Can. J. Math.Vol. 43 (5), 1991 pp. 939–947

THE VITALI INTEGRAL CONVERGENCE THEOREM AND UNIFORM ABSOLUTE CONTINUITY

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ABSTRACT. A geometric version of the Vitali integral convergence theorem is established. Parameterized versions of results on uniform absolute continuity in spaces of measures suggested by the convergence theorem are studied.

1. **Introduction.** In two papers in the 1970's, J. K. Brooks [3] and Brooks and Jewett [4] established marked improvements of the Vitali-Hahn-Saks theorem and the Vitali integral convergence theorem as a consequence of the following fundamental theorem on uniform absolute continuity, a result which has an antecedent in the classical paper by Bartle, Dunford, and Schwartz [1].

THEOREM 1.1 [3]. *Suppose that* (Ω, Σ) *is a measurable space and that* H *is a uniformly countably additive subset of* $ca(\Sigma)$. *If* $\mu \in ca(\Sigma)$ *and* $\nu \ll \mu$ *for all* $\nu \in H$, *then* $\nu \ll \mu$ *uniformly for* $\nu \in H$.

In Section 2 of this paper we establish a geometric version of the Vitali convergence theorem. Our version of the Vitali theorem raises numerous questions about parameterized—or collectionwise—versions of Theorem 1.1. These questions are studied in some detail in Section 3. We also point out a parameterized version of the Vitali-Hahn-Saks theorem which follows immediately from deliberations in this section. We then conclude the paper by establishing a uniform differentiability result for arbitrary continuous convex functions on Banach spaces which is motivated by the measure theoretic results of Section 3.

All Banach spaces X in this paper are defined over the real field $\mathbb{R}$. If $x, y \in X$, then $D(x, y)$ will denote the Gateaux derivative of the norm at x in the direction y provided that this derivative exists, i.e., $D(x, y) = \lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$. We refer the reader to Diestel [5] and Rockafellar [6], [7] for a discussion of Gateaux differentiability of convex functions. We denote the one-sided Gateaux derivatives (which always exist) by $D^+(x, y)$ and $D^-(x, y)$. If Σ is a σ-algebra, then $ca(\Sigma)$ is the Banach space (total variation norm) of all countably additive real valued measures defined on Σ. If $S \subset X$, then the norm closure of S will be denoted by $\bar{S}$ and the weak closure will be denoted by $\bar{S}^w$.

Received by the editors February 26, 1990.
AMS subject classification: 46B20, 46A50, 28A33, 28A99.
Key words and phrases: Vitali integral convergence theorem, uniform absolute continuity, Gateaux differentiability.

https://doi.org/10.4153/CJM-1991-051-3 Published online by Cambridge University Press

2. **Vitali's Convergence Theorem.** We begin with parameterized versions of two results in Bilyeu and Lewis [2]. We sketch the proofs.

LEMMA 2.1. *If X is an abstract L-space and each of (f_n) and (g_n) is a sequence from X, then the following are equivalent:*

(i) $D(f_n, g_n)$ exists uniformly in n,

(ii) $D(|f_n|, |g_n|)$ exists uniformly in n,

(iii) $|kf_n| \wedge |g_n| \underset{k}{\longrightarrow} |g_n|$ uniformly in n.

PROOF. We remark that because of the absolute values which appear in (iii) it clearly suffices to show that (i) $\Leftrightarrow$ (iii). Now

$$-\|kf_n - g_n\| + \|kf_n\| \leq D^-(f_n, g_n) \leq D^+(f_n, g_n) \leq \|kf_n + g_n\| - \|kf_n\|, \quad k \in \mathbb{N}.$$

Therefore $D(f_n, g_n)$ exists uniformly in n iff

$$\|kf_n + g_n\| + \|kf_n - g_n\| - 2\|kf_n\| \longrightarrow 0$$

uniformly in n. Thus by Lemma 2.3 of [2], $D(f_n, g_n)$ exists uniformly in n iff

$$\begin{aligned} &\|kf_n + g_n\| + \|kf_n - g_n\| - 2\|kf_n\| \\ &\quad = 2\|kf_n\| + 2\big\| |g_n| - (|kf_n| \wedge |g_n|)\big\| - 2\|kf_n\| \\ &\quad = 2\| |g_n| - (|kf_n| \wedge |g_n|)\| \underset{k}{\longrightarrow} 0 \text{ uniformly in } n. \end{aligned}$$ ■

DEFINITION. Let Σ be a σ-algebra and let each of (μ_n) and (ν_n) be a sequence in $ca(\Sigma)$. We say that $\nu_n \ll \mu_n$ uniformly in n provided that if $\epsilon > 0$ then there is a $\delta > 0$ such that $|\nu_n(A)| < \epsilon$ whenever $A \in \Sigma$, $n \in \mathbb{N}$, and $|\mu_n|(A) < \delta$.

LEMMA 2.2. *Suppose that Σ is a σ-algebra and each of (μ_n) and (ν_n) is a sequence in $ca(\Sigma)$. If $D(\mu_n, \nu_n)$ exists uniformly in n, then $\nu_n \ll \mu_n$ uniformly in n. Conversely, if (ν_n) is bounded and $\nu_n \ll \mu_n$ uniformly in n, then $D(\mu_n, \nu_n)$ exists uniformly in n.*

PROOF. Suppose that $D(\mu_n, \nu_n)$ exists uniformly in n. Therefore $\| |k\mu_n| \wedge |\nu_n| - |\nu_n| \| \underset{k}{\longrightarrow} 0$ uniformly in n. Let $\epsilon > 0$, and choose $k_0 \in \mathbb{N}$ so that $\| |k_0\mu_n| \wedge |\nu_n| - |\nu_n| \| < \epsilon$ for all n. If $|k_0\mu_n|(A) < \epsilon$, then $|\nu_n(A)| < 2\epsilon$, and the uniform absolute continuity follows.

Conversely, suppose that $\nu_n \ll \mu_n$ uniformly in n, and let $k_0 \in \mathbb{N}$ so that $\|\nu_n\| < k_0$ for all n. Let $\epsilon > 0$, and choose $\delta > 0$ so that if $n \in \mathbb{N}$ and $|\mu_n|(A) < \delta$, then $|\nu_n|(A) < \epsilon$. Now choose $k \in \mathbb{N}$ such that $k\delta \geq k_0$. It follows that $\| |k\mu_n| \wedge |\nu_n| - |\nu_n| \| \leq 2\epsilon$ for all n. Therefore

$$\big\| |k\mu_n| \wedge |\nu_n| - |\nu_n| \big\| \underset{k}{\longrightarrow} 0$$

uniformly in n, and we obtain the desired conclusion by appealing to Lemma 2.1. ■

The following theorem and corollary constitute our geometric interpretation of the Vitali convergence theorem.

https://doi.org/10.4153/CJM-1991-051-3 Published online by Cambridge University Press

THEOREM 2.3. *Suppose that* (Ω, Σ, μ) *is a finite measure space and* (f_n) *is a sequence in* $L^1(\mu)$. *The sequence* (f_n) *converges to an element* $f \in L^1(\mu)$ *if and only if*

(i) (f_n) *is Cauchy in measure, and*

(ii) there is a uniformly integrable sequence (g_n) *in* $L^1(\mu)$ *so that* $D(g_n, f_n)$ *exists uniformly in* n.

PROOF. Suppose that (i) and (ii) hold, and let $\epsilon > 0$. We have that

$$\mu(\{t : |f_n(t) - f_m(t)| > \epsilon\}) \longrightarrow 0$$

as $n, m \longrightarrow \infty$. Since $D(g_n, f_n)$ exists uniformly in n, $\nu_{f_n} \ll \nu_{g_n}$ uniformly in n, where $\nu_{f_n}(A) = \int_A f_n \, d\mu$ and $\nu_{g_n}(A) = \int_A g_n \, d\mu, A \in \Sigma$. Since (g_n) is uniformly integrable, $\nu_{g_n} \ll \mu$ uniformly in n. Choose $\delta > 0$ so that if $\mu(A) < \delta$, then $|\nu_{f_n}|(A) = \int_A |f_n| \, d\mu < \epsilon$ for all n. Let $N \in \mathbb{N}$ so that if $n, m > N$ and $\Omega_{n,m} = \{t : |f_n(t) - f_m(t)| > \epsilon\}$, then $\mu(\Omega_{n,m}) < \delta$. Therefore

$$\begin{aligned} \int_\Omega |f_n - f_m| \, d\mu &= \int_{\Omega \setminus \Omega_{m,n}} |f_n - f_m| \, d\mu + \int_{\Omega_{m,n}} |f_n - f_m| \, d\mu \\ &\leq \int_{\Omega \setminus \Omega_{m,n}} |f_n - f_m| \, d\mu + \int_{\Omega_{m,n}} |f_n| \, d\mu + \int_{\Omega_{m,n}} |f_m| \, d\mu \\ &< \epsilon \mu(\Omega \setminus \Omega_{m,n}) + 2\epsilon. \end{aligned}$$

Thus (f_n) is Cauchy in $L^1(\mu)$ and must converge in $L^1(\mu)$.

Conversely, suppose that $(f_n) \longrightarrow f$ in $L^1(\mu)$. Clearly then (f_n) must be Cauchy in measure and uniformly integrable. Further, $D(f_n, f_n)$ exists uniformly in n. ∎

In the following corollary, $L^p(\mu, X) = L^p(\Omega, \Sigma, \mu, X)$ denotes the Bochner space of X-valued p-th power integrable functions.

COROLLARY 2.4. *Suppose that* (Ω, Σ, μ) *is a finite measure space,* X *is a Banach space, and* (f_n) *is a sequence in* $L^p(\mu, X)$. *The sequence* (f_n) *converges to an element* f *in* $L^p(\mu, X)$ *if and only if*

(i) (f_n) *is Cauchy in measure, and*

(ii) there is a uniformly integrable sequence (g_n) *in* $L^p(\mu, X)$ *so that* $D(\|g_n(\cdot)\|^p, \|f_n(\cdot)\|^p)$ *exists uniformly in* n.

We remark that if (f_n) is an arbitrary sequence in $L^p(\mu, X)$, then obviously $D(\|f_n(\cdot)\|^p, \|f_n(\cdot)\|^p)$ exists uniformly in n. Therefore if (f_n) and f belong to $L^p(\mu, X)$, $(f_n) \longrightarrow f$ *a.e.* $[\mu]$, and (f_n) is uniformly integrable (i.e., $(\int_{(\cdot)} f_n \, d\mu)$ is equicontinuous), then $\|f_n - f\| \longrightarrow 0$. Thus Theorem 3 of [3] follows from 2.3. We also note that the generalization of the Lebesgue convergence theorem which appears in Chapter 4 of Royden [8] follows immediately from 2.3.

3. **Uniform Absolute Continuity.** In this section we study generalizations and interpretations of Theorem 1.1 motivated by the considerations of the preceding section. In general, we are interested in questions of the following type. If $H, K \subset ca(\Sigma)$, $P \subset H \times K$, and $\nu \ll \mu$ for $(\nu, \mu) \in P$, then what conditions on H and/or K will ensure that $\nu \ll \mu$ uniformly for $(\nu, \mu) \in P$? The following very elementary example shows that the most obvious bi-sequential interpretation of Theorem 1.1 (suggested by Lemma 2.2 and Theorem 2.3) is false.

Let Σ be the σ-algebra of Lebesgue measurable subsets of $[0,1]$, and let λ be Lebesgue measure on Σ. Let $\mu_n = (1/n)\lambda$, and let $\nu_n = \lambda$ for all n. Obviously $H = \{\nu_n : n \in \mathbb{N}\}$ and $K = \{\mu_n : n \in \mathbb{N}\}$ are uniformly countably additive subsets of $ca(\Sigma)$, and $\nu_n \ll \mu_m$ for all n and m. However, it is plain that the absolute continuity is not uniform in n, i.e., if $P = \{(\nu_n, \mu_n) : n \in \mathbb{N}\}$, then it is not true that $\nu \ll \mu$ uniformly for $(\nu, \mu) \in P$. We remark that it is not inconsequential that the set K in this example fails to be compact.

The following notation will be useful. If $H, K \subset ca(\Sigma)$, then we write $H \ll K$ if $A \in \Sigma$ and $\inf\{|\mu|(A) : \mu \in K\} = 0$ ensures that $\nu(A) = 0$ for all $\nu \in H$. If $H \ll K$, then certainly $\nu \ll \mu$ for all $(\nu, \mu) \in H \times K$. The remainder of this section will be concerned largely with studying the relative strengths of the following four conditions on H and K in the presence of various topological conditions:

(a) $\nu \ll \mu$ uniformly for $(\nu, \mu) \in H \times K$,
(b) $H \ll K$,
(c) $\nu \ll \mu$ for $(\nu, \mu) \in H \times \bar{K}$,
(d) $\nu \ll \mu$ for $(\nu, \mu) \in H \times K$.

It follows that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. Examples at the end of the section will investigate the reverse implications.

By modifying the sketch of the argument for Theorem 1 in [3], we are able to obtain the following bi-sequential version of Theorem 1.1. A simple corollary of this result shows that $(d) \Rightarrow (a)$ if H is uniformly countably additive and K is compact.

THEOREM 3.1. *If (ν_n) is a uniformly countably additive sequence in $ca(\Sigma)$ and (μ_n) is a sequence in $ca(\Sigma)$ so that*

(i) $\nu_n \ll \mu_n$ for each n, and
(ii) $\big(\nu_n(A_k)\big) \underset{k}{\longrightarrow} 0$ as $\big(\mu_k(A_k)\big) \underset{k}{\longrightarrow} 0$ for each n, then $\nu_n \ll \mu_n$ uniformly in n.

Before we begin the proof of Theorem 3.1, we establish a technical lemma for uniformly countably additive sequences of measures.

LEMMA 3.2. *Suppose that (ν_i) is a uniformly countably additive sequence from $ca(\Sigma)$ and (A_i) is an arbitrary sequence of sets from Σ. If $\delta > 0$ and $j \in \mathbb{N}$, then there is a finite sequence $(j_i)_{i=1}^n$ of positive integers so that $j < j_1 < j_2 < \cdots < j_n$ and if $C_n = A_j \setminus \bigcup_{i=1}^n A_{j_i}$, then*

$$|\nu_i(C_n \cap A_i)| < \delta$$

for $i > j_n$.

https://doi.org/10.4153/CJM-1991-051-3 Published online by Cambridge University Press

PROOF. Deny the conclusion. Choose $j_1 > j$ and $i > j_1$ so that $|\nu_i(C_1 \cap A_i)| \geq \delta$, where $C_1 = A_j \backslash A_{j_1}$. Let j_2 be the smallest positive integer i which satisfies the requirements of the preceding sentence. Continuing, let j_3 be the smallest positive integer so that $j_3 > j_2$ and $|\nu_{j_3}(C_2 \cap A_{j_3})| \geq \delta$, where $C_2 = A_j \backslash (A_{j_1} \cup A_{j_2})$. Continue this process recursively. Note that (C_i) is a decreasing sequence; therefore $(C_i \backslash C_{i+1})$ is clearly a pairwise disjoint sequence in Σ. Further, note that $C_i \backslash C_{i+1} = C_i \cap A_{j_{i+1}}$. Since $|\nu_{j_{i+1}}(C_i \cap A_{j_{i+1}})| \geq \delta$, we contradict the uniform countable additivity of (ν_i). ∎

PROOF OF THEOREM 3.1. Deny the conclusion. Let $\epsilon > 0$ so that if $\delta > 0$, then there is a positive integer n and an element $A_n \in \Sigma$ so that $|\mu_n|(A_n) < \delta$ and $|\nu_n(A_n)| > \epsilon$. Therefore we may (and do) suppose that

$$(*) \qquad |\mu_n|(A_n) < 1/n^2 \text{ and } |\nu_n(A_n)| > \epsilon,\ n \in \mathbb{N}.$$

(Note that $\left(|\nu_k|(A_n)\right)_n \longrightarrow 0$ for each k.) Pass to a subsequence and obtain (ν_i), (μ_i), and (A_i) so that the following two inequalities hold:

(1) $|\nu_i(A_i)| > \epsilon$

(2) $\sum_{i=k+1}^{\infty} |\nu_k|(A_i) < \epsilon / 2^{k+1}$

Now let $\delta = \epsilon/4$, and suppose that $A_n, A_{n_1}, \ldots, A_{n_k}$ satisfy the conclusions of Lemma 3.2 with respect to the sequence (ν_n). Since $n \geq 1$ and

$$\sum_{i=1}^{k} |\nu_n(A_n \cap A_{n_i})| \leq \sum_{i=1}^{k} |\nu_n|(A_{n_i}) < \epsilon / 2^{1+1},$$

it follows that

$$\begin{aligned} |\nu_n(A_n \backslash \bigcup_{i=1}^{k} A_{n_i})| &= |\nu_n(A_n \backslash \bigcup_{i=1}^{k} A_n \cap A_{n_i})| \\ &\geq |\nu_n(A_n)| - \sum_{i=1}^{k} |\nu_n|(A_n \cap A_{n_i}) > 3\epsilon/4. \end{aligned}$$

Put $H_1 = A_n \backslash \bigcup_{i=1}^{k} A_{n_i}$, and recall that $|\nu_p(H_1 \cap A_p)| < \epsilon/4$ if $p > n_k$.

Next we start the process over as follows. Let $p_1 = n_k$, $A_{1,i} = A_{p_1+i} \backslash H_1$, and $\nu_{1,i} = \nu_{p_1+i}$, $i \in \mathbb{N}$. Apply Lemma 3.2 again and find positive integers $n < n_1 < \cdots < n_m$ so that if $H_2 = A_{1,n} \backslash \bigcup_{i=1}^{m} A_{1,n_i}$, then $|\nu_{1,p}(H_2 \cap A_{1,p})| < \epsilon/8$ for $p > n_m$. Note that $|\nu_{1,i}(A_{1,i})| = |\nu_{p_1+i}(A_{p_1+i} \backslash H_1 \cap A_{p_1+i})| > 3\epsilon/4$. Consequently,

$$\begin{aligned} |\nu_{1,n}(H_2)| &\geq |\nu_{1,n}(A_{1,n})| - \sum_{i=1}^{m} |\nu_{1,n}|(A_{1,n} \cap A_{1,n_i}) \\ &\geq |\nu_{1,n}(A_{1,n})| - \sum_{i=1}^{m} |\nu_{1,n_i}|(A_{1,n_i}) \\ &\geq |\nu_{1,n}(A_{1,n})| - \epsilon/2^{p_1+n+1} > \epsilon - \epsilon/4 - \epsilon/8. \end{aligned}$$

Additionally, note that $H_1 \cap H_2 = \emptyset$.

If we continue the construction by putting $p_2 = n_m$, $A_{2,i} = A_{p_2+i} \setminus H_2$, and $\nu_{2,i} = \nu_{1,p_2+i}$, then we obtain H_3, which misses $H_1 \cup H_2$, and a member $\nu_{2,n}$ of the sequence (ν_n) so that

$$|\nu_{2,n}(H_3)| > \epsilon - \epsilon/4 - \epsilon/8 - \epsilon/16.$$

Continuing inductively, we contradict the uniform countable additivity of (ν_n). ∎

COROLLARY 3.3. *Suppose that Σ is a σ-algebra, $H, K \subset ca(\Sigma)$, and H is uniformly countably additive.*

(i) If $\nu \ll \mu$ uniformly for $\mu \in K$ whenever $\nu \in H$, then $\nu \ll \mu$ uniformly for $(\nu, \mu) \in H \times K$.

(ii) If K is compact and $\nu \ll \mu$ for $(\nu, \mu) \in H \times K$, then $\nu \ll \mu$ uniformly for $(\nu, \mu) \in H \times K$.

PROOF. (i) Suppose that the uniform absolute continuity fails in $H \times K$. Let $\epsilon > 0$, let (ν_n) be a sequence in H, let (μ_n) be a sequence in K, and let (A_n) be a sequence in Σ so that $|\nu_n(A_n)| > \epsilon$ for all n and $\big(|\mu_n|(A_n)\big) \longrightarrow 0$. Certainly condition (ii) of Theorem 3.1 must fail for the sequence $(\nu_n, |\mu_n|, A_n)$ of triples. Therefore there exists $p \in \mathbb{N}$ and a subsequence (n_k) of positive integers and $\delta > 0$ so that

$$|\nu_p(A_{n_k})| > \delta$$

for each k. However, this contradicts the hypothesis since $\nu_p \ll \mu$ uniformly for $\mu \in K$.

(ii) In view of (i), it clearly suffices to say that if $\nu \in H$, then $\nu \ll \mu$ uniformly for $\mu \in K$. Suppose to the contrary that this uniformity does not hold. Let $\nu \in H$, let (μ_n) be a sequence from K, let $\epsilon > 0$, and let (A_n) be a sequence from Σ so that $(|\mu_n|(A_n)) \longrightarrow 0$ and $|\nu(A_n)| > \epsilon$ for each n. Certainly (μ_n) must cluster at some point μ in K. Suppose $\|\mu_{n_i} - \mu\| \longrightarrow 0$. Therefore $\big(\mu(A_{n_i})\big) \longrightarrow 0$, and thus $\big(\nu(A_{n_i})\big) \longrightarrow 0$, a clear contradiction. ∎

Our next result shows that (b) and (c) are equivalent for arbitrary subsets H and relatively compact subsets K of $ca(\Sigma)$.

THEOREM 3.4. *Suppose that $H, K \subset ca(\Sigma)$ and that K is relatively compact. The following are equivalent:*

(i) $H \ll K$

(ii) $\nu \ll \mu$ for $(\nu, \mu) \in H \times \bar{K}$.

In addition, if H is uniformly countably additive, then each of (i) and (ii) is equivalent to

(iii) $\nu \ll \mu$ uniformly for $(\nu, \mu) \in H \times K$.

PROOF. Suppose that (i) holds and let $\mu \in \bar{K}$. Let $A \in \Sigma$ so that $|\mu|(A) = 0$, and let (μ_n) be a sequence from K so that $\|\mu_n - \mu\| \longrightarrow 0$. Certainly $|\mu_n|(A) \longrightarrow 0$. Thus $\inf\{|\xi|(A) : \xi \in K\} = 0$, and $\nu(A) = 0$ for all $\nu \in H$.

Conversely, suppose (ii) holds, $A \in \Sigma$, and $\inf\{|\mu|(A) : \mu \in K\} = 0$. Let (μ_n) be a sequence from K so that (μ_n) converges in norm to an element $\mu \in ca(\Sigma)$ and $\big(|\mu_n|(A)\big) \longrightarrow 0$. Therefore $|\mu|(A) = 0$, and the absolute continuity in (ii) forces $\nu(A) = 0$ for all $\nu \in H$.

Now suppose that the set H is uniformly countably additive. If (ii) holds, then we immediately obtain (iii) by an application of Corollary 3.3(ii).

Conversely, suppose (iii) holds, $\mu \in \bar{K}$, $\nu \in H$, $A \in \Sigma$, and $|\mu|(A) = 0$. Let $\epsilon > 0$, and let $\delta > 0$ so that if $B \in \Sigma$, $\xi \in K$, and $|\xi|(B) < \delta$, then $|\nu(B)| < \epsilon$. Let (μ_n) be a sequence from K so that $\|\mu_n - \mu\| \longrightarrow 0$. Thus $\big(|\mu_n|(A)\big) \longrightarrow 0$; hence $|\nu(A)| < \epsilon$. Since ϵ was arbitrary, $\nu(A) = 0$. ∎

If the set K is relatively weakly compact, then condition (b) is implied by

$$(c') \qquad \nu \ll \mu \text{ for all } (\nu, \mu) \in H \times \bar{K}^w.$$

For suppose that $A \in \Sigma$ and $\inf\{|\mu|(A) : \mu \in K\} = 0$. Let (μ_n) be a sequence in K so that $|\mu_n|(A) < 1/n$, $n \in \mathbb{N}$, and let μ be a weak cluster point of (μ_n). If $B \in \Sigma \cap A$, then $\mu(B) = 0$. Therefore $|\mu|(A) = 0$, and $\nu(A) = 0$ for all $\nu \in H$, i.e., $(c') \Rightarrow (b)$. As a consequence of these observations and the preceding theorems, we immediately obtain

PROPOSITION 3.5. *Suppose that $H, K \subset ca(\Sigma)$.*

(A) If K is relatively weakly compact and convex, then $H \ll K$ iff $\nu \ll \mu$ for all $(\nu, \mu) \in H \times \bar{K}$.

(B) If K is weakly compact, then $H \ll K$ iff $\nu \ll \mu$ for all $(\nu, \mu) \in H \times K$.

Before turning to the examples mentioned at the beginning of this section, we briefly discuss parameterized versions of the Vitali-Hahn-Saks theorem. We begin by noting that if r_n denotes the n-th Rademacher function, $\nu_n(A) = \int_A r_n \, d\lambda$ for each measurable subset A of [0,1], and $\mu_n = (1/n)\lambda$ for each n, then $\nu_n \ll \mu_m$ for each n and m, $\big(\nu_n(A)\big) \longrightarrow 0$ for each A, and ν_n is not absolutely continuous with respect to μ_n uniformly in n. However, we do have the following corollary of 3.3 and 3.4.

PROPOSITION 3.6. *(i) If $H = \{\nu_n : n \in \mathbb{N}\} \subset ca(\Sigma)$, $K \subset ca(\Sigma)$, $\nu_n \ll \mu$ for each $(n, \mu) \in \mathbb{N} \times \bar{K}$, and $\big(\nu_n(A)\big)$ converges for each $A \in \Sigma$, then $\nu_n \ll \mu$ uniformly for $(n, \mu) \in \mathbb{N} \times \bar{K}$.*

(ii) If H and K are as above and K is compact, then $\nu_n \ll \mu$ uniformly for $(n, \mu) \in \mathbb{N} \times K$.

To apply 3.3 and 3.4, note that the convergence of $\big(\nu_n(A)\big)$ for each A implies that (ν_n) is uniformly countably additive, e.g., see the proof of Theorem 3.5 in [2].

The elementary example at the beginning of this section shows that $(d) \not\Rightarrow (c)$. More substantial examples are required to show that none of the other reverse implications hold.

EXAMPLE 3.7. Let (r_n) be the sequence of Rademacher functions, let $\mu_n(A) = \int_A \frac{1}{2}(r_n + 1)\, d\lambda$, and let $\lambda_n = \mu_n + (1/n)\lambda$, $n \in \mathbb{N}$. Let $H = \{\lambda\}$, and let $K = \{\lambda_n : n \in \mathbb{N}\}$. Certainly both H and K are uniformly countably additive ((r_n) is uniformly bounded). Now let $B_n = \{t : r_n(t) = -1\}$, $n \in \mathbb{N}$. Therefore $\mu_n(B_n) = 0$ and $\lambda_n(B_n) = 1/2n$ for each n. Clearly we do not have uniform absolute continuity for $H \times K$, i.e., (a) does not hold.

Now suppose that A is a measurable set and that $\inf\{|\lambda_n|(A)\} = 0$. Since $0 \leq (1/n)\lambda(A) \leq \lambda_n(A)$ for all n and $\big(\lambda_n(A)\big) \longrightarrow (1/2)\lambda(A)$, it follows that $\lambda(A) = 0$, Thus (b) holds, i.e., $(b) \not\Rightarrow (a)$.

We remark that Example 3.7 also allows us to distinguish between condition (ii) of Theorem 3.1 and condition (b). Specifically, if $H, K \subset ca(\Sigma)$, then we say that the ordered pair (H, K) satisfies (e) if

$$\big(\nu(A_k)\big) \longrightarrow 0 \text{ whenever } (\mu_k) \text{ is any sequence from } K \text{ and } (A_k)$$
$$\text{is any sequence from } \Sigma \text{ so that } \big(|\mu_k|(A_k)\big) \longrightarrow 0.$$

We note that $(e) \Rightarrow (b)$. For if $\inf\{|\mu|(A) : \mu \in K\} = 0$, then let $A_k = A$ for each k, and choose (μ_k) from K so that $\big(|\mu_k|(A_k)\big) \longrightarrow 0$. Therefore $\nu(A) = \nu(A_k) \longrightarrow 0$. However, if (b) implied (e), then we could apply Theorem 3.1 to the constant sequence (λ) and the sequence (λ_n) of 3.6 and conclude that $\lambda \ll \lambda_n$ uniformly in n.

In addition, we note that Example 3.7 allows us to see that the equivalences of Theorem 3.4 do not hold if we merely assume that K is weakly compact. Specifically, let $K_0 = \{\lambda_n : n \in \mathbb{N}\} \cup \{(1/2)\lambda\}$, and let $H = \{\lambda\}$. It follows that K_0 is weakly compact, H is uniformly countably additive, and $H \ll K_0$. However, since $\big(\lambda_n(B_n)\big) \longrightarrow 0$ and $\lambda(B_n) = 1/2$ for all n, λ is not absolutely continuous with respect to λ_n uniformly in n.

Our next example shows that $(c) \not\Rightarrow (b)$.

EXAMPLE 3.8. As in the preceding example, let (r_n) denote the sequence of Rademacher functions, let $\sigma_n(A) = \int_{A\cap[0,1/2]}(1/2)(r_n + 1)\,d\lambda$, let $\mu_n = \sigma_n + (1/n)\lambda$, let $H = \{\lambda\}$, and let $K = \{\mu_n : n \in \mathbb{N}\}$. Certainly $\lambda \ll \mu_n$ for all n. Clearly $\inf_n\{|\mu_n|([1/2, 1])\} = \inf_n\{1/2n\} = 0$, and $\lambda([1/2, 1]) = 1/2$. Therefore it is not the case that $H \ll K$. Since $\|\sigma_n - \sigma_m\| = 1/4$ for all n, $n \neq m$, it follows that K is closed, i.e., $\nu \ll \mu$ for $(\nu, \mu) \in H \times \bar{K}$.

4. **Uniform Differentiability.** In this section we establish a Gateaux differentiability result for continuous convex functions which is motivated by [2] and Sections 2 and 3 of this paper. In fact, this theorem can be interpreted as generalizing 3.3 to the setting of an arbitrary continuous convex function. If $f: X \longrightarrow \mathbb{R}$ is a continuous convex function, then $Df(x, y)$ is defined to be $\lim_{t\to 0}(f(x + ty) - f(x))/t$, provided that this limit exists. Let $\partial f(a)$ denote the subgradient of f at a, i.e.

$$\partial f(a) = \{x^* \in X^* : x^*(x) \leq f(a + x) - f(a),\ x \in X\}.$$

We note that $\partial f: X \longrightarrow X^*$ is a multivalued mapping which is both monotone and maximal [6], [7]. In the following theorem, we denote w^*-convergence by $\longrightarrow^{w^*}$.

THEOREM 4.1. *If $f: X \longrightarrow \mathbb{R}$ is a continuous convex function, M and N are non-empty subsets of X, M is compact, and N is relatively compact, and $Df(x, y)$ exists for $x \in M$ and $y \in N$, then $Df(x, y)$ exists uniformly for $x \in M$ and $y \in N$.*

PROOF. Suppose that f, M, and N satisfy the hypotheses. Then $\bar{N}$ is compact. Note that $Df(x, y)$ exists for $x \in M$ and $y \in \bar{N}$. Suppose that the conclusion of the theorem is false. Let (a_n) be a sequence from M, (b_n) be a sequence from N, ϵ be a positive number, and (t_n) be a null sequence of positive numbers so that

$$f(a_n + t_nb_n) - f(a_n - t_nb_n) - 2f(a_n) > t_n\epsilon$$

for $n \in \mathbb{N}$. Let $x_n^* \in \partial f(a_n + t_nb_n)$ and $y_n^* \in \partial f(a_n - t_nb_n)$. Then

$$x_n^*(-t_nb_n) \leq f(a_n + t_nb_n - t_nb_n) - f(a_n + t_nb_n)$$

and

$$y_n^*(t_nb_n) \leq f(a_n - t_nb_n + t_nb_n) - f(a_n - t_nb_n)$$

for each n. Combining these inequalities, we see that

$$-t_n\langle x_n^* - y_n^*, b_n\rangle \leq 2f(a_n) - f(a_n + t_nb_n) - f(a_n - t_nb_n) \leq -t_n\epsilon;$$

thus $\langle x_n^* - y_n^*, b_n\rangle > \epsilon$ for all n. Now since ∂f is locally bounded and M and $\bar{N}$ are compact, we may (and do) suppose without loss of generality that $(a_n) \longrightarrow a \in M$, $(b_n) \longrightarrow b \in \bar{N}$, and $(x_{n_\alpha}^*)$ and $(y_{n_\alpha}^*)$ are subnets (respectively) of (x_n^*) and (y_n^*) which are w^*-convergent. Let x^*, $y^* \in X^*$ so that $x_{n_\alpha}^* \longrightarrow^{w^*} x^*$ and $y_{n_\alpha}^* \longrightarrow^{w^*} y^*$. The maximal monotonicity of ∂f ensures that $x^*, y^* \in \partial f(a)$. But then $x^*(b) = y^*(b)$ since $Df(a,b)$ exists. However, $\langle x^* - y^*, b\rangle = \lim_\alpha\langle x_{n_\alpha}^* - y_{n_\alpha}^*, b_{n_\alpha}\rangle \geq \epsilon$. This contradition guarantees that the desired uniformity does hold. ∎

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