REPLACEABLE NETS, NET COLLINEATIONS, AND NET EXTENSIONS

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1. Introduction. A *net* of degree k and order n is a set of n^2 points and nk designated sets of points, called lines, such that

(1) The lines fall into k disjoint parallel classes, i.e. each line occurs in exactly one parallel class.

(2) Lines in the same parallel class have no points in common; lines in different parallel classes have exactly one point in common.

(3) Each point lies on exactly one line of each parallel class.

An *affine plane* of order n is a net of degree $n + 1$ and order n. We shall say that a net N is included in another net N^1 if:

(1) The points of N and N^1 are identical.

(2) Each line of N is a line of N^1.

Thus N may be obtained from N^1 by selecting certain parallel classes of lines in N^1. If N is included in N^1, we shall say that N^1 is an extension of N.

It is well known that a net of degree k is equivalent to a set of $k - 2$ mutually orthogonal Latin squares. Bose **(2)** has pointed out the relation between nets and "Partial Geometries" that are useful in statistical designs. (This is in addition to the more familiar use of Latin squares in statistical designs.) Thus, nets are of considerable interest in their own right. Moreover, knowledge about a net included in a plane gives "local" information about the plane itself. This is particularly the case in situations where the net is what we call "replaceable" (see below).

Two points of a net will lie on at most one line of the net. If two points are on a line of N, they will be said to be joined in N. In an affine plane, every pair of points is joined, but a net of order n whose degree is less than $n + 1$ will contain certain pairs of points that are not joined.

A net N will be said to be *replaceable* if there exists a net N^1 such that:

(1) The points of N and N^1 are identical.

(2) Two points are joined in N if and only if they are joined in N^1.

(3) At least one line of N is not a line of N^1.

In 1954, André **(1)** gave a method for constructing a class of affine planes that are translation planes but are non-Desarguesian. An outline of André's construction follows.

Let K be a field admitting a non-trivial group G of automorphisms. Let F be the subfield of K consisting of the elements of K left fixed by all elements of

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G. For each $m \in K$ define a norm $\mathfrak{N}(m)$, where $\mathfrak{N}(m)$ is the product of all the images of m under the automorphisms in G.

The non-zero elements of K may now be put into equivalence classes by calling a and b equivalent if they have the same norm. Next, assign some element of G to each equivalence class. (Different equivalence classes may or may not be assigned the same element of G. André makes the restriction that the equivalence class containing 1 shall be assigned the identity automorphism.) We may now define an operation "$\cdot$" such that $x \cdot m = x^\sigma m$, where σ is the element of G assigned to the equivalence class containing m.

The points of the André affine plane π^1 consist of ordered pairs (x, y), $x, y \in K$. For each m and b, there is a line consisting of all points (x, y) for which $y = x \cdot m + b$. For each c, there is a line consisting of all points for which $x = c$.

From the point of view of nets, we may start out by considering the Desarguesian plane π co-ordinatized by K. That is, the points of π are ordered pairs (x, y) of elements of K while the lines are defined by linear equations in K. Each equivalence class determines a net consisting of all lines of the Desarguesian plane whose slopes belong to the equivalence class in question. If N is one of these nets, then N is replaceable. The lines of the net N^1 that replace N are defined by the equations $y = x^\sigma m + b$, where m may take on any value in the equivalence class defining N and σ is an element of G.

This suggests a general procedure for constructing new planes. That is, if an affine plane contains a replaceable net N, we can obtain a new plane if we replace the lines of N by the "replacing net" N^1. The new plane might possibly be isomorphic to the original one. This is not in general the case for the André construction. (But, up to an isomorphism, there is only one plane of order 4 or 8.) The author **(4)** has been able to construct a fairly wide class of planes of square order by a procedure that is a special case of the one under discussion.

This suggests two general questions about nets:

(1) Under what circumstances will a net have an extension to a net of larger degree?

(2) Under what circumstances will a net N have extensions N^1 and N^2 such that at least one line of N^1 is not a line of N^2 and at least one line of N^2 is not a line of N^1?

This paper is concerned with a class of nets N having the property that if N has a transversal T, then N has an extension N^1 such that T is one of the lines of N^1. If N has a transversal T^1 which is not a line of N^1, then either (1) T^1 is a transversal of N^1 and N^1 can be extended to a net having T^1 as one of its lines or (2) T^1 is not a transversal of N^1 and N can be extended in more than one way in the sense of question (2) above.

2. The André nets. First let us investigate the André construction and the associated nets in more detail. If $K = \mathrm{GF}(q^r)$ and $F = \mathrm{GF}(q)$, then G

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is generated by the mappings $x \to x^q$. The norm of x is x raised to the power $(1 + q + \ldots + q^{r-1})$. Thus the equivalence class with norm 1 consists of the $(1 + q + \ldots + q^{r-1})$st roots of unity. These roots of unity constitute a subgroup of the multiplicative group in K. Since $\mathfrak{N}(ab) = \mathfrak{N}(a)\mathfrak{N}(b)$, the other equivalence classes are cosets of the multiplicative subgroup mentioned above. Hence the associated nets can be carried into each other by collineations of the Desarguesian plane given by mappings of the form $(x, y) \to (x, ya)$. (These collineations carry lines of slope m into lines of slope ma.) Thus the nets are isomorphic; it will suffice to study the net corresponding to norm 1.

THEOREM 1. *Let $K = \mathrm{GF}(q^r)$ and let π be the affine plane co-ordinatized by K. Let N be the net included in π consisting of all lines whose slopes are $(q^r - 1)/(q - 1)$st roots of unity. Then N is replaceable.*

Proof. Let ρ denote the mapping $x \to x^q$, and let σ be any power of ρ that is different from the identity. Let $\mathfrak{M}$ denote the set of $(q^r - 1)/(q - 1)$st roots of unity in K. Let N^1 be the system of new "lines" defined on the points of π as follows: for each m in $\mathfrak{M}$ and each b in K, the set of points (x, y) for which $y = x^\sigma m + b$ is a line of N^1.

We must show that N^1 is a net and then we must show that two points are joined in N^1 if and only if they are joined in N.

It is elementary to verify that two lines of N^1 given by $y = x^\sigma m + b$ and $y = x^\sigma r + c$ are parallel (have no points in common) if $r = m$ and have exactly one point in common if $r \neq m$. Given (c, d) and $m \in \mathfrak{M}$, the line $y = x^\sigma m + b$ will contain the point (c, d) if and only if $b = d - c^\sigma m$. Thus, each point lies on exactly one line of each parallel class of N^1. It follows that N^1 is indeed a net.

Now two points (a, b) and (c, d) are joined by a line of N if and only if $(b - d)(a - c)^{-1}$ belongs to $\mathfrak{M}$; they are joined by a line of N^1 if and only if $(b - d)(a - c)^{-\sigma}$ belongs to $\mathfrak{M}$. To show that one of these quantities is in $\mathfrak{M}$ if and only if the other one is in $\mathfrak{M}$, we only need show that $(a - c)^{\sigma - 1}$ must be in $\mathfrak{M}$. Since σ may be thought of as representing a power of q, $(a - c)^{\sigma-1}$ is indeed a $(q^r - 1)/(q - 1)$st root of unity, i.e. $(c - d)^{\sigma - 1}$ belongs to $\mathfrak{M}$.

Definition. A collineation of a net N that fixes no points of N will be called a *translation* of N. A translation of N that fixes all lines of some parallel class will be called a *strict translation*. The identity will also be considered a strict translation. (Note that the product of two strict translations is not necessarily a strict translation.)

THEOREM 2. *The net N of Theorem 1 admits the following groups of strict translations. Each of these groups is of order $n = q^r$.*

(1) *For each fixed $m \in \mathfrak{M}$ the set of all mappings of the form*

$$(x, y) \to (x + a, y + am),$$

where a takes on all values in K.

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(2) *For each* σ (defined in the proof of Theorem 1), *the set of mappings of the form* $(x, y) \to (x + a, y + a^{\sigma})$.

Proof. These mappings are all translations of the Desarguesian plane π and induce translations of N. It is fairly immediate that the mappings (1) form a group for each fixed m and that the mappings (2) form a group for each fixed σ. The mappings (1) fix all lines of slope m. For each a, the mapping (2) fixes the lines of slope $a^{\sigma-1}$.

THEOREM 3. *Let the net N of Theorem* 1 *be extended by including the two parallel classes of π that include the lines $x =$* constant *and $y =$* constant. *The extended net admits a group of collineations that* (1) *fixes all lines $x =$* constant, (2) *fixes all points on $y = 0$, and* (3) *is transitive on the parallel classes of the original net N.*

Proof. The collineations in question arise out of the collineations of the plane π which may be represented in the form $(x, y) \to (x \ \ ya), a \in \mathfrak{M}$.

3. Net extensions. If N is any net of order n, a *transversal* of N is a set of n points, no two of which are joined by a line of N. If N is included in an extension N^1, the lines of N^1 that are not lines of N are transversals of N.

LEMMA 1. *Let ρ be any collineation of a net N that fixes all lines of some parallel class. If T is a transversal of N and if P is a point belonging to $T \cap T\rho$, then $P = P\rho$.*

Proof. If Q is any point such that $Q \neq Q\rho$, then Q must be on the fixed line containing Q. Suppose that $P \in (T \cap T\rho)$, then $P = Q\rho$ for some point Q in T. We must either have $Q = P$ or we have two points, Q and $Q\rho = P$, in T which are joined by a line of N. Since no two points of a transversal are joined by a line of N, we must have $Q = P$, i.e., $P = P\rho$.

THEOREM 4. *Let N be a net of order n. Suppose that N admits a group H of strict translations, where H is of order n. Then, if N has a transversal T, N is included in a net N^1 that admits H and has T for one of its lines.*

Proof. It follows from Lemma 1 and the definition of a strict translation that the n images of T under H are disjoint. The images of T are transversals; they can be taken as the lines of a new parallel class in an extension N^1 of N.

COROLLARY. *Under the hypotheses of Theorem* 4, *N is included in a net that* (1) *admits H as a group of collineations*, (2) *has T for one of its lines, and* (3) *has no transversals.*

Proof. If the net N^1 of Theorem 4 has any transversals, then N^1 satisfies the hypotheses of Theorem 4. Thus we can continue to make extensions until we reach a net with no transversals.

COROLLARY. *Let T be any transversal of the net N of Theorem 1. Then N can be extended to a net with no transversals that has T for one of its lines.*

LEMMA 2. *Let N be a net with at least four parallel classes. Suppose that N admits a collineation σ that fixes all lines of some parallel class and all points on some line. Then σ fixes no other points unless σ is the identity.*

Proof. Let Z denote the fixed parallel class. If any line L not in Z is fixed, then each point of L is the intersection of two fixed lines. Hence L must be pointwise fixed. If L is any line (possibly in Z) that is pointwise fixed and if some point P not on L is fixed, then every line through P is fixed. In this case, every parallel class other than Z contains at least one line that is pointwise fixed. Let W be a parallel class different from Z. With at most one possible exception, every line of W must intersect at least two lines that are pointwise fixed. It follows that every line of W must be fixed, and hence pointwise fixed. Hence all points of N are fixed and σ must be the identity.

LEMMA 3. *Let ρ_1 and ρ_2 be two strict translations that leave the same parallel class Z linewise fixed. Then, if N contains at least four parallel classes, $\rho_1 \rho_2$ is a strict translation leaving Z linewise fixed.*

Proof. A net is usually understood to contain at least three parallel classes. However, the argument of Lemma 1 still holds for a net consisting of a single parallel class. Thus we may consider the lines of N that are not in Z to be transversals of the trivial net whose sole parallel class is Z. By Lemma 1, both ρ_1 and ρ_2 must fix all parallel classes of N (i.e., each line is carried into a parallel line). Thus $\rho_1 \rho_2$ must fix all parallel classes of N and must fix Z linewise. If $\rho_1 \rho_2$ should fix some point, then every parallel class would contain a fixed line. The parallel classes distinct from Z will each contain at least one line that is pointwise fixed. Applying Lemma 2, either $\rho_1 \rho_2$ fixes no points or is the identity.

THEOREM 5. *Let N be a net of order n whose degree is at least four. Suppose that N admits a group H of collineations such that:*

(1) *There is a parallel class Z in N such that all lines of Z are fixed by every member of H.*

(2) *H contains a subgroup of n strict translations.*

(3) *If an element ρ of H has any fixed points, then ρ fixes all points on some line of N.*

Then, if N has a transversal T, N can be extended to a net N^1 such that (a) *N^1 admits the group H as a group of collineations,* (b) *T is one of the lines of N^1,* (c) *N^1 has no transversals.*

Proof. Suppose that N has a transversal T. The images of T under H are also transversals. Consider the intersection of T with $T\rho$, where ρ is a non-identity element of H. If ρ has no fixed points, then T and $T\rho$ are parallel by Lemma 1. By Lemma 2 and condition (3), if ρ has any fixed points, they all lie on some line of N. Since T is a transversal, T cannot contain two fixed points. Hence $T \cap T\rho$ contains exactly one point in this case. (The n points of a transversal must each lie on a separate line of any given parallel class so that each line of N intersects T exactly once.)

The same argument applies to any two images of T, so that any pair of images of T will have at most one point in common.

The n translations assumed in condition (2) will be transitive on the points of any one line of Z. If we have two translations ρ_1 and ρ_2, both of which carry a point A into a point B, then $\rho_1 \rho_2^{-1}$ will fix A. By Lemma 3, $\rho_1 \rho_2^{-1}$ is the identity and $\rho_1 = \rho_2$. Thus H must contain exactly n translations.

Hence T has exactly n images (counting itself) that are parallel to T. More generally, the images of T fall into disjoint parallel classes with n transversals in each parallel class. Since each transversal contains n points, the n^2 points of N each lie on exactly one transversal of each parallel class.

Thus we may extend N to a larger net N^1 by taking T and its images, together with the lines of N, as the lines of N^1.

From the method of construction, N^1 still admits H as a group of collineations and the hypotheses of Theorem 5 still apply. Hence the process can be repeated until we have extended N to a net with no transversals.

COROLLARY. *Let N be the net of Theorem* 3. *Let H be the group of collineations given by mappings of the form* $(x, y) \rightarrow (x, ya + b)$, $a \in \mathfrak{M}$. *Let T be any transversal of N. Then N can be extended to a net that* (1) *admits H,* (2) *has T for one of its lines, and* (3) *has no transversals.*

Remark. At the risk of being obvious, perhaps we should point out some differences between Theorems 4 and 5. In Theorem 4 we add only one parallel class at a time; in Theorem 5 we add several parallel classes at a time. Theorem 4 fails to be a special case of Theorem 5 only in that Theorem 4 does not require the same parallel class to be linewise fixed by all elements of the strict translation group.

4. Direct products of nets. The most interesting aspect of Theorems 4 and 5 is undoubtedly what they tell us about the nets of Theorem 1. Yet it may be worth while to point out the existence of more extensive classes of nets satisfying the conditions of these theorems.

Bruck **(3)** has defined a direct product of nets by a procedure equivalent to the following.

Let N_1 and N_2 be two nets of the same degree, but not necessarily of the same order. Let τ denote a 1–1 correspondence between the parallel classes of N_1 and N_2. Specifically, if m denotes a parallel class in N_1, let $m\tau$ denote the corresponding parallel class in N_2. Let $N_1 \times N_2$ denote the system of "points" and "lines" as indicated:

1. The points of $N_1 \times N_2$ are the ordered pairs (p_1, p_2) where p_1 is a point of N_1 and p_2 is a point of N_2.

2. Let L_1 be any line of N_1 belonging to the parallel class m. Let L_2 be any line of N_2 belonging to the parallel class $m\tau$. For each such pair of lines (L_1, L_2), the set of points (p_1, p_2) such that p_1 is on L_1 and p_2 is on L_2 is a line of $N_1 \times N_2$.

THEOREM 6. *Let ρ_1 be a collineation of N_1 and let ρ_2 be a collineation of N_2. Suppose that, for each parallel class m of N_1, $m\rho_1 \tau = m\tau\rho_2$. Then the mapping $(p_1, p_2) \rightarrow (p_1 \rho_1, p_2 \rho_2)$ is a collineation of $N_1 \times N_2$.*

Proof. Note that if L_1 belongs to the parallel class m and L_2 belongs to the parallel class $m\tau$, then $L_1 \rho_1$ belongs to $m\rho_1$ while $L_2 \rho_2$ belongs to $m\tau\rho_2 = m\rho_1 \tau$. Hence, if (L_1, L_2) defines a line of $N_1 \times N_2$, $(L_1 \rho_1, L_2 \rho_2)$ also defines a line. This is sufficient to establish that the mapping is a collineation.

Remarks. The easiest applications of Theorem 6 would be to the cases where $N_1 = N_2$ and τ is the identity. Thus if N is a net satisfying the hypotheses of Theorem 5, $N \times N$ can be defined so as to satisfy these same hypotheses. If N is a Desarguesian affine plane and T is any transversal of $N \times N$, then we can apply Theorem 5 to get a plane. (This particular case is investigated from another point of view in a forthcoming paper of the author.) In a net that is sufficiently small so that all of its transversals can be found by a computer, it would appear possible to find all possible extensions of the types given by Theorems 4 and 5. It would be interesting to see whether any new planes can be obtained in this manner.

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