J. Aust. Math. Soc. **101** (2016), 29–53
doi:10.1017/S1446788715000531

THE LEVEL 12 ANALOGUE OF RAMANUJAN'S FUNCTION k

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(Received 27 March 2014; accepted 29 October 2015; first published online 22 January 2016)

Communicated by W. Zudilin

Abstract

We provide a comprehensive study of the function $h = h(q)$ defined by

$$h = q \prod_{j=1}^{\infty} \frac{(1-q^{12j-1})(1-q^{12j-11})}{(1-q^{12j-5})(1-q^{12j-7})}$$

and show that it has many properties that are analogues of corresponding results for Ramanujan's function $k = k(q)$ defined by

$$k = q \prod_{j=1}^{\infty} \frac{(1-q^{10j-1})(1-q^{10j-2})(1-q^{10j-8})(1-q^{10j-9})}{(1-q^{10j-3})(1-q^{10j-4})(1-q^{10j-6})(1-q^{10j-7})}.$$

2010 *Mathematics subject classification*: primary 11F11; secondary 05A10, 33C75, 33E05.

Keywords and phrases: Dedekind eta function, Domb numbers, Eisenstein series, elliptic integral, hypergeometric function, Jacobi triple product identity, modular equation, quintuple product identity, Ramanujan's cubic continued fraction, Rogers–Ramanujan continued fraction.

1. Introduction

The function $h = h(q)$ defined for $|q| < 1$ by

$$h = h(q) = q \prod_{j=1}^{\infty} \frac{(1-q^{12j-1})(1-q^{12j-11})}{(1-q^{12j-5})(1-q^{12j-7})}$$

has been studied only recently. A continued fraction for h was derived by Mahadeva Naika *et al.* [22], and several modular equations for h have been given in [18, 21, 22, 26]. The signs of the coefficients in the q-series expansions of h and its reciprocal, along with the 2-, 3-, 4-, 6- and 12-dissections, have been studied by Lin [20]. The objective of this work is to provide a comprehensive and systematic study of the function h and prove many properties that are new.

Ramanujan's function $k = k(q)$ is defined by

$$k = k(q) = q\prod_{j=1}^{\infty}\frac{(1-q^{10j-1})(1-q^{10j-2})(1-q^{10j-8})(1-q^{10j-9})}{(1-q^{10j-3})(1-q^{10j-4})(1-q^{10j-6})(1-q^{10j-7})}.$$

We will show that h and k have analogous properties. For example, if $r = r(q)$ is Ramanujan's cubic continued fraction defined by

$$r(q) = \cfrac{q^{1/3}}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\cfrac{q^3+q^6}{1+\cdots}}}} \tag{1.1}$$

then it will be shown in Corollary 3.4 that

$$r^3(q) = h\Big(\frac{1-h}{1+h^2}\Big)^2 \quad \text{and} \quad r^3(q^2) = \frac{h^2}{(1-h)^2(1+h^2)}.$$

This is an analogue of the following result of Ramanujan: if $R(q)$ is the Rogers–Ramanujan continued fraction defined by

$$R(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}}$$

then

$$R^5(q) = k\Big(\frac{1-k}{1+k}\Big)^2 \quad \text{and} \quad R^5(q^2) = k^2\Big(\frac{1+k}{1-k}\Big).$$

More information about Ramanujan's result is provided in [7, page 167] and [15, Theorem 3.3].

As another example, in Theorem 5.4 we will show that the following identity holds:

$$q\frac{d}{dq}\log\Big(\frac{h(1+h^2)}{(1-4h+h^2)(1-h+h^2)}\Big) = \sum_{n=0}^{\infty}\Big\{\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}\Big\}\Big(\frac{h(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-h^2)^4}\Big)^n. \tag{1.2}$$

It is an analogue of the identity [16, Theorem 4.2]

$$q\frac{d}{dq}\log\Big(\frac{k(1-k^2)}{(1+k-k^2)(1-4k-k^2)}\Big) = \sum_{n=0}^{\infty}\Big\{\sum_{j=0}^{n}\binom{n}{j}^4\Big\}\Big(\frac{k(1-k^2)(1+k-k^2)(1-4k-k^2)}{(1+k^2)^4}\Big)^n. \tag{1.3}$$

In Section 5 we will describe how (1.2) and (1.3) are analogues of the identity

$$q\frac{d}{dq}\log\Big(\frac{w}{1-16w}\Big) = \sum_{n=0}^{\infty}\binom{2n}{n}^3 (w(1-16w))^n$$

where

$$w = q\prod_{j=1}^{\infty}\frac{(1-q^j)^8(1-q^{4j})^{16}}{(1-q^{2j})^{24}}.$$

This is a reformulation of an identity used by Ramanujan [23, Section 13, (25)] as the starting point in his study of rapidly converging series for $1/\pi$.

This work is organized as follows. Notation and background results are set out in Section 2. The basic properties involving h are proved in Section 3. The logarithmic derivative $z = q(d/dq)\log h$ is introduced in Section 4. The functions h and z are used to parameterize several functions involving Eisenstein series and the Dedekind eta function. Differential equations are derived in Section 5. The function z is expressed as a function of h in terms of the ${}_3F_2$ hypergeometric function.

2. Definitions and preliminary results

Let τ be a complex number that satisfies $\mathrm{Im}(\tau) > 0$ and let $q = e^{2\pi i\tau}$. Let h and z be defined by

$$h = h(q) = q\prod_{j=1}^{\infty}(1-q^j)^{(12/j)} = q\prod_{j=1}^{\infty}\frac{(1-q^{12j-1})(1-q^{12j-11})}{(1-q^{12j-5})(1-q^{12j-7})} \tag{2.1}$$

and

$$z = z(q) = q\frac{d}{dq}\log h = 1 - \sum_{j=1}^{\infty}\Big(\frac{12}{j}\Big)\frac{jq^j}{1-q^j} \tag{2.2}$$

where (m/n) denotes the Jacobi symbol. Dedekind's eta function is defined by

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j).$$

For any positive integer m, let η_m be defined by

$$\eta_m = \eta(m\tau) = q^{m/24}\prod_{j=1}^{\infty}(1-q^{mj}).$$

We will occasionally use the compact notation for q-infinite products given by

$$(x_1, x_2, \ldots, x_n; q)_\infty = \prod_{j=0}^{\infty}(1-q^jx_1)(1-q^jx_2)\cdots(1-q^jx_n).$$

Frequent use will be made of Jacobi's triple product identity [7, page 10]

$$\sum_{j=-\infty}^{\infty} q^{j^2} t^j = \prod_{j=1}^{\infty} (1+q^{2j-1}t)(1+q^{2j-1}t^{-1})(1-q^{2j}).$$

Another form of Jacobi's triple product identity is given by [17, page 2305]

$$(tq, t^{-1}q, q; q)_\infty = 1 + \sum_{n=1}^{\infty} (-1)^n (t^n + t^{n-1} + \cdots + t^{-n}) q^{n(n+1)/2}. \tag{2.3}$$

A special case of Jacobi's triple product identity is Euler's identity [7, page 12]

$$\prod_{j=1}^{\infty} (1-q^j) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j-1)/2}. \tag{2.4}$$

The quintuple product identity will be used in the forms [12, (1.9)]

$$\sum_{j=-\infty}^{\infty} q^{j(3j+1)/2} (t^{3j} - t^{-3j-1}) = \prod_{j=1}^{\infty} \frac{(1-q^j t^2)(1-q^{j-1}t^{-2})(1-q^j)}{(1+q^j t)(1+q^{j-1}t^{-1})} \tag{2.5}$$

and [12, (1.10)]

$$\sum_{j=-\infty}^{\infty} q^{j(3j+2)} (t^{3j} - t^{-3j-2}) = \prod_{j=1}^{\infty} \frac{(1-q^{2j} t^2)(1-q^{2j-2}t^{-2})(1-q^{2j})}{(1+q^{2j-1} t)(1+q^{2j-1}t^{-1})}. \tag{2.6}$$

Ramanujan's Eisenstein series P, Q and R are defined by

$$P = P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \quad Q = Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1-q^j}$$

and

$$R = R(q) = 1 - 504 \sum_{j=1}^{\infty} \frac{j^5 q^j}{1-q^j}.$$

For brevity, we will sometimes write P_m, Q_m and R_m for $P(q^m)$, $Q(q^m)$ and $R(q^m)$, respectively. The derivative of P is given by

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}; \tag{2.7}$$

see, for example, [7, page 92]. There are similar formulas for the derivatives of Q and R, but they will not be required.

3. The modular function h

In this section we establish some of the basic properties of the function h. We begin by proving that eight linear combinations of $\sqrt{h}$ and $1/\sqrt{h}$ have simple expressions as infinite products. The corresponding results for Ramanujan's function k were given in [17, Theorem 2.1].

THEOREM 3.1. *Let* $\omega = \exp(2\pi i/3)$, $\xi = \exp(\pi i/6)$, $a = 2 - \sqrt{3}$ *and* $b = 2 + \sqrt{3}$. *The following identities hold:*

$$\frac{1}{\sqrt{h}} + \sqrt{h} = \left(\frac{\eta_2^7\eta_3}{\eta_1^3\eta_4^2\eta_6\eta_{12}^2}\right)^{1/2}, \tag{3.1}$$

$$\frac{1}{\sqrt{h}} - \sqrt{h} = \left(\frac{\eta_1\eta_4^2\eta_6^9}{\eta_2^3\eta_3^3\eta_{12}^6}\right)^{1/2}, \tag{3.2}$$

$$\frac{1}{\sqrt{h}} + i\sqrt{h} = \left(\frac{\eta_2\eta_3^3}{\eta_1\eta_6\eta_{12}^2}\right)^{1/2} \times \frac{1}{q^{1/12}} \prod_{j=1}^{\infty}(1 - iq^{3j-1})(1 + iq^{3j-2}), \tag{3.3}$$

$$\frac{1}{\sqrt{h}} - i\sqrt{h} = \left(\frac{\eta_2\eta_3^3}{\eta_1\eta_6\eta_{12}^2}\right)^{1/2} \times \frac{1}{q^{1/12}} \prod_{j=1}^{\infty}(1 + iq^{3j-1})(1 - iq^{3j-2}), \tag{3.4}$$

$$\frac{1}{\sqrt{h}} + \omega\sqrt{h} = \left(\frac{\eta_1\eta_4^4\eta_6^3}{\eta_2^3\eta_3\eta_{12}^4}\right)^{1/2} \times \frac{1}{q^{1/24}} \prod_{j=1}^{\infty}(1 - \omega q^{4j-1})(1 - \omega^2 q^{4j-3}), \tag{3.5}$$

$$\frac{1}{\sqrt{h}} + \omega^2\sqrt{h} = \left(\frac{\eta_1\eta_4^4\eta_6^3}{\eta_2^3\eta_3\eta_{12}^4}\right)^{1/2} \times \frac{1}{q^{1/24}} \prod_{j=1}^{\infty}(1 - \omega^2 q^{4j-1})(1 - \omega q^{4j-3}), \tag{3.6}$$

$$\frac{1}{\sqrt{h}} - a\sqrt{h} = \left(\frac{\eta_1^3\eta_4^2\eta_6^3}{\eta_2^3\eta_3\eta_{12}^4}\right)^{1/2} \times q^{1/12} \prod_{j=1}^{\infty}(1 - \xi^5 q^j)(1 - \xi^7 q^j) \tag{3.7}$$

and

$$\frac{1}{\sqrt{h}} - b\sqrt{h} = \left(\frac{\eta_1^3\eta_4^2\eta_6^3}{\eta_2^3\eta_3\eta_{12}^4}\right)^{1/2} \times q^{1/12} \prod_{j=1}^{\infty}(1 - \xi q^j)(1 - \xi^{11} q^j). \tag{3.8}$$

It will be convenient to break the proof into three parts.

PROOF OF (3.1)–(3.4). Replace q with $-q$ in Euler's identity (2.4), and then separate the terms in the series according to whether the summation index j is even or odd, to get

$$\prod_{j=1}^{\infty}(1 - (-q)^j) = \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-n} + \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n+1}.$$

Applying Jacobi's triple product identity to each series on the right gives

$$\prod_{j=1}^{\infty}(1 - (-q)^j) = \prod_{j=1}^{\infty}(1 - q^{12j-5})(1 - q^{12j-7})(1 - q^{12j}) + q\prod_{j=1}^{\infty}(1 - q^{12j-1})(1 - q^{12j-11})(1 - q^{12j}). \tag{3.9}$$

It is straightforward to verify the infinite product identities

$$\prod_{j=1}^{\infty}(1-(-q)^j) = \prod_{j=1}^{\infty}\frac{(1-q^{2j})^3}{(1-q^j)(1-q^{4j})} = q^{-1/24}\frac{\eta_2^3}{\eta_1\eta_4} \tag{3.10}$$

and

$$\begin{aligned}\prod_{j=1}^{\infty}&(1-q^{12j-1})(1-q^{12j-5})(1-q^{12j-7})(1-q^{12j-11})\\ &= \prod_{j=1}^{\infty}\frac{(1-q^j)(1-q^{6j})}{(1-q^{2j})(1-q^{3j})} = q^{-1/12}\frac{\eta_1\eta_6}{\eta_2\eta_3}.\end{aligned} \tag{3.11}$$

Identity (3.1) may be obtained by dividing both sides of (3.9) by

$$q^{1/2}\prod_{j=1}^{\infty}(1-q^{12j-1})^{1/2}(1-q^{12j-5})^{1/2}(1-q^{12j-7})^{1/2}(1-q^{12j-11})^{1/2}(1-q^{12j})$$

and using (3.10) and (3.11).

The identities (3.2)–(3.4) may be proved by a similar procedure, starting with the identities

$$\sum_{n=-\infty}^{\infty}(-q)^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n+1},$$

$$\sum_{n=-\infty}^{\infty} i^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-n} + i\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n+1}$$

and

$$\sum_{n=-\infty}^{\infty}(-i)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-n} - i\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n+1},$$

respectively, and using Jacobi's triple product identity to convert each series to a product. We omit the details as they are similar to those given above. □

Proof of (3.5) and (3.6). Apply the series rearrangement

$$\sum_{m,n} c_{m,n} = \sum_{m,n} c_{m+2n,m-n} + \sum_{m,n} c_{m+2n+1,m-n} + \sum_{m,n} c_{m+2n-1,m-n}$$

with

$$c_{m,n} = (-1)^n q^{2m^2-m+4n^2+2n}\omega^{m+2n}$$

to get

$$\begin{aligned}\Big(\sum_{m=-\infty}^{\infty} q^{2m^2-m}\omega^m\Big)\Big(\sum_{n=-\infty}^{\infty} q^{4n^2+2n}(-\omega^2)^n\Big)\\ = \Big(\sum_{m=-\infty}^{\infty}(-1)^m q^{6m^2+m}\Big)\Big(\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-4n}\Big)\\ + \omega q\Big(\sum_{m=-\infty}^{\infty}(-1)^m q^{6m^2+5m}\Big)\Big(\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2+4n}\Big)\\ + \omega^2 q^3\Big(\sum_{m=-\infty}^{\infty}(-1)^m q^{6m^2-3m}\Big)\Big(\sum_{n=-\infty}^{\infty}(-1)^n q^{12n^2-12n}\Big).\end{aligned}$$

Now use Jacobi's triple product identity to convert each series to a product. The result is

$$\begin{aligned}(-q\omega, -q^3\omega^2, q^4; q^4)_\infty(q^6\omega^2, q^2\omega, q^8; q^8)_\infty\\ = (q^5, q^7, q^{12}; q^{12})_\infty(q^8; q^8)_\infty + \omega q(q, q^{11}, q^{12}; q^{12})_\infty(q^8; q^8)_\infty + 0,\end{aligned} \quad (3.12)$$

the last term on the right being identically zero. The left-hand side of (3.12) simplifies to

$$\prod_{j=1}^{\infty}\frac{(1-q^j)(1-q^{4j})^2(1-q^{6j})^2(1-q^{8j})}{(1-q^{2j})^2(1-q^{3j})(1-q^{12j})}\times(1-\omega q^{4j-1})(1-\omega^2 q^{4j-3}). \quad (3.13)$$

Identity (3.5) may be obtained by dividing both sides of (3.12) by

$$q^{1/2}\prod_{j=1}^{\infty}(1-q^{12j-1})^{1/2}(1-q^{12j-5})^{1/2}(1-q^{12j-7})^{1/2}(1-q^{12j-11})^{1/2}(1-q^{12j})(1-q^{8j})$$

and using (3.13). Identity (3.6) may be deduced from (3.5) by replacing ω by its complex conjugate ω^2. □

Proof of (3.7) and (3.8). If $t = \xi = e^{\pi i/6}$ or $t = \xi^5 = e^{5\pi i/6}$, then elementary calculations show that

$$t^n + t^{n-1} + \cdots + t^{-n} = \begin{cases} 1 & \text{if } n \equiv 0, 5 \bmod 12,\\ -1 & \text{if } n \equiv 6, 11 \bmod 12,\\ 1 \pm \sqrt{3} & \text{if } n \equiv 1, 4 \bmod 12,\\ -1 \mp \sqrt{3} & \text{if } n \equiv 7, 10 \bmod 12,\\ 2 \pm \sqrt{3} & \text{if } n \equiv 2, 3 \bmod 12,\\ -2 \mp \sqrt{3} & \text{if } n \equiv 8, 9 \bmod 12,\end{cases}$$

where the upper part of the ± or ∓ symbol is used if $a = \xi$ and the lower part is used if $a = \xi^5$. Substituting these values into Jacobi's triple product identity (2.3) and breaking the series into terms according to the power of q modulo 12, we get

$$(\xi q, \xi^{11} q, q; q)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+3j} - (1+\sqrt{3}) \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+9j+1} + (2+\sqrt{3}) \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+15j+3} \tag{3.14}$$

and

$$(\xi^5 q, \xi^7 q, q; q)_\infty = \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+3j} - (1-\sqrt{3}) \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+9j+1} + (2-\sqrt{3}) \sum_{j=-\infty}^{\infty} (-1)^j q^{18j^2+15j+3}. \tag{3.15}$$

Now, replacing q with q^{12} and letting $x = -q^{-1}$ in (2.5) gives

$$\sum_{j=-\infty}^{\infty} (-1)^j (q^{18j^2+3j} + q^{18j^2+9j+1}) = \prod_{j=1}^{\infty} \frac{(1-q^{12j-2})(1-q^{12j-10})(1-q^{12j})}{(1-q^{12j-1})(1-q^{12j-11})}, \tag{3.16}$$

and replacing q with q^6 and letting $x = -q^{-1}$ in (2.6) gives

$$\sum_{j=-\infty}^{\infty} (-1)^j (q^{18j^2+9j+1} - q^{18j^2+15j+3}) = q \prod_{j=1}^{\infty} \frac{(1-q^{12j-2})(1-q^{12j-10})(1-q^{12j})}{(1-q^{12j-5})(1-q^{12j-7})}. \tag{3.17}$$

Using (3.16) and (3.17) in (3.14) gives

$$(\xi q, \xi^{11} q, q; q)_\infty = \frac{(q^2, q^{10}, q^{12}; q^{12})_\infty}{(q, q^{11}; q^{12})_\infty} - (2+\sqrt{3}) q \frac{(q^2, q^{10}, q^{12}; q^{12})_\infty}{(q^5, q^7; q^{12})_\infty}.$$

Multiplying both sides by

$$\frac{(q, q^5, q^7, q^{11}; q^{12})_\infty^{1/2}}{q^{1/2}(q^2, q^{10}, q^{12}; q^{12})_\infty} \quad \left(= \frac{(q;q)_\infty^{1/2} (q^4;q^4)_\infty (q^6;q^6)_\infty^{3/2}}{q^{1/2} (q^2;q^2)_\infty^{3/2} (q^3;q^3)_\infty^{1/2} (q^{12};q^{12})_\infty^2} \right)$$

gives (3.8). In a similar way, using (3.16) and (3.17) in (3.15), we get (3.7). □

The next result shows that certain rational functions of h may be expressed as eta quotients. The analogous results involving Ramanujan's function k were given in [15, Theorem 3.5].

THEOREM 3.2.

$$\frac{1}{h} - h = \frac{\eta_2^2\eta_6^4}{\eta_1\eta_3\eta_{12}^4}, \tag{3.18}$$

$$\frac{1}{h} + h = \frac{\eta_3^3\eta_4}{\eta_1\eta_{12}^3}, \tag{3.19}$$

$$\frac{1}{h} - 1 + h = \frac{\eta_4^4\eta_6^2}{\eta_2^2\eta_{12}^4}, \tag{3.20}$$

$$\frac{1}{h} - 4 + h = \frac{\eta_1^3\eta_4\eta_6^2}{\eta_2^2\eta_3\eta_{12}^3}, \tag{3.21}$$

$$\frac{1}{h} - 2 + h = \frac{\eta_1\eta_4^2\eta_6^9}{\eta_2^3\eta_3^3\eta_{12}^6}, \tag{3.22}$$

$$\frac{1}{h} + 2 + h = \frac{\eta_2^7\eta_3}{\eta_1^3\eta_4^2\eta_6\eta_{12}^2}. \tag{3.23}$$

PROOF. Identity (3.18) follows by multiplying the first two identities in Theorem 3.1. Similarly, identities (3.19)–(3.21) may be obtained by multiplying the other conjugate pairs of identities in Theorem 3.1. Identities (3.22) and (3.23) are follow by squaring identities (3.1) and (3.2), respectively. □

COROLLARY 3.3. *As q increases from 0 to 1, the function $h(q)$ defined by (2.1) strictly increases from 0 to $2 - \sqrt{3}$.*

PROOF. For $0 < q < 1$, let $f(q)$ be the function defined by

$$f(q) = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^3(1-q^{4j})(1-q^{6j})^2}{(1-q^{2j})^2(1-q^{3j})(1-q^{12j})^3}.$$

By simple algebra,

$$f(q) = \frac{1}{q}\prod_{j=1}^{\infty}\frac{1}{(1+q^j)^2(1+q^j+q^{2j})(1+q^{4j}+q^{8j})(1+q^{6j})^2}.$$

It follows that f is strictly decreasing on the interval $0 < q < 1$ and

$$\lim_{q\to 0^+} f(q) = +\infty, \quad \lim_{q\to 1^-} f(q) = 0.$$

By (3.21),

$$\frac{1}{h(q)} - 4 + h(q) = f(q).$$

On solving the quadratic equation and using the fact that $h(0) = 0$ to determine the sign, we find that

$$h(q) = \tfrac{1}{2}\Big(4 + f(q) - \sqrt{(4+f(q))^2 - 4}\Big).$$

It is easy to check that as y increases from 0 to $+\infty$, the function

$$\tfrac{1}{2}\Big(4+y-\sqrt{(4+y)^2-4}\Big)$$

strictly decreases from $2-\sqrt{3}$ to 0. The result follows. □

The results of Theorem 3.2 can be used to derive the quadratic transformation formula for Ramanujan's cubic continued fraction stated in the introduction:

Corollary 3.4. *Let $r(q)$ be Ramanujan's cubic continued fraction defined by (1.1). Then*

$$r^3(q)=h\Big(\frac{1-h}{1+h^2}\Big)^2 \quad \text{and} \quad r^3(q^2)=\frac{h^2}{(1-h)^2(1+h^2)},$$

and hence

$$r(q)r(q^2)=\frac{h}{1+h^2}.$$

Proof. It is well known (see, for example, [6, page 346, (1.1)]) that

$$r(q)=\frac{\eta_1\eta_6^3}{\eta_2\eta_3^3}.$$

Hence, by the results in Theorem 3.2, we have

$$h\Big(\frac{1-h}{1+h^2}\Big)^2=\frac{\frac{1}{h}-2+h}{\Big(\frac{1}{h}+h\Big)^2}=\frac{\eta_1^3\eta_6^9}{\eta_2^3\eta_3^9}=r^3(q)$$

and

$$\frac{h^2}{(1-h)^2(1+h^2)}=\frac{1}{\Big(\frac{1}{h}-2+h\Big)\Big(\frac{1}{h}+h\Big)}=\frac{\eta_2^3\eta_{12}^9}{\eta_4^3\eta_6^9}=r^3(q^2).$$ □

Several modular equations for h appear in the literature, and they may be summarized by the following table:

Degree	Mathematicians	References
3, 5	Mahadeva Naika, Dharmendra and Shivashankara	[22]
2, 7, 9, 11, 13	Vasuki, Kahtan, Sharath and Sathish Kumar	[26]
3/2, 5/2, 7/2	Mahadeva Naika, Chandankumar and Bairy	[21]
6, 10, 14, 18	Dharmendra, Rajesh Kanna and Jagadeesh	[18]

The results in Theorem 3.2 can be used to deduce an algebraic relation between $h(q)$ and $h(-q)$, and to provide simple proofs of modular equations of degrees 2, 3, 4 and 6.

THEOREM 3.5. *Let $h = h(q)$ be defined by (2.1).*

(1) *If $H = h(-q)$ then $(H-1)^2 h + (1+h^2)H = 0$.*
(2) *If $H = h(q^2)$ then $(h+H)^2 = H(1+h^2)$.*
(3) *If $H = h(q^3)$ then $(H-h)^3 = H(H-1)(1+h)(1-4h+h^2)$.*
(4) *If $H = h(q^4)$ then $(H-h)^4 = H(1-H+H^2)(1+h^2)(1-4h+h^2)$.*
(5) *If $H = h(q^6)$, then $(H-h)^6 = H(1-4h+h^2)b(h,H)$, where*

$$\begin{aligned} b(h,H) = {} & h^4H^4 + h^4H - h^4H^2 + 6h^3H^3 - 2h^3H^4 - 2h^3H - 4h^3H^2 \\ & - 3h^2H - 3h^2H^3 + 12h^2H^2 + 6hH - 2hH^3 - 2h - 4hH^2 \\ & - H^2 + H^3 + 1. \end{aligned}$$

PROOF. We will give a detailed proof of (2) and then outline how to prove the other results. By the identities in Theorem 3.2,

$$h\left(\frac{1-h}{1+h^2}\right)^2 = \frac{\eta_1^3\eta_6^9}{\eta_2^3\eta_3^9} \quad \text{and} \quad \frac{h^2}{(1-h)^2(1+h^2)} = \frac{\eta_2^3\eta_{12}^9}{\eta_4^3\eta_6^9}.$$

It follows that

$$H\left(\frac{1-H}{1+H^2}\right)^2 = \frac{h^2}{(1-h)^2(1+h^2)}.$$

Clearing factions and factorizing the resulting polynomial, we get

$$(h^2H^2 - H + 1 + 2hH - h^2H)(H^2 - H + 2hH + h^2 - h^2H) = 0.$$

Noting that $h = H = 0$ when $q = 0$, it follows that the first factor does not vanish when $q = 0$. It follows that $H^2 - H + 2hH + h^2 - h^2H$ is zero in a neighborhood of $q = 0$ and hence is identically zero in the domain $|q| < 1$ by analytic continuation. On rearranging, we obtain (2).

The identities in (1), (3), (4) and (5) can be proved by the same method, by observing that

$$\frac{h^2}{(1-h)^2(1+h^2)} = \frac{\eta_2^3\eta_{12}^9}{\eta_4^3\eta_6^9}, \quad \text{an even function of } q;$$

$$\frac{h(1-h+h^2)^3}{(1+h^2)(1-4h+h^2)^3} = \frac{\eta_4^8}{\eta_1^8} \quad \text{and} \quad \frac{h^3(1-h+h^2)}{(1+h^2)^3(1-4h+h^2)} = \frac{\eta_{12}^8}{\eta_3^8};$$

$$\frac{h(1-h)^2(1+h^2)^4}{(1+h)^2(1-h+h^2)(1-4h+h^2)^4} = \frac{\eta_3^{12}}{\eta_1^{12}} \quad \text{and}$$

$$\frac{h^4(1-h)^2(1+h^2)}{(1+h)^2(1-h+h^2)^4(1-4h+h^2)} = \frac{\eta_{12}^{24}}{\eta_4^{24}};$$

and

$$\frac{h(1-h)^2(1+h)^6(1-h+h^2)^3}{(1+h^2)^2(1-4h+h^2)^6} = \frac{\eta_2^{24}}{\eta_1^{24}} \quad \text{and}$$

$$\frac{h^6(1-h+h^2)^2}{(1-h)^6(1+h)^2(1+h^2)^3(1-4h+h^2)} = \frac{\eta_{12}^{24}}{\eta_6^{24}}. \qquad \square$$

4. Weight-two modular forms

In this section we study the weight-two modular form z. We begin by establishing its representation as an infinite product. It is an analogue of [15, Theorem 3.1].

THEOREM 4.1. *The function z defined by (2.2) has a representation as an infinite product given by*

$$z = \frac{\eta_1\eta_3\eta_4^2\eta_6^2}{\eta_{12}^2}.$$

PROOF. By the definition (2.2), the identity we are required to prove is

$$1 - \sum_{j=1}^{\infty}\left(\frac{12}{j}\right)\frac{jq^j}{1-q^j} = \frac{\eta_1\eta_3\eta_4^2\eta_6^2}{\eta_{12}^2}. \tag{4.1}$$

This is a special case of the identity

$$\wp(u) - \wp(v) = -\frac{\sigma(u-v)\sigma(u+v)}{\sigma^2(u)\sigma^2(v)} \tag{4.2}$$

where $\wp$ is the Weierstrass elliptic function and σ is the associated sigma function. An explicit form of identity (4.2) is given by, for example, [14, Lemma 3.7],

$$\begin{aligned}1 &+ \frac{(1-a)^2(1-b)^2}{(1-ab)(a-b)}\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}(a^j+a^{-j}-b^j-b^{-j})\\ &= \prod_{j=1}^{\infty}\frac{(1-abq^j)(1-ab^{-1}q^j)(1-a^{-1}bq^j)(1-a^{-1}b^{-1}q^j)(1-q^j)^4}{(1-aq^j)^2(1-bq^j)^2(1-a^{-1}q^j)^2(1-b^{-1}q^j)^2},\end{aligned} \tag{4.3}$$

which holds for $|q| < |a|, |b| < |q|^{-1}$. Identity (4.1) may be obtained by taking $a = \exp(i\pi/6)$ and $b = \exp(5\pi i/6)$ in (4.3) and simplifying. The full details are given in [2, 14]. □

The next result provides parameterizations of various eta functions in terms of z and h. It is an analogue of [15, Theorem 3.6].

THEOREM 4.2. *The following identities hold:*

$$\begin{aligned}\eta^{24}(\tau) &= z^6\frac{h(1-4h+h^2)^6}{(1-h)^4(1+h^2)^2(1-h+h^2)^3},\\ \eta^{24}(2\tau) &= z^6\frac{h^2(1+h)^6}{(1-h)^2(1+h^2)^4},\\ \eta^{24}(3\tau) &= z^6\frac{h^3(1+h^2)^6}{(1-h+h^2)^5(1+h)^4(1-4h+h^2)^2},\\ \eta^{24}(4\tau) &= z^6\frac{h^4(1-h+h^2)^6}{(1+h^2)^5(1-h)^4(1-4h+h^2)^3},\end{aligned}$$

$$\eta^{24}(6\tau) = z^6 \frac{h^6(1-h)^6}{(1+h)^2(1-h+h^2)^4(1-4h+h^2)^4},$$
$$\eta^{24}(12\tau) = z^6 \frac{h^{12}}{(1+h)^4(1+h^2)^3(1-h+h^2)^2(1-4h+h^2)^5}.$$

Proof. These are immediate from the results of Theorems 3.2 and 4.1. □

The results in the next theorem provide parameterizations of various series in terms of z and h, along with their representations as eta quotients. The first result is in fact equivalent to Theorem 4.1.

Theorem 4.3. *The following identities hold:*

$$1 - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{12}{j}\right) j q^{jk} = \frac{\eta_1\eta_3\eta_4^2\eta_6^2}{\eta_{12}^2} = z,$$
$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{12}{k}\right) j q^{jk} = \frac{\eta_1^2\eta_4\eta_6^2\eta_{12}}{\eta_3^2} = \frac{zh}{1-4h+h^2},$$
$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{-3}{j}\right)\left(\frac{-4}{k}\right) j q^{jk} = \frac{\eta_2^2\eta_3^2\eta_4\eta_{12}}{\eta_1^2} = \frac{zh}{1+h^2},$$
$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{-4}{j}\right)\left(\frac{-3}{k}\right) j q^{jk} = \frac{\eta_1\eta_2^2\eta_3\eta_{12}^2}{\eta_4^2} = \frac{zh}{1-h+h^2}.$$

Proof. The first equality in each identity is given in [14, Theorem 4.1]. The second equality in each identity follows from the results in Theorem 4.2. □

We now show that $P(q^\ell)$, where $\ell \mid 12$, can be parameterized in terms of z, h and the derivative dz/dh. We begin with the following theorem.

Theorem 4.4. *The following identities hold:*

$$h\frac{dz}{dh} = \frac{1}{24}(P(q) + 3P(q^3) + 8P(q^4) + 12P(q^6) - 24P(q^{12})),$$
$$\frac{1-h^2}{1+h^2} z = \frac{1}{24}(P(q) - 9P(q^3) - 4P(q^4) + 36P(q^{12})),$$
$$\frac{1-h^2}{1-h+h^2} z = \frac{1}{24}(4P(q^2) - 16P(q^4) - 12P(q^6) + 48P(q^{12})),$$
$$\frac{1-h^2}{1-4h+h^2} z = \frac{1}{24}(-3P(q) + 4P(q^2) + 3P(q^3) - 4P(q^4) - 12P(q^6) + 36P(q^{12})),$$
$$\frac{1-h^2}{1-2h+h^2} z = \frac{1}{24}(-P(q) + 6P(q^2) + 9P(q^3) - 8P(q^4) - 54P(q^6) + 72P(q^{12})),$$
$$\frac{1-h^2}{1+2h+h^2} z = \frac{1}{24}(3P(q) - 14P(q^2) - 3P(q^3) + 8P(q^4) + 6P(q^6) + 24P(q^{12})).$$

PROOF. By the chain rule, we have

$$h\frac{dz}{dh} = q\frac{dz}{dq} \Big/ q\frac{d}{dq}\log h.$$

On applying (2.2) and Theorem 4.1, this becomes

$$h\frac{dz}{dh} = \left(q\frac{dz}{dq}\right)\times\left(\frac{1}{z}\right) = q\frac{d}{dq}\log z = q\frac{d}{dq}\log\left(\frac{\eta_1\eta_3\eta_4^2\eta_6^2}{\eta_{12}^2}\right).$$

On computing the logarithmic derivative on the right-hand side, we obtain the first result in the theorem.

To prove the remaining results, take the logarithmic derivative (with respect to q) of each of (3.19)–(3.23), respectively, and apply (2.2) and Theorem 4.1. □

The next result gives parameterizations of $P(q^\ell)$, for $\ell \mid 12$, in terms of z, h and dz/dh. It is an analogue of [15, Theorem 5.3].

THEOREM 4.5. *The following identity, expressed as a matrix product, holds:*

$$\begin{pmatrix} P(q) \\ P(q^2) \\ P(q^3) \\ P(q^4) \\ P(q^6) \\ P(q^{12}) \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 & -6 & 2 & 0 \\ 3 & 2 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 2 & -2 & \frac{5}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{3}{2} & \frac{5}{4} & -\frac{3}{2} & \frac{3}{4} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} h\dfrac{dz}{dh} \\ \dfrac{1-h^2}{1+h^2}z \\ \dfrac{1-h^2}{1-h+h^2}z \\ \dfrac{1-h^2}{1-4h+h^2}z \\ \dfrac{1-h^2}{1-2h+h^2}z \\ \dfrac{1-h^2}{1+2h+h^2}z \end{pmatrix}.$$

PROOF. This is immediate from Theorem 4.4. □

The Eisenstein series $Q(q^\ell)$ and $R(q^\ell)$ can be parameterized in terms of z and h. The results may be stated as:

THEOREM 4.6. *For each* $\ell \in \{1, 2, 3, 4, 6, 12\}$, *there are polynomials* $s_\ell(h)$ *and* $t_\ell(h)$ *of degrees* 16 *and* 24, *respectively, such that*

$$Q(q^\ell) = \frac{z^2}{(h-1)^2(h+1)^2(h^2+1)^2(h^2-h+1)^2(h^2-4h+1)^2} \times s_\ell(h)$$

and

$$R(q^\ell) = \frac{z^3}{(h-1)^3(h+1)^3(h^2+1)^3(h^2-h+1)^3(h^2-4h+1)^3} \times t_\ell(h).$$

Proof. It is known, for example, [13, (3.19) and Lemma 4.5], that

$$Q(q) - Q(q^2) = 240\frac{\eta_2^{16}}{\eta_1^8}$$

and

$$Q(q) + 4Q(q^2) = 5(2P(q^2) - P(q))^2.$$

Hence,

$$Q(q) = (2P(q^2) - P(q))^2 + 192\frac{\eta_2^{16}}{\eta_1^8}$$

and

$$Q(q^2) = (2P(q^2) - P(q))^2 - 48\frac{\eta_2^{16}}{\eta_1^8}.$$

The claimed results for $Q(q^\ell)$ follow by replacing q with an appropriate power of q in either of the last two identities, and making use of Theorems 4.2 and 4.5. We omit the computational details.

On using the results for $Q(q^\ell)$ and the formulas for η_ℓ in Theorem 4.2, we find by direct calculation, that for each $\ell \in \{1, 2, 3, 4, 6, 12\}$,

$$\frac{Q^3(q^\ell) - 1728\eta_\ell^{24}}{z^6} \times (h-1)^6(h+1)^6(h^2+1)^6(h^2-h+1)^6(h^2-4h+1)^6$$

is the square of a 24th degree polynomial in h. On taking square roots and using the well-known identity, for example, [7, page 92],

$$Q^3(q^\ell) - R^2(q^\ell) = 1728\eta_\ell^{24},$$

we obtain parameterizations for $R(q^\ell)$ in terms of z and h. The fact that t_ℓ turns out to be a polynomial follows by direct calculation for each value of ℓ. □

We end this section by relating the functions h and z to two functions κ and p studied by Alaca *et al.* [1]. In that work it was shown how many identities involving theta functions, Eisenstein series and the eta function can be verified by parameterizing in terms of κ and p. The corresponding parameterizations in terms of h and z often involve simpler algebraic expressions.

Theorem 4.7. *Let Ramanujan's theta function $\varphi = \varphi(q)$ be defined by*

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}$$

and let $\kappa = \kappa(q)$ and $p = p(q)$ be defined by

$$\kappa = \kappa(q) = \frac{\varphi^3(q^3)}{\varphi(q)} \quad \text{and} \quad p = p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}. \tag{4.4}$$

Then

$$\kappa = \left(z \times \frac{(1-h)^7}{(1+h)(1+h^2)(1-h+h^2)(1-4h+h^2)}\right)^{1/2} \quad \text{and} \quad p = \frac{2h}{(1-h)^2}.$$

PROOF. By Jacobi's triple product identity and some infinite product manipulations, we have, for example, [27, page 26],

$$\varphi(q) = \frac{\eta_2^5}{\eta_1^2\eta_4^2}.$$

Hence, on applying Theorem 4.2 we get

$$\varphi^4(q) = \frac{\eta_2^{20}}{\eta_1^8\eta_4^8} = z \times \frac{(1-h)(1+h)^5}{(1+h^2)(1-h+h^2)(1-4h+h^2)}$$

and

$$\varphi^4(q^3) = \frac{\eta_6^{20}}{\eta_3^8\eta_{12}^8} = z \times \frac{(1-h)^5(1+h)}{(1+h^2)(1-h+h^2)(1-4h+h^2)}.$$

On using these in (4.4) we obtain the required results after simplification. □

5. Differential equations

In this section we define modular functions x_A, x_B, x_C, y_A, y_B, y_C and weight-two modular forms z_A, z_B and z_C, and find a third-order differential equation that is satisfied by z_A with respect to y_A. We also provide some expressions for z and z_A in terms of hypergeometric functions involving h and y_A, respectively. All of the properties presented below are analogues of [16, Section 3].

Let x_A, x_B and x_C be defined by

$$x_A = \left(\frac{\eta_2\eta_3\eta_{12}}{\eta_1\eta_4\eta_6}\right)^4, \quad x_B = \left(\frac{\eta_1\eta_{12}}{\eta_3\eta_4}\right)^4, \quad x_C = \left(\frac{\eta_4\eta_{12}}{\eta_1\eta_3}\right)^2. \tag{5.1}$$

By Theorem 4.2, the following representations in terms of h hold:

$$x_A = \frac{h(1+h^2)}{(1-4h+h^2)(1-h+h^2)},$$
$$x_B = \frac{h(1-4h+h^2)}{(1-h+h^2)(1+h^2)},$$
$$x_C = \frac{h(1-h+h^2)}{(1+h^2)(1-4h+h^2)}.$$

It follows that x_A, x_B and x_C are algebraically related, and in fact

$$\frac{x_B}{(1+x_B)^2} = \frac{x_C}{(1+4x_C)^2}, \quad \frac{x_A}{(1-3x_A)^2} = \frac{x_C}{(1-4x_C)^2}$$

and

$$\frac{x_A}{(1+3x_A)^2} = \frac{x_B}{(1-x_B)^2}.$$

The above relations prompt us to define y_A, y_B and y_C by

$$y_A = \frac{x_B}{(1+x_B)^2} = \frac{x_C}{(1+4x_C)^2}, \tag{5.2}$$

$$y_B = \frac{x_A}{(1-3x_A)^2} = \frac{x_C}{(1-4x_C)^2}, \tag{5.3}$$

$$y_C = \frac{x_A}{(1+3x_A)^2} = \frac{x_B}{(1-x_B)^2}. \tag{5.4}$$

In terms of h, we have

$$y_A = \frac{h(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-h^2)^4}, \tag{5.5}$$

$$y_B = \frac{h(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-8h+6h^2-8h^3+h^4)^2}, \tag{5.6}$$

$$y_C = \frac{h(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-2h+6h^2-2h^3+h^4)^2}. \tag{5.7}$$

From (5.2)–(5.4),

$$\frac{1}{y_A} - \frac{1}{y_B} = 16, \quad \frac{1}{y_C} - \frac{1}{y_B} = 12 \quad \text{and} \quad \frac{1}{y_A} - \frac{1}{y_C} = 4. \tag{5.8}$$

Let z_A, z_B and z_C be defined by

$$z_A = q\frac{d}{dq}\log x_A, \quad z_B = q\frac{d}{dq}\log x_B, \quad z_C = q\frac{d}{dq}\log x_C. \tag{5.9}$$

By direct calculation using (5.1),

$$z_A = \tfrac{1}{6}(-P_1 + 2P_2 + 3P_3 - 4P_4 - 6P_6 + 12P_{12}),$$
$$z_B = \tfrac{1}{6}(P_1 - 3P_3 - 4P_4 + 12P_{12}),$$
$$z_C = \tfrac{1}{12}(-P_1 - 3P_3 + 4P_4 + 12P_{12}).$$

On applying Theorem 4.5, it follows that

$$z_A = z\frac{(1-h^2)^3}{(1+h^2)(1-h+h^2)(1-4h+h^2)}, \tag{5.10}$$

$$z_B = z\frac{(1-h^2)(1-8h+6h^2-8h^3+h^4)}{(1+h^2)(1-h+h^2)(1-4h+h^2)}, \tag{5.11}$$

$$z_C = z\frac{(1-h^2)(1-2h+6h^2-2h^3+h^4)}{(1+h^2)(1-h+h^2)(1-4h+h^2)}. \tag{5.12}$$

On combining (5.5)–(5.7) with (5.10)–(5.12) we immediately deduce that

$$z_A y_A^{1/2} = z_B y_B^{1/2} = z_C y_C^{1/2}$$
$$= z\frac{h^{1/2}(1-h^2)}{(1+h^2)^{1/2}(1-h+h^2)^{1/2}(1-4h+h^2)^{1/2}}.$$

The next result provides differentiation formulas for y_A, y_B and y_C.

THEOREM 5.1. *The following formulas hold:*

$$q\frac{d}{dq}\log y_A = z_A\sqrt{1-20y_A+64y_A^2},$$
$$q\frac{d}{dq}\log y_B = z_B\sqrt{1+28y_B+192y_B^2},$$
$$q\frac{d}{dq}\log y_C = z_C\sqrt{1-8y_C-48y_C^2}.$$

PROOF. From (5.5)–(5.7) and (5.8), we may deduce that

$$1-4y_A = \frac{y_A}{y_C} = \frac{(1-2h+6h^2-2h^3+h^4)^2}{(1-h^2)^4} \tag{5.13}$$

and

$$1-16y_A = \frac{y_A}{y_B} = \frac{(1-8h+6h^2-8h^3+h^4)^2}{(1-h^2)^4}. \tag{5.14}$$

On the other hand, by (2.2) and (5.5) and a calculation using the chain rule, we find that

$$\begin{aligned} q\frac{d}{dq}\log y_A &= \Big(q\frac{d}{dq}\log h\Big)\times h\frac{d}{dh}\log\Big(\frac{h(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-h^2)^4}\Big) \\ &= z\frac{(1-2h+6h^2-2h^3+h^4)(1-8h+6h^2-8h^3+h^4)}{(1+h^2)(1-h^2)(1-h+h^2)(1-4h+h^2)}. \end{aligned} \tag{5.15}$$

On using (5.10), (5.13) and (5.14) in (5.15) we obtain the first result. The other two results may be proved in a similar way. □

The next result shows that y_A and z_A have beautiful representations in terms of eta products. The other functions y_B, y_C, z_B and z_C do not have simple expressions as eta products.

THEOREM 5.2. *Let y_A and z_A be defined by (5.2) and (5.9); or equivalently, be given by (5.5) and (5.10). The following identities hold:*

$$y_A = \frac{\eta_1^6\eta_3^6\eta_4^6\eta_{12}^6}{\eta_2^{12}\eta_6^{12}} \quad \textit{and} \quad z_A = \frac{\eta_2^{10}\eta_6^{10}}{\eta_1^4\eta_3^4\eta_4^4\eta_{12}^4}. \tag{5.16}$$

PROOF. Start with the parameterizations for y_A and z_A in terms of h and z given by (5.5) and (5.10), then apply Theorems 3.2 and 4.1 to obtain the representations in terms of eta functions. □

From now on we shall focus attention on the functions y_A and z_A. Similar results can be obtained for the other functions y_B, y_C, z_B and z_C but the results are more complicated, so we will not record them.

The next goal will be to show that z_A satisfies a third-order linear differential equation with respect to y_A. Let D be the differential operator defined by

$$D = \frac{1}{z_A} q \frac{d}{dq}. \tag{5.17}$$

By Theorem 5.1 we also have

$$D = y_A \sqrt{1 - 20y_A + 64y_A^2}\, \frac{d}{dy_A}.$$

Theorem 5.3. *Let y_A and z_A be defined by (5.2) and (5.9). Then z_A satisfies the second-order nonlinear differential equation with respect to y_A given by*

$$D^2 z_A - \frac{1}{2z_A}(Dz_A)^2 = 4z_A y_A(1 - 8y_A). \tag{5.18}$$

Moreover, the following third-order linear differential equation holds:

$$D^3 z_A - 8y_A(1 - 8y_A)Dz_A = 4z_A(1 - 16y_A)Dy_A. \tag{5.19}$$

Proof. By a direct calculation using the eta product for z_A in (5.16),

$$q\frac{d}{dq} \log z_A = \frac{1}{6}(-P_1 + 5P_2 - 3P_3 - 4P_4 + 15P_6 - 12P_{12}). \tag{5.20}$$

Applying the operator $q(d/dq)$ and using the differentiation formula (2.7) gives

$$\begin{aligned} q\frac{d}{dq}\Big(q\frac{d}{dq}\log z_A\Big) &= \frac{1}{72}(-P_1^2 + 10P_2^2 - 9P_3^2 - 16P_4^2 + 90P_6^2 - 144P_{12}^2) \\ &\quad + \frac{1}{72}(Q_1 - 10Q_2 + 9Q_3 + 16Q_4 - 90Q_6 + 144Q_{12}). \end{aligned} \tag{5.21}$$

It is known, for example, [13, Lemma 4.5], that

$$9Q_{3\ell} + Q_\ell = \tfrac{5}{2}(3P_{3\ell} - P_\ell)^2.$$

Using this to eliminate $Q_1, \ldots, Q_{12}$ from (5.21), and then simplifying, we obtain

$$\begin{aligned} q\frac{d}{dq}\Big(q\frac{d}{dq}\log z_A\Big) &= \frac{1}{48}(P_3 - P_1)(9P_3 - P_1) - \frac{5}{24}(P_6 - P_2)(9P_6 - P_2) \\ &\quad + \frac{1}{3}(P_{12} - P_4)(9P_{12} - P_4). \end{aligned} \tag{5.22}$$

Hence, by (5.17), (5.20) and (5.22),

$$\begin{aligned} &z_A \times \Big(D^2 z_A - \frac{1}{2z_A}(Dz_A)^2\Big) \\ &\quad = \frac{1}{48}(P_3 - P_1)(9P_3 - P_1) - \frac{5}{24}(P_6 - P_2)(9P_6 - P_2) + \frac{1}{3}(P_{12} - P_4)(9P_{12} - P_4) \\ &\qquad - \frac{1}{72}(-P_1 + 5P_2 - 3P_3 - 4P_4 + 15P_6 - 12P_{12})^2. \end{aligned}$$

By Theorem 4.5, the right-hand side can be expressed in terms of z, h and hdz/dh. On simplification, all of the terms that involve hdz/dh cancel. On comparing the remaining terms in the simplified result with the parameterizations in (5.5) and (5.10), we complete the proof of (5.18).

Now we will prove (5.19). Applying the differential operator D to both sides of (5.18) and using the basic rules of calculus, we get

$$D^3 z_A + \frac{Dz_A}{z_A}\left\{\frac{(Dz_A)^2}{2z_A} - D^2 z_A\right\} = 4y_A(1-8y_A)Dz_A + 4z_A(1-16y_A)Dy_A.$$

The differential equation (5.18) may be used to simplify the term in braces, and identity (5.19) follows on further simplification. □

THEOREM 5.4. *The following series expansion holds:*

$$z_A = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}\right\}y_A^n.$$

PROOF. The differential equation (5.19) may be written in the form

$$\begin{aligned} y_A^2(1-4y_A)(1-16y_A)\frac{d^3 z_A}{dy_A^3} + 3y_A(1-30y_A+128y_A^2)\frac{d^2 z_A}{dy_A^2} \\ + (1-68y_A+448y_A^2)\frac{dz_A}{dy_A} - 4(1-16y_A)z_A = 0. \end{aligned}$$

On writing

$$z_A = \sum_{n=0}^{\infty} t(n) y_A^n,$$

we deduce that the coefficients $t(n)$ satisfy the recurrence relation

$$(n+1)^3 t(n+1) = 2(2n+1)(5n^2+5n+2)t(n) - 64n^3 t(n-1)$$

and initial conditions

$$t(0) = 1, \quad t(1) = 4.$$

The method of creative telescoping may be used to show that sequence whose nth term is given by

$$\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}$$

satisfies the same recurrence relation and initial conditions. It follows that

$$t(n) = \sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}.$$

This completes the proof. □

The result of Theorem 5.4 was first given, with $-q$ in place of q, in [9, (4.10), (4.13)]. The numbers $t(n)$ are called the Domb numbers. They have numerous applications and have been extensively studied; see, for example, [3, 5, 8, 10, 11, 19, 24].

By Theorem 5.2, together with the elementary identity

$$\prod_{j=1}^{\infty}(1-(-q)^j) = \prod_{j=1}^{\infty} \frac{(1-q^{2j})^3}{(1-q^j)(1-q^{4j})}, \tag{5.23}$$

we have

$$-y_A(-q) = \frac{\eta_2^6\eta_6^6}{\eta_1^6\eta_3^6} \quad \text{and} \quad z_A(-q) = \frac{\eta_1^4\eta_3^4}{\eta_2^2\eta_6^2}.$$

The functions $-y_A(-q)$ and $z_A(-q)$ were denoted by X and Z, respectively, in [9, Section 4].

Identity (1.2) in the introduction is the restatement of Theorem 5.4 in which (5.2) and (5.9) are used to express y_A and z_A in terms of h and z.

It is also instructive to compare the result of Theorem 5.4 with the identity

$$\left(\frac{2K}{\pi}\right)^2 = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 4k^2(1-k^2)\right) \tag{5.24}$$

where k and K denote the modulus[1] and complete elliptic integral of the first kind, respectively. The standard results from elliptic functions, for example, [7, pages 119, 120] or [27, (3.69), (3.79)], can be used to rewrite (5.24) in the form given by

$$q\frac{d}{dq}\log\left(\frac{w}{1-16w}\right) = \sum_{n=0}^{\infty}\binom{2n}{n}^3 (w(1-16w))^n \tag{5.25}$$

where $w = k^2/16$, which is an analogue of (1.2) and (1.3). Identity (5.24) may also be expressed in the form

$$\varphi^4(q) = \sum_{n=0}^{\infty}\binom{2n}{n}^3\left(\frac{\eta_1^4\eta_4^4}{\eta_2^4} \times \frac{1}{\varphi^4(q)}\right)^{2n}. \tag{5.26}$$

In a similar way, the result of Theorem 5.4 may be presented in the form

$$\varphi^2(q)\varphi^2(q^3) = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}\right\}\left(\frac{\eta_1^2\eta_3^2\eta_4^2\eta_{12}^2}{\eta_2^2\eta_6^2} \times \frac{1}{\varphi^2(q)\varphi^2(q^3)}\right)^n,$$

which is clearly an analogue of (5.26).

The next result gives two expressions for z in terms of h in terms of the hypergeometric function ${}_3F_2$. It also gives a cubic hypergeometric transformation formula.

[1] The modulus k in (5.24) is different from Ramanujan's function $k = k(q)$ defined in the introduction.

THEOREM 5.5. *Let u and v be defined by*

$$u = \frac{h(1+h^2)}{(1-h)^4} \quad \text{and} \quad v = \frac{(1-h+h^2)(1-4h+h^2)}{(1+h)^4}.$$

Then

$$\begin{aligned} z &= \frac{(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-h)(1+h)^5}\, {}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 16uv^3\Big) \\ &= \frac{(1+h^2)(1-h+h^2)(1-4h+h^2)}{(1-h)^5(1+h)}\, {}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; 16u^3v\Big). \end{aligned}$$

PROOF. By the formulas in [7, Ch. 5] or [13],

$$\frac{1}{3}(4P(q^4) - P(q)) = \varphi^4(q) = \Big({}_2F_1\Big(\frac{1}{2}, \frac{1}{2}; 1; \frac{16\eta_1^8\eta_4^{16}}{\eta_2^{24}}\Big)\Big)^2.$$

Applying Clausen's formula [4, page 116, Example 13 and page 125 (3.13)]

$${}_2F_1(2a, 2b; a+b+\tfrac{1}{2}; h) = {}_3F_2(2a, 2b, a+b; 2a+2b, a+b+\tfrac{1}{2}; 4h(1-h)), \tag{5.27}$$

we get[1]

$$\frac{1}{3}(4P(q^4) - P(q)) = {}_3F_2\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; \frac{64\eta_1^8\eta_4^{16}}{\eta_2^{24}}\Big(1 - \frac{16\eta_1^8\eta_4^{16}}{\eta_2^{24}}\Big)\Big). \tag{5.28}$$

The results of Theorems 4.2 and 4.5 can now be used to express the left-hand side and the eta functions on the right-hand side in terms of z and h. The first result in the theorem follows by a straightforward, but tedious, calculation. The second result in the theorem can be obtained in a similar way be first replacing q with q^3 in (5.28) and then using Theorems 4.2 and 4.5 to obtain the required expressions in terms of z and h. □

REMARK 5.6. Identity (5.28) is equivalent to each of the identities (5.24), (5.25) and (5.26).

The next result gives two expressions for z_A in terms of y_A in terms of the hypergeometric function ${}_3F_2$, along with a quadratic transformation formula.

THEOREM 5.7. *Let y_A and z_A be defined by (5.2) and (5.9), or equivalently by the eta products in Theorem 5.2. Then*

$$\begin{aligned} z_A &= \frac{1}{1-16y_A}\, {}_3F_2\Big(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1, 1; \frac{-108y_A}{(1-16y_A)^3}\Big) \\ &= \frac{1}{1-4y_A}\, {}_3F_2\Big(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1, 1; \frac{108y_A^2}{(1-4y_A)^3}\Big). \end{aligned}$$

[1] By a result of Jacobi, it is known that $1 - (16\eta_1^8\eta_4^{16})/\eta_2^{24} = (\eta_1^{16}\eta_4^8)/\eta_2^{24}$, but this will not be required.

Proof. By the identities in [13],

$$\frac{1}{2}(3P(q^3)-P(q)) = \left({}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{27\eta_3^{12}}{\eta_1^{12}+27\eta_3^{12}}\right)\right)^2.$$

Clausen's formula (5.27) may be applied to the right-hand side to give

$$\frac{1}{2}(3P(q^3)-P(q)) = {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;\frac{108\eta_1^{12}\eta_3^{12}}{(\eta_1^{12}+27\eta_3^{12})^2}\right). \tag{5.29}$$

Replace q with $-q$ in (5.29) and make use of identity (5.23) and its logarithmic derivative

$$P(-q) = -P(q)+6P(q^2)-4P(q^4)$$

to get

$$\begin{aligned}\frac{1}{2}&(P_1-6P_2-3P_3+4P_4+18P_6-12P_{12})\\ &= {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;\frac{-108\eta_1^{12}\eta_2^{36}\eta_3^{12}\eta_4^{12}\eta_6^{36}\eta_{12}^{12}}{(\eta_2^{36}\eta_3^{12}\eta_{12}^{12}-27\eta_1^{12}\eta_4^{12}\eta_6^{36})^2}\right).\end{aligned} \tag{5.30}$$

By Theorem 4.5, together with (5.10) and (5.14),

$$\begin{aligned}\frac{1}{2}&(P_1-6P_2-3P_3+4P_4+18P_6-12P_{12})\\ &= z\times\frac{(1-8h+6h^2-8h^3+h^4)^2}{(1-h^2)(1+h^2)(1-h+h^2)(1-4h+h^2)}\\ &= z_A\times(1-16y_A).\end{aligned} \tag{5.31}$$

On the other hand, by Theorem 4.2 together with (5.10) and (5.14) we have

$$\begin{aligned}&\frac{-108\eta_1^{12}\eta_2^{36}\eta_3^{12}\eta_4^{12}\eta_6^{36}\eta_{12}^{12}}{(\eta_2^{36}\eta_3^{12}\eta_{12}^{12}-27\eta_1^{12}\eta_4^{12}\eta_6^{36})^2}\\ &\qquad= \frac{-108h(1+h^2)(1-h+h^2)(1-4h+h^2)(1-h^2)^8}{(1-8h+6h^2-8h^3+h^4)^6}\\ &\qquad= \frac{-108y_A}{(1-16y_A)^3}.\end{aligned} \tag{5.32}$$

On substituting (5.31) and (5.32) into (5.30), we obtain the first result.

The second result can be obtained in a similar way, by replacing q with q^2 in (5.29) to get

$$\frac{1}{2}(3P(q^6)-P(q^2)) = {}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;\frac{108\eta_2^{12}\eta_6^{12}}{(\eta_2^{12}+27\eta_6^{12})^2}\right),$$

and then using Theorems 4.2 and 4.5, together with (5.10) and (5.13), to express each side in terms of y_A and z_A. We omit the details of the calculation. □

COROLLARY 5.8 [25]. *In a neighborhood of $y = 0$, the following identities hold:*

$$\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-2j}{n-j}\binom{2j}{j}\right\}y^n = \frac{1}{1-16y}\,{}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;\frac{-108y}{(1-16y)^3}\right)$$
$$= \frac{1}{1-4y}\,{}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;\frac{108y^2}{(1-4y)^3}\right).$$

PROOF. The is immediate from Theorems 5.4 and 5.7. □

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