup with constant trace is triangularizable. (As shown in [**4**], the spectrum of every member of such a semigroup is contained in $\{0,1\}$.) Watters [**17**] showed that the theorem is also true if the characteristic of $F$ is larger than $n/2$, where $n$ is the size of the matrices. The analogue of Kaplansky's result for trace-class operators on Hilbert space is given in [**12**]. Other related results are proved in [**10**] and [**12**]. In Section 1 of this paper we give an extension of these theorems to a necessary and sufficient condition in terms of trace, from which several corollaries are deduced. These include a strengthening of a theorem of McCoy on triangularizing pairs of matrices [**9**] as well as the result that a semigroup of idempotents is triangularizable. In Section 2 we prove similar results for trace-class operators on a Hilbert space.

In what follows $\mathcal{S}$ will always denote a semigroup of matrices or of linear operators on a vector space $\mathcal{V}$ over $F$. We say that *trace is permutable on* $\mathcal{S}$ if for every $k$, every word $A_1A_2 \dots A_k$ in $\mathcal{S}$, and every permutation $s$ of $\{1, 2, \dots, k\}$ the equation

$$\mathrm{tr}(A_{s(1)}A_{s(2)} \dots A_{s(k)}) = \mathrm{tr}(A_1A_2 \dots A_k)$$

holds. It is easy to see that this is the case if and only if

$$\mathrm{tr}(ABC) = \mathrm{tr}(CBA) \quad \text{for all } A, B, C \text{ in } \mathcal{S}.$$

Received September 5, 1984 and in revised form March 12, 1985. The author was supported by the NSERC. This paper is dedicated to Gerhard K. Kalisch on his seventieth birthday.

A collection of linear operators on $\mathscr{V}$ is called *transitive* (= *irreducible*) if there is no nontrivial subspace $\mathscr{M}$ (i.e., $0 \neq \mathscr{M} \neq \mathscr{V}$) invariant under (every member of) the collection. The corresponding collection of matrices relative to a fixed basis is also called irreducible or transitive. The spectrum of $A$ will be denoted by $\sigma(A)$.

**1. The finite-dimensional case.** In this section we prove that, in the case of an algebraically closed field of characteristic zero, a semigroup $\mathscr{S}$ is triangularizable if and only if trace is permutable on $\mathscr{S}$. As in [**17**], a suitably large characteristic would also do. In order to obtain certain corollaries we state and prove a somewhat more general theorem.

The well-known Burnside's theorem [**2**] will be used more than once in the sequel: An irreducible algebra of $n \times n$ matrices over an algebraically closed field contains all $n \times n$ matrices. Equivalently, an irreducible semigroup over such a field contains a basis for the linear space of all $n \times n$ matrices [**3**].

THEOREM 1. *Let $\mathscr{S}$ be a semigroup of $n \times n$ matrices over a field that contains all the eigenvalues of the members of $\mathscr{S}$ and whose characteristic is either zero or greater than $n/2$. Let $\mathscr{E}$ be a generating subset of $\mathscr{S}$. Then the following assertions are mutually equivalent.*

(i) *$\mathscr{S}$ is triangularizable.*

(ii) *trace is permutable on $\mathscr{S}$.*

(iii) *trace is permutable on $\mathscr{E}$.*

(iv) $\operatorname{tr}(ABC) = \operatorname{tr}(CBA)$ *for all $A$, $B$, $C$ in $\mathscr{S}$.*

(v) $\operatorname{tr}(ABC) = \operatorname{tr}(CBA)$ *for all $A$, $B$ in $\mathscr{E}$ and $C \in \mathscr{S}$.*

*Proof.* With no loss of generality we can assume that the field is algebraically closed. Since every assertion in the sequence clearly implies the next, we must only prove the implication (v) $\Rightarrow$ (i). Assume (v) holds, and rewrite it as

$$\operatorname{tr} C(AB - BA) = 0.$$

Since the algebra $\mathfrak{A}$ generated by $\mathscr{S}$ coincides with the linear span of $\mathscr{S}$, the equation

$$\operatorname{tr} C(AB - BA) = 0$$

holds for all $C$ in $\mathfrak{A}$. Now $\mathfrak{A}$ admits a block triangularization with irreducible diagonal blocks, whose number $k$ we shall show to be $n$ (but could a priori be any integer from 1 to $n$). For each $A$ in the block-triangularized $\mathfrak{A}$, let $A_j$ denote the $j^{\text{th}}$ diagonal block of $A$, $1 \leqq j \leqq k$, and let $\mathfrak{A}_j$ be the irreducible algebra consisting of all the matrices $A_j$. By [**17**], we can assume that the $\mathfrak{A}_j$ satisfy the following property. For each $j$, there exists a subset $R_j$ of $\{1, \ldots, k\}$ and a subalgebra $\mathfrak{C}$ of $\mathfrak{A}$ whose $j^{\text{th}}$ diagonal block is $\mathfrak{A}_j$ and such that a member $A$ of $\mathfrak{A}$ belongs to $\mathfrak{C}$ if and only if $A_i = A_j$ for all $i$ in $R_j$ and $A_i = 0$ for all $i$ out of $R_j$.

We must show that $A_j$ is $1 \times 1$ for every $A$. Consider $\mathfrak{A}_j$ for a fixed $j$, and assume it is a subset of $m \times m$ matrices with $m \geqq 2$. Then $\mathfrak{A}_j$ is an irreducible algebra and, in particular, it is not commutative, which implies that the set $\{A_j : A \in \mathscr{E}\}$ is not commutative. Pick $A$ and $B$ in $\mathscr{E}$ with $A_jB_j - B_jA_j \neq 0$. Then, denoting the number of elements in $R_j$ by $r$, we obtain the following equalities for every $C$ in $\mathfrak{A}$.

$$r \operatorname{tr} C_j(A_jB_j - B_jA_j) = \operatorname{tr} C(AB - BA) = 0.$$

Thus $\operatorname{tr} C_j(A_jB_j - B_jA_j) = 0$ for every $C_j$ in $\mathfrak{A}_j$. (The inequality $r \leqq n/2$ is used in the case of nonzero characteristic.) Since $\mathfrak{A}_j$ is a full algebra of matrices by Burnside's theorem, we arrive at the contradiction $A_jB_j - B_jA_j = 0$. Thus $m = 1$ and the proof is complete.

The above theorem yields, at least in the case of zero characteristic, the results of Kolchin, Levitzki, their unification by Kaplansky, and, in the nonzero characteristic case, their extension by Watters [**17**]: constant trace on a semigroup is sufficient for triangularizability. We give other corollaries below.

COROLLARY 1. *A semigroup of idempotent matrices over a field of zero (or sufficiently large) characteristic is triangularizable.*

*Proof.* Let $r(A)$ denote the rank of $A$, and observe that $\operatorname{tr}(A) = r(A)$ for idempotent $A$. Now for $A$ and $B$ in the semigroup,

$$r(BAB) = \operatorname{tr}(BAB) = \operatorname{tr}(B^2A) = \operatorname{tr}(BA) = r(BA).$$

Since the (column) range of $BAB$ is clearly contained in the range of $BA$, the above equation shows that $BAB$ and $BA$ have the same range. This implies that

$$r(CBAB) = r(CBA)$$

for $A$, $B$, $C$ in $\mathscr{S}$. Similarly,

$$r(ABC) = r(ABCB).$$

Hence, for arbitrary $A$, $B$, $C$ in $\mathscr{S}$,

$$\begin{aligned} \operatorname{tr}(ABC) &= r(ABC) = r(ABCB) = \operatorname{tr}(ABCB) \\ &= \operatorname{tr}(CBAB) = r(CBAB) = r(CBA) \\ &= \operatorname{tr}(CBA). \end{aligned}$$

In the following let $F$ stand for a field of zero or suitably large characteristic that is algebraically closed, or merely contains $\sigma(A)$ for every $A$ in the semigroup.

The next corollary extends a result of McCoy [**9**] stating that a necessary and sufficient condition for the simultaneous triangularization of a pair $\{A, B\}$ is that

$$p(A, B)(AB - BA)$$

be nilpotent for every (noncommutative) polynomial $p$. It should be mentioned that McCoy assumes only algebraic closure for the underlying field.

COROLLARY 2. *Matrices $A$ and $B$ are simultaneously triangularizable over $F$ if and only if*

$$\operatorname{tr} W(AB - BA) = 0$$

*for every word $W$ in $A$ and $B$.*

Here is another immediate corollary that extends the well-known result on the triangularizability of commutative semigroups.

COROLLARY 3. *A semigroup $\mathcal{S}$ over $F$ is triangularizable if for all $A$, $B$, $C$ in $\mathcal{S}$ we have either $ABC = ACB$ or $ABC = CBA$.*

COROLLARY 4. *If trace is multiplicative on a semigroup $\mathcal{S}$ over $F$, then $\mathcal{S}$ is triangularizable.*

When trace is multiplicative, every member of the triangularized $\mathcal{S}$ will have at most one nonzero entry on its diagonal. This follows from $\operatorname{tr} A^k = (\operatorname{tr} A)^k$ for each $A \in \mathcal{S}$ and every integer $k$: Let $a_1, \dots, a_n$ be the diagonal entries of $A$. If $\Sigma a_i = 0$, we have $\Sigma a_i^k = 0$ for all $k$; if $\Sigma a_i = t \neq 0$, we have

$$\Sigma(a_i/t)^k = 1.$$

It follows from [**4**] that in the first case $a_i = 0$ for all $i$, and in the second only one $a_i$ is nonzero and equals $t$.

COROLLARY 5. *If $\mathcal{S}$ is a group of matrices over $F$, then $\mathcal{S}$ is triangularizable if and only if trace is constant on each coset of $\mathcal{S}$ relative to the commutator subgroup.*

*Proof.* Let $\mathcal{C}$ be the commutator subgroup of $\mathcal{S}$. First assume that each coset has constant trace. Then, for $A$, $B$, $C$ in $\mathcal{S}$,

$$\begin{aligned}\operatorname{tr}(ABC) &= \operatorname{tr}(ABA^{-1}B^{-1}BAC) \\ &= \operatorname{tr}(BAC) = \operatorname{tr}(CBA),\end{aligned}$$

and $\mathcal{S}$ is triangularizable by Theorem 1. Conversely, assume that trace is permutable on $\mathcal{S}$. Then, for every $A \in \mathcal{S}$ and $B \in \mathcal{C}$,

$$\operatorname{tr}(AB) = \operatorname{tr}(AI) = \operatorname{tr} A,$$

which shows the constancy of trace on the coset $A\mathscr{C}$.

If $F$ is the field of complex numbers, then triangularization can be effected by a unitary similarity. A semigroup over this field is said to be *self-adjoint* if $A \in \mathscr{S}$ implies $A^* \in \mathscr{S}$.

COROLLARY 6. *A self-adjoint semigroup $\mathscr{S}$ of matrices over the field of complex numbers is commutative if and only if trace is permutable on $\mathscr{S}$.*

*Proof.* A self-adjoint set of matrices is triangularizable if and only if it is (unitarily) diagonalizable.

Before presenting further corollaries we make a few remarks.

(1) If $\mathscr{S}$ is any semigroup of matrices over $F$, and if trace is permutable on $\mathscr{S}$, then any block triangularization of $\mathscr{S}$ has the following property: The semigroup $\mathscr{S}_j$ consisting of the $j^{\text{th}}$ diagonal blocks of the members of $\mathscr{S}$ has permutable trace for each $j$. This follows easily from the proof of Theorem 1.

(2) Let $\mathscr{S}$ be a semigroup of matrices over an arbitrary field. If $\mathscr{S}$ is triangularizable, then every diagonal block in any block triangularization of $\mathscr{S}$ is also triangularizable. In the language of linear transformations: if $\mathscr{C}$ is any chain of invariant subspaces for a triangularizable semigroup of operators on an $n$-dimensional space $\mathscr{V}$, then $\mathscr{C}$ is contained in a subspace chain

$$\mathscr{V}_0 = 0 \subset \mathscr{V}_1 \subset \ldots \subset \mathscr{V}_n = \mathscr{V},$$

where each $\mathscr{V}_i$ is invariant under (every member of) $\mathscr{S}$ and has codimension 1 in $\mathscr{V}_{i+1}$. This is implicit in the result of [**17**].

(3) Let $\mathscr{S}$ be a semigroup of operators on $\mathscr{V}$ (over an arbitrary field), and let $\mathscr{J}$ be an ideal in $\mathscr{S}$, i.e., $JS \in \mathscr{S}$ and $SJ \in \mathscr{S}$ for every $J \in \mathscr{J}$ and $S \in \mathscr{S}$. Assume $\mathscr{J} \neq \{0\}$. It is well known that the irreducibility of $\mathscr{S}$ implies that of $\mathscr{J}$. (Short proof: Let $\mathscr{M}$ be a nontrivial subspace invariant under $\mathscr{J}$. Then the linear span of $\{J\mathscr{M}:J \in \mathscr{J}\}$ is invariant under $\mathscr{S}$; so is the intersection of the null spaces of all $J \in \mathscr{J}$. At least one of these two spaces is nontrivial.) We observe that the corresponding assertion for triangularizability is false. Simply, consider the semigroup $\mathscr{S}$ of all operators on $\mathscr{V}$ leaving a fixed nontrivial subspace $\mathscr{M}$ of $\mathscr{V}$ invariant. Let $\mathscr{J}$ be the set of all operators on $\mathscr{V}$ such that $J\mathscr{M} = \{0\}$ and $J^2 = 0$. Then $\mathscr{J}$ is a triangularizable ideal in $\mathscr{S}$, but $\mathscr{S}$ is not triangularizable if the dimension of $\mathscr{V}$ is larger than 2.

In what follows we use the symbol $\mathscr{S}^k$ to denote the ideal of $\mathscr{S}$ consisting of all words of length $k$ or more.

COROLLARY 7. *Let $\mathscr{S}$ be a semigroup of matrices over any field. If $\mathscr{S}^k$ is triangularizable for some $k$, then $\mathscr{S}$ is triangularizable.*

*Proof.* Assume $\mathscr{S}$ is block-triangularized with irreducible diagonal blocks. Let $\mathscr{S}_j$ be the semigroup of all $j^{\text{th}}$ diagonal blocks. It follows from

Remark (2) above that $\mathcal{S}_j^k$ is triangularizable. Now if $\mathcal{S}_j^k = \{0\}$, then every member of $\mathcal{S}_j$ is nilpotent; hence $\mathcal{S}_j$ is triangularizable. If $\mathcal{S}_j^k \neq \{0\}$, since it is an ideal of $\mathcal{S}_j$, its irreducibility follows from that of $\mathcal{S}_j$, by Remark (3). Thus, in both cases, we deduce that the members of $\mathcal{S}_j$ are $1 \times 1$, for every $j$, and $\mathcal{S}$ is triangularized.

COROLLARY 8. *Let $\mathcal{E}$ be an arbitrary set of $n \times n$ matrices over a field of characteristic zero (or larger than $n/2$). If there exists an integer $k$ such that trace is permutable on all words in $\mathcal{E}$ of length $\geqq k$, then it is permutable on all words in $\mathcal{E}$.*

*Proof.* Let $\mathcal{S}$ be the semigroup generated by $\mathcal{E}$. Then every member of $\mathcal{S}^k$ is a word of length $\geqq k$ in $\mathcal{E}$. Thus $\mathcal{S}^k$ is triangularizable over the algebraic closure of the field at hand, by Theorem 1. So is $\mathcal{S}$ by Corollary 7. The desired permutability on $\mathcal{S}$ follows by Theorem 1.

**2. The case of trace-class operators.** In this section we assume $\mathcal{S}$ is a semigroup of trace-class operators on the complex Hilbert space $\mathcal{H}$. Certain properties of the Banach algebra $\mathcal{C}_1$ of all trace-class operators on $\mathcal{H}$ will be used. (See, e.g., [**1**].) The trace norm of an operator $A$ in $\mathcal{C}_1$ will be denoted by $|A|$.

By a *subspace* of $\mathcal{H}$ we shall mean a closed subspace. A collection $\mathcal{E}$ of bounded (linear) operators on $\mathcal{H}$ is *triangularizable* if there exists a chain $\Delta$ of subspaces of $\mathcal{H}$ such that (a) $\Delta$ is maximal, i.e., not properly contained in any chain of subspaces of $\mathcal{H}$, and (b) every member of $\Delta$ is invariant under every member of $\mathcal{E}$. The maximality of $\Delta$ implies that if $\mathcal{M} \in \Delta$, and if $\mathcal{M}$ is the (closed linear) span of $\{\mathcal{N} \in \Delta : \mathcal{N} \subsetneq \mathcal{M}\}$, then the orthogonal complement $\mathcal{M} \ominus \mathcal{M}_-$ of $\mathcal{M}_-$ in $\mathcal{M}$ has dimension 0 or 1.

We shall use the fact that an ideal of a transitive algebra of operators is transitive. (The proof is the same as in the finite-dimensional case given above; just read "closed span" for "span".)

Our main result in this section parallels that in the preceding one: $\mathcal{S}$ is triangularizable if and only if

$$\operatorname{tr}(ABC) = \operatorname{tr}(CBA) \quad \text{for all } A, B, C \text{ in } \mathcal{S}.$$

Theorem 2 below is a little more general and enables us to deduce the appropriate corollaries as in Section 1. We shall use, among other things, Lomonosov's lemma [**8**, **15**] that if $\mathfrak{A}$ is a transitive algebra of bounded operators on $\mathcal{H}$, and if $K$ is a nonzero compact operator, then there is an $A$ in $\mathfrak{A}$ such that $AK$ has an eigenvalue 1. One corollary, among others, is that a commutative collection of compact operators is triangularizable. In the case of trace-class operators, Theorem 2 below is an extension of this result.

We also need the following lemmas.

https://doi.org/10.4153/CJM-1986-018-1 Published online by Cambridge University Press

LEMMA 1. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be transitive algebras of bounded operators on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, each containing nonzero finite-rank operators, and let $\phi$ be an algebra isomorphism of $\mathfrak{A}$ onto $\mathfrak{C}$. Then there is an injective, closed linear transformation $T$ from a dense linear manifold $\mathscr{D}$ of $\mathscr{H}_2$ into $\mathscr{H}_1$ such that $T\mathscr{D}$ is dense in $\mathscr{H}_1$, $AT\mathscr{D} \subseteq T\mathscr{D}$, and*

$$\phi(A) = T^{-1}AT \quad \text{on } \mathscr{D}$$

*for every $A$ in $\mathfrak{A}$.*

*Proof.* This is proved in [**11**].

LEMMA 2. *Let $A$ be a trace-class operator on $\mathscr{H}_2$ and $T$ an injective, closed linear transformation with dense domain $\mathscr{D}$ in $\mathscr{H}_1$ and dense range in $\mathscr{H}_2$. If $AT\mathscr{D} \subseteq T\mathscr{D}$ and $T^{-1}AT$ is (extendible to) a trace-class operator, then*

$$\operatorname{tr}(T^{-1}AT) = \operatorname{tr} A.$$

*Proof.* Let $B = T^{-1}AT$, so that $TB = AT$. We use the polar decomposition $T = UH$, where $U$ is unitary and $H$ is positive, closed, and defined on $\mathscr{D}$. Replacing $U^{-1}AU$ by $A$, we can assume, with no loss of generality, that $T = H$. Since $H$ is injective, there is a sequence $\{P_n\}$ of spectral projections for $H$ such that $\lim P_n = I$ in the strong operator topology, and $P_nHP_n$ is bounded and invertible on $P_n\mathscr{H}$ for every $n$. Since

$$\begin{aligned}(P_nHP_n)(P_nBP_n) &= P_nHBP_n = P_nAHP_n \\ &= (P_nAP_n)(P_nHP_n),\end{aligned}$$

the operators $P_nAP_n$ and $P_nBP_n$ are similar and

$$\operatorname{tr}(P_nBP_n) = \operatorname{tr}(P_nAP_n).$$

But $P_nBP_n$ and $P_nAP_n$ tend to $B$ and $A$, respectively, in the $\mathscr{C}_1$ norm; the continuity of trace then implies $\operatorname{tr} B = \operatorname{tr} A$ and completes the proof.

Let $\mathfrak{A}$ be any algebra of bounded operators on $\mathscr{H}$, and let $A \in \mathfrak{A}$. If $P$ is any orthogonal projection on $\mathscr{H}$, then the restriction of $PAP$ to $P\mathscr{H}$ is called the *compression* of $A$ to $P\mathscr{H}$, and the set of compressions of all $A \in \mathfrak{A}$ is called the compression of $\mathfrak{A}$ to $P\mathscr{H}$. Let $\Delta$ be a maximal chain of invariant subspaces for $\mathfrak{A}$ (which could be very far from a triangularizing chain). Let $\mathscr{M} \in \Delta$ with $\mathscr{M} \ominus \mathscr{M}_- \neq 0$. Then the compression of $\mathfrak{A}$ to $\mathscr{M} \ominus \mathscr{M}_-$, which is easily seen to be an algebra in its own right, is transitive by the maximality of $\Delta$. This compression is an irreducible "diagonal block" of $\mathfrak{A}$.

The following technical lemma is similar to a result of [**14**] used in dealing with reductive algebras. In the absence of reductivity, however, the transitive "diagonal blocks" of an algebra $\mathfrak{A}$ that are unboundedly similar need not be unitarily similar.

LEMMA 3. *Let $\mathfrak{A}$ be a closed subalgebra of $\mathscr{C}_1$ and let $\Delta$ be a maximal chain of invariant subspaces for $\mathfrak{A}$. Let $\mathscr{M} \in \Delta$ such that ($\mathscr{M} \ominus \mathscr{M}_- \neq \{0\}$ and) the compression of $\mathfrak{A}$ to $\mathscr{M} \ominus \mathscr{M}_-$ is nonzero. Then $\mathfrak{A}$ has an ideal $\mathfrak{C}$ with the following property: There exist a finite subset of $\Delta$, say*

$$\Delta_0 = \{\mathscr{M}_0 = \mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n\}$$

*and injective closed linear transformations $T_1, \dots, T_n$ such that*

(1) *$T_i$ has dense domain $\mathscr{D}_i$ in $\mathscr{M}_0 \ominus \mathscr{M}_{0-}$ and dense range in $\mathscr{M}_i \ominus \mathscr{M}_{i-}$;*

(2) *denoting the compression of $A$ to $\mathscr{M}_i \ominus \mathscr{M}_{i-}$ by $A_i$ we have*

$$A_0 T_i \mathscr{D}_i \subseteq T_i \mathscr{D}_i \quad \text{and}$$

$$A_i = T_i^{-1} A_0 T_i \quad \text{on } \mathscr{D}_i$$

*for every $A$ in $\mathfrak{C}$ and $i = 1, \dots, n$; and*

(3) *the compression of $\mathfrak{C}$ to $\mathscr{N} \ominus \mathscr{N}_-$ is zero if $\mathscr{N}$ is in $\Delta \backslash \Delta_0$.*

*Proof.* The compression of $\mathfrak{A}$ to $\mathscr{M} \ominus \mathscr{M}_-$ is transitive and hence, by **[8]**, contains an operator $K_0$ with an eigenvalue 1. Let $K$ be a member of $\mathfrak{A}$ whose compression is $K_0$, and observe that $\sigma(K_0) \subseteq \sigma(K)$. Since $\mathfrak{A}$ is closed, it contains the closed algebra $\mathscr{K}$ generated by the single operator $K$. Let $\gamma$ be a circle centred at 1 that does not meet $\sigma(K)$ and whose interior intersects $\sigma(K)$ in $\{1\}$. Then, as in **[14]**, the finite-rank projection

$$P = \frac{1}{2\pi i} \int_\gamma (z - K)^{-1}\, dz$$

is in $\mathscr{K}$. Since $(z - K_0)^{-1}$ is the compression of $(z - K)^{-1}$ to $\mathscr{M} \ominus \mathscr{M}_-$, we see that the compression $P_0$ of $P$ satisfies

$$P_0 = \frac{1}{2\pi i} \int_\gamma (z - K_0)^{-1}\, dz.$$

The fact that $1 \in \sigma(K_0)$ now implies $P_0 \neq 0$.

Since $P$ has finite rank and since $\mathscr{M} \ominus \mathscr{M}_-$ is orthogonal to $\mathscr{N} \ominus \mathscr{N}_-$ for distinct $\mathscr{M}$ and $\mathscr{N}$ in $\Delta$, it follows that $P$ has only finitely many nonzero compressions. Let $F$ be any finite-rank operator in $\mathfrak{A}$ with the following property: there exists a finite subset

$$\Delta_0 = \{\mathscr{M}_0 = \mathscr{M}, \mathscr{M}_1, \dots, \mathscr{M}_n\}$$

such that the compression of $F$ to $\mathscr{N} \ominus \mathscr{N}_1$ is zero for $\mathscr{N} \in \Delta \backslash \Delta_0$, and $\Delta_0$ is a minimal such subset containing $\mathscr{M}$. For each $i$ let $\mathfrak{A}_i$ be the algebra of compressions $A_i$ of $A \in \mathfrak{A}$ to $\mathscr{M}_i \ominus \mathscr{M}_{i-}$.

Let $\mathfrak{C}$ be the (two-sided) ideal of $\mathfrak{A}$ generated by $F$, so that (3) of the lemma is satisfied. The ideal $\mathfrak{C}_i$ of $\mathfrak{A}_i$ generated by $F_i$ is transitive for every $i$, and consists of finite-rank operators. The existence of $T_i$ satisfying (1) and (2) will be established by Lemma 1 if we show that, for $B \in \mathfrak{C}$, $B_i = 0$ if and only if $B_0 = 0$.

By the minimality of $\Delta_0$ it is impossible to have $B_0 \neq 0$ and $B_i = 0$ for any $B \in \mathfrak{C}$. Suppose the existence of $B \in \mathfrak{C}$ with $B_0 = 0$, $B_i \neq 0$. Then the ideal $\mathscr{J}$ of $\mathfrak{C}$ generated by $B$ has the property that its compression $\mathscr{J}_i$ is transitive and $\mathscr{J}_0 = 0$. The transitivity of $F_i\mathscr{J}_iF_i$ implies, because of finite dimensionality, that $F_i \in \mathscr{J}_i$. Thus $\mathscr{J}$ contains a finite-rank operator $G$ with $G_0 = 0$ and $G_i = F_i$. Then

$$(F - G)_i = 0, \quad (F - G)_0 \neq 0,$$

which contradicts the minimality of $\Delta_0$.

In the following by a *generating* subset of a semigroup $\mathscr{S} \subseteq \mathscr{C}_1$ we mean a subset $\mathscr{E}$ such that the set of all words in $\mathscr{E}$ is $\mathscr{C}_1$-dense in $\mathscr{S}$.

THEOREM 2. *Let $\mathscr{S}$ be a semigroup of trace-class operators. Let $\mathscr{E}$ be a generating subset of $\mathscr{S}$. Then the conclusions of Theorem* 1 *hold.*

*Proof.* The implication (i) $\rightarrow$ (ii), in this case, is a consequence of the result of Ringrose [**16**] that the "diagonal elements" of a compact operator $K$ together with 0 form $\sigma(K)$. If $\Delta$ is a triangularizing chain for $K$, then the diagonal elements of $K$ are those scalars occurring as compressions of $K$ to 1-dimensional spaces $\mathscr{M} \ominus \mathscr{M}_-$ with $\mathscr{M}$ in $\Delta$. If $K \in \mathscr{C}_1$, it is easily seen that $\operatorname{tr} K$ is the sum of all such scalars occurring with distinct spaces $\mathscr{M}$ in $\Delta$.

Again, as in the proof of Theorem 1, we must only show the implication (v) $\rightarrow$ (i). Thus assume

$$\operatorname{tr} A(BC - CB) = 0$$

for all $A$ in $\mathscr{S}$ and $B$, $C$ in $\mathscr{E}$. Since trace is continuous, we can assume, with no loss, that the above equation holds for all $A$ in the closed subalgebra $\mathfrak{A}$ of $\mathscr{C}_1$ generated by $\mathscr{S}$. Let $\Delta$ be a maximal chain of invariant subspaces for $\mathfrak{A}$. We shall show that $\Delta$ is triangularizing, i.e., for every $\mathscr{M} \in \Delta$ the space $\mathscr{M} \ominus \mathscr{M}_-$ has dimension 0 or 1.

Let $\mathscr{M}$ be any member of $\Delta$ with $\mathscr{M} \ominus \mathscr{M}_- \neq \{0\}$. If the corresponding compression of $\mathfrak{A}$ is zero, then $\mathscr{M} \ominus \mathscr{M}_-$ has dimension 1, because otherwise any maximal subspace chain inserted in this gap would yield a chain properly containing $\Delta$ and consisting of invariant subspaces for $\mathfrak{A}$. Thus assume that the compression $\mathfrak{A}_0$ of $\mathfrak{A}$ to $\mathscr{M} \ominus \mathscr{M}_-$ is nonzero. Now let $\mathfrak{C}$, $\Delta_0$, and $T_i$ be as described in Lemma 3.

Suppose $\dim(\mathscr{M} \ominus \mathscr{M}_-) > 1$. Then the transitivity of $\mathfrak{A}_0$ implies that it is not commutative. Thus there exist $B$ and $C$ in $\mathscr{E}$ whose corresponding compressions $B_0$ and $C_0$ do not commute. Using Lemma 2 and the notation of Lemma 3 we deduce that for $A \in \mathfrak{C}$,

$$0 = \operatorname{tr} A(BC - CB) = \sum_{i=0}^{n} \operatorname{tr} A_i(B_iC_i - C_iB_i)$$

$$= \operatorname{tr} A_0(B_0C_0 - C_0B_0) + \sum_{i=1}^{n} \operatorname{tr} T_i^{-1}A_0(B_0C_0 - C_0B_0)T_i$$

$$= (n + 1) \operatorname{tr} A_0(B_0C_0 - C_0B_0).$$

Since $B_0C_0 - C_0B_0 \neq 0$, this contradicts the fact that $\mathfrak{C}_0$ is transitive, because, by Lomonov's theorem, a transitive algebra of trace-class operators is dense in $\mathscr{C}_1$. (See e.g., [**14**].)

One corollary of Theorem 2 is that a semigroup of trace-class operators with constant trace is triangularizable [**12**]. There are other corollaries paralleling those of Theorem 1.

COROLLARY 9. *A semigroup of compact idempotents on $\mathscr{H}$ is triangularizable.*

This is a result of [**12**], its proof is exactly that of Corollary 1, because compact idempotents have finite rank.

COROLLARY 10. *Trace-class operators $A$ and $B$ are simultaneously triangularizable if and only if* tr $W(AB - BA) = 0$ *for every word $W$ (of finite length) in $A$ and $B$.*

This is an extension of the analogue of McCoy's result given in [**6**].

Analogues of Corollaries 3 and 4 also hold, together with the remark following Corollary 4; for a proof that

$$\left(\sum_{i=1}^{\infty} a_i\right)^k = \sum_{i=1}^{\infty} a_i^k \quad \text{for all } k$$

implies at most one $a_i$ is nonzero see [**12**]. The corresponding version of Corollaries 6, 7 and 8 also hold; we omit the proofs, which are not unlike those given in the finite dimensional case. It is still true that the compression of a triangularizable algebra $\mathfrak{A}$ of compact operators to a subspace of the type $\mathscr{M} \ominus \mathscr{N}$, where $\mathscr{M}$ and $\mathscr{N}$ are invariant for $\mathfrak{A}$, is also triangularizable [**6**]. Thus the analogue of Corollary 7 is true for any semigroup of compact operators.

## REFERENCES

**1.** N. Dunford and J. T. Schwartz, *Linear operators*, Part II: *Spectral theory* (Interscience, New York, 1963).

**2.** N. Jacobson, *Lectures in abstract algebra II: linear algebra* (van Nostrand, Princeton, 1953).

**3.** I. Kaplansky, *Fields and rings*, 2nd Ed. (University of Chicago Press, Chicago, 1972).

**4.** I. Kaplansky, *The Engel-Kolchin theorem revisited*, in *Contributions to algebra* (Academic Press, New York, 1977).

**5.** E. Kolchin, *On certain concepts in the theory of algebraic matric groups*, Ann. of Math. *49* (1948), 774-789.

**6.** C. Laurie, E. Nordgren, H. Radjavi and P. Rosenthal, *On triangularization of algebras of operators*, J. Reine Angew. Math. *327* (1981), 143-155.

**7.** J. Levitzki, *Uber nilpotente Unterringe*, Math. Ann. *105* (1931), 620-627.

**8.** V. J. Lomonosov, *Invariant subspaces for operators commuting with compact operators*, Funkcional Anal. i Prilozen 7 (1973), (Russian); Functional Anal. Appl. *7* (1973), 213-214.

**9.** N. H. McCoy, *On the characteristic roots of matric polynomials*, Bull. Amer. Math. Soc. *42* (1936), 592-600.

**10.** E. Nordgren, H. Radjavi and P. Rosenthal, *Triangularizing semigroups of compact operators*, Indiana Univ. Math. J. *33* (1984), 271-275.

**11.** H. Radjavi, *Isomorphisms of transitive operator algebras*, Duke Math. J. *41* (1974), 555-564.

**12.** ——— *On the reduction and triangularization of semigroups of operators*, J. Operator Theory *13* (1985), 63-71.

**13.** H. Radjavi and P. Rosenthal, *On transitive and reductive operator algebras*, Math. Ann. *209* (1974), 43-56.

**14.** ——— *Invariant subspaces* (Springer-Verlag, New York, 1973).

**16.** J. R. Ringrose, *Compact non-self-adjoint operators* (Van Nostrand, Princeton, 1971).

**17.** J. F. Watters, *Block triangularization of algebras of matrices*, Linear Alg. App. *32* (1980), 3-7.

*Dalhousie University,*
*Halifax, Nova Scotia*