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SUMS AND PRODUCTS OF NORMAL FUNCTIONS

DAVID W. BASH

1. Let D be the unit disk $\{z: |z| < 1\}$ in the complex plane. Let $\rho(z, z')$ denote the hyperbolic distance between z and z' in $D(\rho(z, z') = \frac{1}{2}\log((1 + u)/(1 - u)) = \tanh^{-1} u$, $u = |z_1 - z_2|/|1 - \bar{z}_1 z_2|$ [**6**, chapter 15]). Let W be the Riemann sphere with the chordal metric. A complex valued function $f(z)$ in D is a *normal function* if for each pair of sequences $\{z_n\}$ and $\{z_n'\}$ of points in D such that $\rho(z_n, z_n') \to 0$, the convergence of $\{f(z_n)\}$ to a value α in W implies the convergence of $\{f(z_n')\}$ to α. Two sequences $\{z_n\}$ and $\{z_n'\}$ of points in D are called *close sequences* if $\rho(z_n, z_n') \to 0$. (There are several equivalent definitions of normality if the functions are meromorphic.) The definition of a normal function implies that a normal function is continuous at each point of D when using the Euclidean metric in the domain and the chordal metric in the range.

We wish to study the sums and products of normal functions. Some functions, such as a function in a Hardy p-class, $p > 0$, (or really any function of bounded characteristic) can be written as a sum or product of two normal functions, but sums and products of normal functions need not be normal (see Lappan [**7**]). Hence it is desirable to know when the sum or product of two or more normal functions is normal. In this paper we prove a necessary and sufficient condition for the sum and for the product of two normal functions to be normal. Then several sufficient conditions for the sum of normal functions to be normal are proved which involve looking at the arguments of the individual functions. An example illustrates the usefullness of the method. Then the product theorem is used to obtain several of the basic normal function results for pseudo-analytic functions of the first kind that we have for holomorphic functions.

In the sequel we will require the following:

Definition. (Lappan [**8**, p. 44].) The meromorphic function $f(z)$ in D is said to have *property* (D) on a sequence $\{z_n\}$ of points in D if $\{f(z_n)\}$ converges and for each complex number δ, there exists a sequence $\{z_n'\}$ close to $\{z_n\}$ such that $f(z_n') \to \delta$.

THEOREM. (Lappan [**8**, p. 45].) *The meromorphic function $f(z)$ in D is not normal if and only if there exists a sequence of points in D on which $f(z)$ has property (D).*

2. We start with some theorems on sums and products of normal functions

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whose proofs employ the definition of normalcy through convergence on close sequences.

THEOREM 1. *Let $f(z)$ and $g(z)$ be normal functions in D. Then $f(z) + g(z)$ is normal in D if and only if for each sequence $\{z_n\}$ in D such that $f(z_n) \to \infty$, $g(z_n) \to \infty$, and $\{f(z_n) + g(z_n)\}$ converges to a complex value α (possibly ∞), the sum $\{f(z_n') + g(z_n')\}$ converges to α for each sequence $\{z_n'\}$ close to $\{z_n\}$.*

Proof. The necessity is obvious from the definition of normal functions. For sufficiency, let $\{z_n\}$ be a sequence where $f(z_n) + g(z_n) \to \alpha$. By considering appropriate subsequences, if necessary, we may assume that $f(z_n) \to \beta$ and $g(z_n) \to \gamma$, β and γ complex numbers. Let $\{z_n'\}$ be close to $\{z_n\}$. If $\beta = \gamma = \infty$, then $f(z_n') + g(z_n') \to \alpha$ by the condition of the theorem. Otherwise, $\beta + \gamma$ is a well-defined complex number (possibly ∞) and by the normalcy of $f(z)$ and $g(z)$ we get that $f(z_n') + g(z_n') \to \beta + \gamma$. Hence $f(z) + g(z)$ is normal.

Using Theorem 1 we can prove several sufficient conditions for $f(z) + g(z)$ to be normal in D where $f(z)$ and $g(z)$ are normal in D. The three corollaries say, roughly speaking, that when one function is big and the other function is about the same size, if the function arguments are never quite π apart then the sum is normal.

COROLLARY 1. *If $f(z)$ and $g(z)$ are normal in D and there is a number M, $0 < M < \infty$, such that the sets $\{z: |f(z)| > M\}$ and $\{z: |g(z)| > M\}$ are disjoint, then $f(z) + g(z)$ is normal in D.*

Proof. The hypotheses of the condition of Theorem 1 are satisfied vacuously.

Let $f(z)$ and $g(z)$ be continuous finite-valued functions in D which omit the value zero, and let $Q_n(f) = \{z: n < |f(z)| < 2n\}$, $R_n(f, g) = Q_n(f) \cap \{z: 1 - 1/n^{1/2} < |f(z)/g(z)| < 1 + 1/n^{1/2}\}$, and $T_n(f, g, \mu) = \{z: |\arg f(z) - \arg g(z)| < \pi - \mu(n)\}$ where $\mu(n)$ is a nonincreasing function on the positive integers such that $0 < \mu(n) < \pi$. For each z the arguments of $f(z)$ and $g(z)$ are chosen so as to have the difference, in absolute values, less than or equal to π. Hence the "arguments" in this paper are not continuous functions of z.

COROLLARY 2. *Let $f(z)$ and $g(z)$ be normal finite-valued functions in D, each of which omits the value zero and $\mu(n)$ such that $(n \sin \mu(n))/(2 \cos(\mu(n)/2))$ increases to infinity monotonically as n increases to infinity. If there is a positive integer N such that $n \geqq N$ implies $R_n(f, g)$ is contained in $T_n(f, g, \mu)$, then $f(z) + g(z)$ is a normal function in D.*

Proof. It suffices to show that if $\{z_m\}$ is a sequence of points in D and $f(z_m) \to \infty$, then $f(z_m) + g(z_m) \to \infty$. Let $\epsilon > 0$ be given and let N satisfy the hypotheses, and let $m > P \geqq N$ and both $(P \sin \mu(P))/(2 \cos(\mu(P)/2)) > 1/\epsilon$ and $P(1/(P^{1/2} + 1)) > 1/\epsilon$ be true.

We assume without loss of generality that z_m is in $Q_m(f)$. Either z_m is in R_m or $|f(z_m)/g(z_m)| \leqq 1 - 1/m^{1/2}$, or $|f(z_m)/g(z_m)| \geqq 1 + 1/m^{1/2}$. If z_m is in R_m, then z_m is in $T_m(f, g, \mu)$. Since $|g(z_m)| > (1/2)m$ for $m > 4$, by computing the length

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of the diagonal of the parallelogram of sides m and $(1/2)m$, with interior angles $\mu(m)$ and $\pi - \mu(m)$, and noting that the diagonal is shorter than $f(z_m) + g(z_m)$, we see that

$$|f(z_m) + g(z_m)| > (m \sin \mu(m))/(2 \cos (\mu(m)/2)),$$

which in turn is greater than $(P \sin \mu(P))/(2 \cos (\mu(P)/2))$ or $1/\epsilon$.

If $|f(z_m)/g(z_m)| \leqq 1 - 1/m^{1/2}$, then $|g(z_m)| > (m^{1/2}/(m^{1/2} - 1))|f(z_m)|$. So

$$\begin{aligned}|f(z_m) + g(z_m)| &\geqq |g(z_m)| - |f(z_m)| \\ &\geqq |f(z_m)|((m^{1/2}/(m^{1/2} - 1)) - 1) \\ &> m(1/(m^{1/2} - 1)) \\ &> P(1/(P^{1/2} - 1)) \\ &> P(1/(P^{1/2} + 1)) > 1/\epsilon.\end{aligned}$$

If $|f(z_m)/g(z_m)| \geqq 1 + 1/m^{1/2}$, then $|f(z_m) + g(z_m)| > 1/\epsilon$, similarly (where one uses $P(1/P^{1/2} + 1)) > 1/\epsilon$ more directly).

We note that the $\frac{1}{2}$ in the definition of $R_n(f, g, \mu)$ could be replaced by any number between zero and one to obtain a similar result. In the proof we showed that if $f(z_m) \to \infty$, then $f(z_m) + g(z_m) \to \infty$ also. In doing this, we use the arguments of $f(z)$ and $g(z)$ only when $|f(z)|$ and $|g(z)|$ are large and hence we don't need to require that $f(z)$ and $g(z)$ are non-zero. If $f(z)$ and $g(z)$ are meromorphic, we consider separately those z_m in the sequence $\{z_m\}$ from the above proof which are poles of either $f(z)$ or $g(z)$ and call them a sequence $\{p_n\}$ (possibly finite in number). For each p_n there is a neighborhood U_n in D of p_n such that $P < |f(z)| < \infty$, $g(z) \neq \infty$ for all z in $U_n - \{p_n\}$, where P is from the above proof. As before with given $\epsilon > 0$, $|f(z) + g(z)| > 1/\epsilon$ for every z in $U_n - \{p_n\}$. Since $f(z)$ and $g(z)$ are continuous at p_n, $|f(p_n) + g(p_n)| \geqq 1/\epsilon$ also, and so the corollary is valid for meromorphic functions too.

There exist normal holomorphic functions $f(z)$ and $g(z)$ whose sum is normal but the functions fail to satisfy the hypotheses of Corollary 2. Hence the hypotheses of Corollary 2 are not necessary conditions. For example, let $f(z) = (z + 1)/(z - 1)$, and $g(z) = (-1)f(z) + z$ in D. Then $f(z) + g(z) = z$ which is normal; but, as evident, there is no positive integer N such that $n \geqq N$ implies $R_n(f, g)$ is contained in $T_n(f, g, \mu)$ for any $\mu(n)$ satisfying the hypotheses.

COROLLARY 3. *Let $f(z)$ and $g(z)$ be normal finite-valued functions in D. If there exist δ, M, R where $0 < \delta < \pi$, $0 < M < \infty$, $0 \leqq R < 1$ such that the set*

$$\{z: M < |f(z)|\} \cap \{z: M < |g(z)|\} \cap \{z: R < |z| < 1\}$$

is contained in $\{z: |\arg f(z) - \arg g(z)| \leqq \pi - \delta\}$, then $f(z) + g(z)$ is a normal function in D.

One use of the corollary is when one knows the total cluster sets of the normal functions f and g. For example, the cluster sets, sufficiently far out in the

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extended plane containing the cluster sets, may be contained in infinite strips (possibly more than one strip for each function) such that no side of any strip containing in part the cluster set of f is parallel to the side of any strip containing in part the cluster set of g. Then $f + g$ is normal by the corollary.

One could consider linear functional combinations of normal functions also. See [**2**, p. 7].

3. We obtain variations of the previous summation theorems by considering property (D) of non-normal functions. Again in the following theorem the arguments of the functions $f(z)$ and $g(z)$ are chosen so as to have the difference in absolute values less than or equal to π.

THEOREM 2. *Let $f(z)$ and $g(z)$ be two normal meromorphic functions in D. Let $\delta > 0$ be arbitrary and for each sequence $\{z_n\}$ for which $f(z_n) \to \infty$, $f(z_n) \neq \infty$, $g(z_n) \neq \infty$, let there exist $\alpha(n)$, a non-increasing function on the positive integers, such that $0 \leqq \alpha \leqq \pi$ and $|f(z_n)| \sin \alpha(n) \geqq \delta$ for all n greater than some positive integer N. If*

$$1 - 1/|f(z_n)| < |f(z_n)/g(z_n)| < 1 + 1/|f(z_n)|$$

implies

$$|\arg f(z_n) - \arg g(z_n)| \leqq \pi - \alpha(n)$$

for $n > N$, then $f(z) + g(z)$ is a normal meromorphic function in D.

Proof. It is clear that if $\{z_n\}$ is a sequence where $\{f(z_n)\}$ has no unbounded subsequences and $\{f(z_n) + g(z_n)\}$ converges, then $\{z_n\}$ is not a sequence for which $f(z) + g(z)$ has property (D). Therefore let $\{z_n\}$ be a sequence for which $\{f(z_n)\}$ diverges to ∞ and $\{f(z_n) + g(z_n)\}$ converges to a finite complex value or diverges to ∞. Let $\{z_n'\}$ be any sequence close to $\{z_n\}$ so that $f(z_n') \to \infty$ also. Fix $n > N$ and consider the following cases.

Case I. $f(z_n') \neq \infty$.

(a) If $g(z_n') = \infty$, then $|g(z_n') + f(z_n')| = \infty$.

(b) If $g(z_n') \neq \infty$ and

$$1 - 1/|f(z_n')| < |f(z_n')/g(z_n')| < 1 + 1/|f(z_n')|,$$

then $|f(z_n') + g(z_n')| \geqq |f(z_n')| \sin \alpha(n) \geqq \delta > 0$ for $n > N$, the δ and N in the hypotheses of the theorem.

(c) If $g(z_n') \neq \infty$ and $|f(z_n')/g(z_n')| \geqq 1 + 1/|f(z_n')|$, then

$$|g(z_n')| \leqq (|f(z_n')|/(1 + |f(z_n')|))|f(z_n')|.$$

Hence

$$\begin{aligned}|f(z_n') + g(z_n')| &\geqq |f(z_n')| - |g(z_n')| \\ &\geqq |f(z_n')|(1 - |f(z_n')|/(1 + |f(z_n')|)) \\ &= 1/(1 + 1/|f(z_n')|) \geqq \tfrac{1}{2} > 0\end{aligned}$$

for n sufficiently large.

(d) If $g(z_n') \neq \infty$ and $|f(z_n')/g(z_n')| \leqq 1 - 1/|f(z_n')|$, then $|f(z_n') + g(z_n')| > 1$ for n sufficiently large by computations similar to those in part (c).

Case II. $f(z_n') = \infty$.

(a) If $g(z_n') \neq \infty$, then $|f(z_n') + g(z_n')| = \infty$.

(b) If $g(z_n') = \infty$, then there exists a neighborhood U containing z_n' such that $f(z) \neq \infty$ and $g(z) \neq \infty$ for all z in $U - \{z_n'\}$. Then there exists $\{z_{n,m}\}^{\infty}{}_{m=1}$ contained in U, $z_{n,m} \to z_n'$ as $m \to \infty$, where $|f(z_{n,m})| > m$ and we may proceed as in Case I.

4. It is well-known that a meromorphic function bounded above (or below) by a normal meromorphic function is the product and sum of normal meromorphic functions. We can say more than this if the function is appropriately bounded both above and below.

THEOREM 3. *Let $f(z)$, $g(z)$, and $h(z)$ be meromorphic functions in D such that $g(z)$ and $h(z)$ are normal, and let K_1 and K_2 be non-negative constants such that $|g(z)| - K_1 \leqq |f(z)| \leqq |h(z)| + K_2$ for each z in D. If* $\liminf |g(z_n)| > K_1$ *for each sequence $\{z_n\}$ in D for which $h(z_n) \to \infty$, then $f(z)$ is normal in D.*

Proof. Assume $f(z)$ is not normal. Then there exists a sequence $\{z_n\}$ such that $f(z)$ has property (D) on $\{z_n\}$. Then $\{z_n\}$ has a subsequence $\{z_{n,k}\}$ for which there exist sequences $\{z'_{n,k}\}$ and $\{z''_{n,k}\}$, both close to $\{z_{n,k}\}$, satisfying $f(z'_{n,k}) \to 0$, $f(z''_{n,k}) \to \infty$. But then

$$\liminf_{n,k\to\infty} |g(z'_{n,k})| \leqq K_1$$

and $h(z''_{n,k}) \to \infty$. Since $g(z)$ and $h(z)$ are normal, we have

$$\liminf_{n,k\to\infty} |g(z_{n,k})| \leqq K_1$$

and $h(z_{n,k}) \to \infty$ in violation of the hypotheses. Thus there is no sequence $\{z_n\}$ on which $f(z)$ has property (D), and the theorem is proved.

COROLLARY 4. *Let $f(z)$ and $h(z)$ be meromorphic functions in D and $|h(z)| - K_1 \leqq |f(z)| \leqq |h(z)| + K_2$, $K_i \geqq 0$. Then $f(z)$ is normal if and only if $h(z)$ is normal.*

Remark. Since there are non-normal functions of bounded characteristic, the condition above on sequences is needed. For example, Bagemihl and Seidel [**1**, p. 7] construct a holomorphic function $f(z)$ of bounded characteristic such that $f(z_n) = 0$ for $z_n = 1 - 1/n^2$ and $f(z_n') \to \infty$ where $z_n' = (1/2)(z_n + z_{n+1})$ and an elementary calculation shows that $\{z_n\}$ and $\{z_n'\}$ are close. Hence $f(z)$ is not normal. The function is $B(z)/\exp((-1 - z)/(1 - z))$ where

$$B(z) = \prod_{n=1}^{\infty} (z_n - z)/(1 - z_n z).$$

But $|B(z)| \leqq |f(z)| \leqq |1/\exp((-1 - z)/(1 - z))|$, and $B(z)$ and

$$1/\exp((-1 - z)/(1 - z))$$

are normal holomorphic functions. Since $B(z_n') \to 0$ and

$$1/\exp((-1 - z_n')/(1 - z_n')) \to \infty,$$

there exists a sequence, namely $\{z_n'\}$, not fullfilling the hypotheses of Theorem 3.

THEOREM 4. *Let $f(z) = h_1(z)/h_2(z)$ be the quotient of two bounded holomorphic functions in D with no common zeros. If $f(z)$ is not normal, then there is a sequence $\{z_n\}$ of points in D such that $h_1(z_n) \to 0$ and $h_2(z_n) \to 0$.*

Proof. We prove the contrapositive. Let $|h_1(z)| \leqq M_1$ and $|h_2(z)| \leqq M_2$. Then $|h_1(z)/M_2| \leqq |f(z)| \leqq |M_1/h_2(z)|$ and $h_1(z)/M_2$ and $M_1/h_2(z)$ are normal. Since there is no sequence $\{z_n\}$ where $h_1(z_n)/M_2 \to 0$ and $M_1/h_2(z) \to \infty$, $f(z)$ is normal by Theorem 3.

COROLLARY 5. *Let $h_1(z) = B_1(z)\exp(g_1(z))$, $h_2(z) = B_2(z)\exp(g_2(z))$ be holomorphic functions with $B_1(z)$ and $B_2(z)$ Blaschke products, and choose M, $0 < M < \infty$, such that $-M \leqq \mathrm{Re}(g_1(z) - g_2(z)) \leqq M$. Let*

$$L_i(\delta) = \{z: |B_i(z)| < \delta\}, \quad i = 1, 2.$$

If there exists δ' such that

$$\rho(L_1(\delta'), L_2(\delta')) = \inf\{\rho(z_1, z_2): z_1 \text{ in } L_1(\delta'), z_2 \text{ in } L_2(\delta')\} \geqq \eta > 0,$$

then $f(z) = h_1(z)/h_2(z)$ is normal.

COROLLARY 6. (Cima [**5**, p. 769].) *Let $F(z) = B_1(z, a_n)/B_2(z, b_n)$, where $B_1(z, a_n)$ and $B_2(z, b_n)$ are Blaschke products with zeros $\{a_n\}$ and $\{b_n\}$, respectively. If the cluster points of the zeros of $B_1(z, a_n)$ and $B_2(z, b_n)$ are disjoint, then $F(z)$ is normal.*

5. We can also get a necessary and sufficient condition for a product of normal functions to be normal by considering again convergence on close sequences.

THEOREM 5. *Let $f(z)$ and $g(z)$ be normal functions in D. Then $f(z)g(z)$ is normal in D if and only if for each sequence $\{z_n\}$ in D such that $f(z_n) \to 0$, $g(z_n) \to \infty$ (or if $f(z_n) \to \infty$, $g(z_n) \to 0$) and $\{f(z_n)g(z_n)\}$ converges to a complex value α (possibly ∞), the product $\{f(z_n')g(z_n')\}$ converges to α for each sequence $\{z_n'\}$ close to $\{z_n\}$.*

Proof. The necessity is immediate. The sufficiency follows much as in the proof of Theorem 1.

The hypotheses of the following corollaries vacuously satisfy the hypotheses of Theorem 2.

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COROLLARY 7. *Let $f(z)$ and $g(z)$ be normal functions in D. If there exist finite positive constants K_f, K_g, M_f, M_g such that the sets $D_1 = \{z: |f(z)| > K_f; |g(z)| > K_g\}$ and $D_2 = \{z: |f(z)| < M_f; |g(z)| < M_g\}$ cover D (i.e., $D = D_1 \cup D_2$), then $f(z)g(z)$ is a normal function in D.*

COROLLARY 8. (Zinno [**9**, pp. 160–161].) *Let $f(z)$ and $g(z)$ be two normal meromorphic functions in D. Let a_ν and a_ν' be zeros of $f(z)$ and $g(z)$, respectively, and let b_ν and b_ν' be poles of $f(z)$ and $g(z)$, respectively. Suppose that*

(1) $\inf \rho(a_\nu, b_\mu') > 0$ *and* $\inf \rho(a_\nu', b_\mu) > 0$ ($\nu = 1, 2, \ldots, \mu = 1, 2, \ldots$, inf *over an empty set is infinite*) *and*

(2) *for any positive number ρ there exists a positive number m_ρ such that*

$$|f(z)| < m_\rho \quad \text{for } z \text{ in } D - \bigcup U(b_v, \rho),$$

$$|g(z)| < m_\rho \quad \text{for } z \text{ in } D - \bigcup U(b_v', \rho),$$

$$|f(z)| > \frac{1}{m_\rho} \quad \text{for } z \text{ in } D - \bigcup U(a_v, \rho), \text{ and}$$

$$|g(z)| > \frac{1}{m_\rho} \quad \text{for } z \text{ in } D - \bigcup U(a_v', \rho)$$

($U(z, \delta) = \{\zeta: \rho(z, \zeta) < \delta\}$, *unions over all positive integers*). *Then the product $f(z)g(z)$ is a normal meromorphic function in D.*

6. Theorem 5 may be applied to obtain some properties of normal pseudoanalytic functions in D. The following definitional material with background information may be found in detail in [**3**] and [**4**]. Let D_0 be a domain containing D.

Definition A [**3**, p. 18]. Let $w(z)$ be a function from D into the complex w-plane which possesses continuous partial derivatives. If there exists a constant K such that $|w_{\bar{z}}(z)| \leqq K|w(z)|$, we say $w(z)$ is *approximately analytic* in D.

Definition B [**4**, p. 215]. Two continuous functions, $F(z)$ and $G(z)$, defined in D_0 are said to form a *generating pair* if $\operatorname{Im}\{\overline{F(z)}G(z)\} > 0$ for z in D.

Every function $w(z)$ defined in D admits the unique representation $w(z) = \phi(z)F(z) + \psi(z)G(z)$ with real functions $\phi(z)$, $\psi(z)$ [**4**, p. 216].

Definition C [**4**, p. 217]. The function $w(z) = \phi(z)F(z) + \psi(z)G(z)$ is said to possess at the point z_0 in D the *(F, G)-derivative*, denoted $\dot{w}(z_0)$, if the limit

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \phi(z_0)F(z) - \psi(z_0)G(z)}{z - z_0}$$

exists and is finite.

Definition D [**4**, p. 219]. A function $w(z)$ possessing an (F, G)-derivative at all points of the domain D is called *regular (F, G)-pseudoanalytic of the first kind in D* or simply *pseudoanalytic*.

THEOREM A [**3**, p. 18]. *Every pseudoanalytic function is also approximately analytic.*

Definition E [**3**, p. 24]. We call two functions $w(z)$ and $f(z)$ defined in D *similar* if there exists a function $S(z)$ which is continuous and different from zero on the closure of D and such that $f(z) = S(z)w(z)$ in D.

THEOREM B [**3**, p. 24–25] (Similarity Principle). *Every approximately analytic function $w(z)$ in D possesses a similar function $f(z)$ which is analytic in D.*

THEOREM 6. *If $f(z)$ and $w(z)$ are similar functions in D, and if one of them is normal, then the other one is normal also.*

Proof. Let $S(z)$ be such that $f(z) = S(z)w(z)$ where $S(z)$ is continuous and non-zero on the closure of D. Then there exists an M such that $1/M < |S(z)| < M$ for all z in the closure of D. Hence Theorem 5 implies $f(z)$ is normal if and only if $w(z)$ is normal since $S(z)$ and $1/S(z)$ are normal in D.

COROLLARY 9. *If $w(z)$ is approximately analytic and bounded (or bounded from zero) in D, then $w(z)$ is normal in D.*

Proof. Let $f(z)$ be the analytic function in D similar to $w(z)$. Then $f(z) = S(z)w(z)$ is bounded (or bounded from zero) in D and therefore $f(z)$ is normal.

COROLLARY 10. *If $w(z)$ is a normal approximately analytic function in D and has an asymptotic value α at ζ on C, then $w(z)$ has a Fatou value α at ζ on C.*

Proof. Let Γ be the Jordan arc in $D \cup \{\zeta\}$ on which $w(z)$ tends to the limit α. Let $f(z)$ be the analytic function similar to $w(z)$. Since $S(z)$ in $f(z) = S(z)w(z)$ is uniformly continuous on $D \cup C$, $S(z)$ tends to a limit, say β (finite and non-zero), on Γ also. Hence $f(z)$ tends to a limit, $\beta\alpha$, on Γ and so has a Fatou value $\beta\alpha$ at ζ on C. $S(z)$ also has β as a Fatou value at ζ. Therefore, since $w(z) = f(z)/S(z)$, $w(z)$ has a Fatou value α at ζ.

We could show, with proofs similar to those above, that normal approximately analytic functions have many of the properties related to Fatou values of normal analytic functions and that normal pseudoanalytic functions have some of the same identity and uniqueness properties as normal analytic functions.

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Purdue University at Fort Wayne,
Fort Wayne, Indiana