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OSCILLATION CRITERIA FOR $y'' + p(t)f(y, y') = 0$ WITH f HOMOGENEOUS OF DEGREE ONE

G. B. GUSTAFSON

1. Introduction. Let $\mathscr{F}$ be the class of functions $f : \mathbf{R}^2 \to \mathbf{R}$ satisfying (1) f is continuous, (2) $xf(x, y) > 0$ for $x \neq 0$, (3) $f(tx, ty) = tf(x, y)$ for all $t, x, y \in \mathbf{R}$, (4) f is locally Lipschitzian. The classical Sturm theorems, the Leighton-Wintner oscillation theorem, and perturbation theorems have been established by Bihari in [**1**; **2**; **3**] for the equation $y'' + p(t)f(y, y') = 0$, $p(t) \geqq 0, f \in \mathscr{F}$.

This paper investigates the question of strong oscillation of

$$y'' + p(t)f(y, y') = 0 \tag{1.1}$$

for $p \in C[0, \infty)$, $f \in \mathscr{F}$. The equation is viewed as "almost linear" because of the homogeneity and sign conditions on f. There appears to be great hope that oscillation criteria for $y'' + p(t)y = 0$ will carry over to the nonlinear equation, and it is shown in section 2 that this is indeed the case for $p(t) \geqq 0$.

The case when $p(t)$ changes sign is considered in sections 3 and 4. It is shown that for $\int^\infty p = \infty$, virtually all $f \in \mathscr{F}$ built from standard elementary functions will force $y'' + p(t)f(y, y') = 0$ to be strongly oscillatory. The encouraging direction given by these results suggest that $\int^\infty p = \infty$ and $f \in \mathscr{F}$ forces strong oscillation, but this is not true: a pathological example is constructed in section 6 to show that additional hypotheses are needed.

A technical problem occurs when one talks about oscillation of (1.1) because a positive constant multiplying $p(t)$ can be absorbed into f. For this reason, one considers, with Nehari [**7**], the notion of strong oscillation: all solutions of $y'' + \lambda p(t)f(y, y') = 0$ oscillate for every $\lambda > 0$.

To the best knowledge of the author, the results here do not overlap general theorems given in the excellent survey on nonlinear oscillation by J. S. W. Wong [**8**]. The proofs presented here use standard Riccati equation methods. In particular, the starting point for the investigation herein is the following lemma, the proof of which is left to the reader.

LEMMA 1.1. *Let $f \in \mathscr{F}$, $p \in C[0, \infty)$. The equation* (1.1) *has a solution $y(t) > 0$ on $t \geqq T$ if and only if the nonlinear Riccati equation*

$$z' = z^2 + p(t)f(1, -z)$$

has a solution $z \in C^1[T, \infty)$.

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The proofs borrow freely from basic properties of $y'' + p(t)f(y, y') = 0$ developed in [**1**], [**2**], and [**3**].

Many of the results herein are valid without the local Lipschitz condition on $f \in \mathscr{F}$. In particular, 2.2(a), 3.1, 4.1, 4.3, 4.4, 4.5 are valid without this assumption. However, one must assume that the given solutions extend to the half-line $[0, \infty)$.

It should be noted that the term $p(t)f(y, y')$ has the sign of $p(t)y(t)$, since $f(1, -z) > 0$.

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2. Oscillation and nonoscillation criteria for $p(t) \geqq 0$. In this section, strong oscillation of the nonlinear equation $y'' + p(t)f(y, y') = 0$ is shown to be equivalent to strong oscillation of the corresponding linear equation.

LEMMA 2.1. *Assume $p \in C$, $f \in \mathscr{F}$ and let u and v be positive solutions of $y'' + p(t)f(y, y') = 0$ on $[a, b]$. If $-u'(a)/u(a) < -v'(a)/v(a)$ then $-u'(t)/u(t) \leqq -v'(t)/v(t)$ for $t \in [a, b]$.*

Proof. Let $r = -u'/u$, $s = -v'/v$. Then r and s satisfy a first order Riccati differential equation and, by virtue of the relation $r(a) < s(a)$, one can apply Corollary 4.2 in [**4**, p. 27].

BIHARI'S SEPARATION THEOREM. *If $f \in \mathscr{F}$, $p \in C$, then $y'' + p(t)f(y, y') = 0$ has the interlacing zero property.*

Proof. Let u and v be two linearly independent solutions with $u(c) = u(d) = 0$, $u(t) > 0$ on (c, d). If v does not vanish in (c, d), then Lemma 2.1 can be applied on a subinterval $[a, b] \subseteq (c, d)$ to obtain a contradiction.

THEOREM 2.2. *Let $p \in C$, $f \in \mathscr{F}$ and $p(t) \geqq 0$.*

(a) *If $y'' + p(t)f(y, y') = 0$ is nonoscillatory, then for some $\lambda > 0$ the linear equation $y'' + \lambda p(t)y = 0$ is nonoscillatory.*

(b) *If the linear equation $y'' + p(t)y = 0$ is nonoscillatory, then for some $\mu > 0$ the nonlinear equation $y'' + \mu p(t)f(y, y') = 0$ is nonoscillatory.*

(c) *Strong oscillation of $y'' + p(t)f(y, y') = 0$ is equivalent to strong oscillation of $y'' + p(t)y = 0$.*

Remark. In the case $p(t) \geqq 0$, conclusion (c) allows one to formulate a hierarchy of criteria for strong oscillation of the nonlinear equation. In particular, $\int^\infty p = \infty$ or $t^2p(t) \to \infty$ will suffice (see P. Hartman [**4**, p. 362], and [**3**; **6**]). The results in (a) and (b) of Theorem 2.2 cannot be improved, as is shown by the functions $p(t) = at^{-2}$, $f(y, y') = by$, $a, b > 0$. The result in (c) is, of course, the main result, and it is called to the reader's attention that (c) is equivalent to the condition

$$\limsup_{t\to\infty} t \int_t^\infty p(r)dr = \infty;$$

see [7, page 429].

Proof of 2.2(a). Let $y(t)$ be the given positive solution and put $u(t) = -y'(t)/y(t)$, $t \geqq T$. Then

$$u'(t) = u^2(t) + p(t)f(1, -u(t)).$$

The sign conditions on $p(t)$ and $f(x, y)$ insure that $u'(t) \geqq 0$. Let us show that $u(t)$ increases to a finite limit. If not, then $u(t) \to \infty$ and $u'(t) \geqq u^2(t)$ gives

$$\frac{1}{u(t_0)} - \frac{1}{u(t)} \geqq t - t_0$$

for large t, $t \geqq t_0$. This is impossible, and hence $u(t)$ increases to a finite limit u_0. It is easy to check that $u \leqq 0$, and hence $u(t)$ increases to 0. Select a positive number λ such that $f(1, -u(t)) \geqq \lambda$ for large t. Then

$$u' \geqq u^2 + \lambda p(t)$$

and by Theorem 7.2 of Hartman's text [4, page 362], it follows that $z'' + \lambda p(t)z = 0$ is eventually disconjugate, hence nonoscillatory.

Proof of 2.2(b). Let $z(t)$ be the positive solution of the linear equation and let $u = -z'/z$, so that $u' = u^2 + p(t)$. As in the previous proof, $u(t)$ increases to zero as $t \to \infty$. Select $\lambda > 0$ such that $2\lambda f(1, 0) \leqq 1$. Then for large t

$$\lambda f(1, -u(t)) \leqq 2\lambda f(1, 0) \leqq 1$$

because $u(t) \to 0$. Therefore,

$$z'' + \lambda p(t)f(z, z') = z'' + \lambda p(t)zf(1, -u) \leqq z'' + p(t)z = 0.$$

Given a closed interval $[a, b]$ on which the preceding inequalities are valid, define $\beta(t) = z(t)$, $\alpha(t) = \min_{a\leqq t\leqq b} z(t) = z(a)$. Then $\alpha(t) \leqq \beta(t)$ for $a \leqq t \leqq b$. Further,

$$\beta'' + \lambda p(t)f(\beta, \beta') \leqq 0 \leqq \alpha'' + \lambda p(t)f(\alpha, \alpha'),$$

by the preceding differential inequality and the sign conditions on α, p, f.

For $|r| \geqq 1$ and $\alpha(t) \leqq s \leqq \beta(t)$ one has

$$|\lambda p(t)f(s, r)| = |\lambda rp(t)f(s/r, 1)| \leqq k|r|$$

for some constant $k = k(\alpha, \beta, p, f, \lambda)$. For $|r| \leqq 1$, $\alpha(t) \leqq s \leqq \beta(t)$, $|\lambda p(t)f(s, r)| \leqq L$ for some constant $L = L(\alpha, \beta, p, f, \lambda)$. Therefore, $|\lambda p(t)f(s, r)| \leqq k|r| + L$ on the set $\{(t, s, r) : \alpha(t) \leqq s \leqq \beta(t), r \in \mathbf{R}\}$.

The conditions for the Jackson-Schrader theorem [5, p. 354], are satisfied, and therefore there exists a solution $y(t)$ of the differential equation $y'' + \lambda p(t)f(y, y') = 0$ satisfying $\alpha(t) \leqq y(t) \leqq \beta(t)$ for $a \leqq t \leqq b$.

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Suppose now that the theorem fails. Then for large t there exists a solution of the nonlinear differential equation having two zeros. By the Bihari separation theorem, the solution $y(t)$ constructed from the Jackson-Schrader theorem must have at least one zero in the open interval. However, $y(t) \geqq \alpha(t) > 0$ for $a \leqq t \leqq b$, a contradiction. Therefore, all solutions of the nonlinear equation are eventually one-signed, and the equation is non-oscillatory.

3. Oscillation criteria for $f \in \mathscr{F}$, p not one-signed. When $p(t)$ is non-positive, two integrations of $y'' + p(t)f(y, y') = 0$ on $[a, b]$ show that there cannot exist a solution $y(t)$ with $y(a) = y(b) = 0$, $y(t) > 0$ in (a, b). Hence oscillation is impossible when $p(t) \leqq 0$ for large t.

In analogy with the linear case, it is assumed throughout the present section that $\int^\infty p(t)dt = \infty$, but $p(t)$ is allowed to change sign infinitely often. Resulting criteria shall be independent of $f \in \mathscr{F}$. As usual, $p_+ = (|p| + p)/2$, $p_- = (|p| - p)/2$.

THEOREM 3.1. *Any one of the following conditions is sufficient for strong oscillation of* (1.1) *for all* $f \in \mathscr{F}$:

(i) $\displaystyle\int^\infty p_+ = \infty, \int^\infty p_- < \infty.$

(ii) $\displaystyle\int^\infty p = \infty$, $p(t)$ *bounded.*

(iii) $\displaystyle\int^\infty p = \infty, \liminf_{t\to\infty} p_-(t) > -\infty.$

(iv) $\displaystyle\int^\infty p = \int^\infty \exp\left(-\epsilon \int_0^t p_-\right)dt = \infty$ *for some* $\epsilon > 0$.

Proof. Suppose the conclusion fails. Then there is a solution $y(t)$ of $y'' + \lambda p(t)f(y, y') = 0$, for some $\lambda > 0$, which is positive on a half-line $t \geqq T$. By Lemma 1.1, $z = -y'/y$ satisfies

$$z' = z^2 + \lambda p(t)f(1, -z)$$

and we can divide by $f(1, -z)$, integrate over $[T, t]$, and obtain the following integral equation:

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} = \int_T^t \frac{z^2(s)ds}{f(1, -z(s))} + \lambda \int_T^t p(s)ds.$$

Since $f(1, -u) > 0$ and $\int^\infty p = \infty$ in (i)–(iv), the left side of the integral equation has limit ∞ as $t \to \infty$. But $f(1, -u) > 0$ for all u, whereby it follows that $z(t)$ must eventually leave every subset of the form $(-\infty, k]$, i.e., $z(t) \to \infty$ as $t \to \infty$.

Let us return now to the nonlinear Riccati equation and divide by $z(t)$:

$$z'/z = z + \lambda p(t) f(1/z(t), -1).$$

Since $z(t) \to \infty$, and $f(0, -1) = 0$, one has $f(1/z(t), -1) \to 0$ as $t \to \infty$. In particular, if one writes $p = p_+ - p_-$, then the following inequality is valid for arbitrary $\epsilon > 0$ and sufficiently large t:

$$z'/z \geqq z - \epsilon p_-(t).$$

If (i), (ii), or (iii) holds, then $\int^\infty z = \infty$ implies that

$$\log \frac{z(t)}{z(a)} \geqq \frac{1}{2} \int_a^t z(s) ds$$

for a large, $t \geqq a$. Therefore, $u' \geqq k \exp(\frac{1}{2}u)$, where

$$u = \int_a^t z(s) ds, \ k > 0.$$

Integration of this equation gives

$$e^{-u(b)/2} - e^{-u(t)/2} \geqq k(\mathrm{t} - b)/2, \ (a \geqq b \geqq t)$$

which is impossible.

If (iv) holds, then one uses the $\epsilon > 0$ supplied by (iv) in the preceding inequality to get

$$\log \frac{z(t)}{z(a)} \geqq \int_a^t z(s) ds - \int_a^t p_-(s) ds,$$

or

$$u'(t) \exp[-u(t)] \geqq z(a) \exp\left(-\epsilon \int_a^t p_-(s) ds\right)$$

where

$$u(t) = \int_a^t z(s) ds.$$

Integration gives a contradiction to (iv).

4. Oscillation criteria depending on $f \in \mathscr{F}$**.** Let $p(t)$ be continuous, not necessarily one-signed, with $\int^\infty p = \infty$. Criteria are sought involving $f \in \mathscr{F}$ such that $y'' + p(t) f(y, y') = 0$ is strongly oscillatory. The results show that $f \in \mathscr{F}$ exhaust most of the elementary functions in $\mathscr{F}$. Furthermore, this is the best result of this kind, as will be shown in the next section.

LEMMA 4.1. *Let* $\int^\infty p = \infty$ *and suppose* $f \in \mathscr{F}$. *If* (1.1) *has a positive solution* $y(t)$ *for* $t \geqq T$, *then*

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(a) $-y'(t)/y(t) \to 0$ *as* $t \to \infty$,

(p) $\lim_{t\to\infty} y(t) = 0$ *and* $\limsup_{t\to\infty} y'(t) = 0$,

(c) $\liminf_{t\to\infty} \int_T^t p(s)f(-y(s)/y'(s), -1)ds = -\infty$.

THEOREM 4.2. *Let* $\int^\infty p = \infty$. *If* $f \in \mathscr{F}$ *is a rational function, then* (1.1) *is strongly oscillatory.*

THEOREM 4.3. *Assume* $f \in \mathscr{F}$ *and* $\int^\infty p = \infty$. *If for some constant* $c > 0$ *and integer* $m \geqq -1$, *the inequality* $1/c \leqq u^m f(1, -u) \leqq c$ *is valid for* $u \geqq u_0 > 0$, *then* (1.1) *is strongly oscillatory.*

COROLLARY 4.4. *Assume* $f \in \mathscr{F}$ *and* $\int^\infty p = \infty$. *If*

$$\liminf_{u\to\infty} u^m f(1, -u) > 0 \quad \textit{and} \quad \limsup_{u\to\infty} u^m f(1, -u) < \infty$$

for some integer $m \geqq -1$, *then equation* (1.1) *is strongly oscillatory.*

THEOREM 4.5. *Assume* $f \in \mathscr{F}$ *and* $\int^\infty p = \infty$. *Let*

$$G(u) = \int_0^u \frac{ds}{f(1, -s)},$$

and put

$$Q(t, \lambda, a) = p_-(t) \left[\sup\left\{ f(u, -1) : u > 0, uG^{-1}\left(\lambda \int_a^t p(s)ds\right) \leqq 1 \right\} \right]$$

for $\lambda > 0$, $t \geqq a$. *Any one of the following conditions is sufficient for strong oscillation of* (1.1):

(i) $\liminf_{t\to\infty} Q(t, \lambda, a) > 0$, $\lambda > 0$, *a large.*

(ii) $\int_a^\infty Q(t, \lambda, a)dt$ *exists for each* $\lambda > 0$, *a large.*

(iii) $\int_a^\infty \exp\left(-\int_a^t Q(s, \lambda, a)ds\right)dt = \infty$, $\lambda > 0$, *a large.*

Example 4.6. If $\int^\infty p = \infty$, then $y'' + p(t)[y + y|\sin(y'/y)|] = 0$ is strongly oscillatory, by Corollary 4.4.

Example 4.7. If $\int^\infty p = \infty$, then $y'' + p(t)[y|y|/(|y| + |y'|)] = 0$ is strongly oscillatory, by Corollary 4.4.

Proof of Lemma 4.1. Part (a) was established in 3.1. To demonstrate (b), write $z(t) = -y'(t)/y(t)$,

$$y(t) = y(a) \exp\left(-\int_a^t z(s)ds\right).$$

From part (a), $y(t) \to 0$. Since $y(t) > 0$, (a) gives $y'(t) < 0$, and from

$$y(t) = y(a) + \int_a^t y'(s)ds$$

it is evident that $\limsup_{t\to\infty} y'(t) < 0$ cannot happen. Thus

$$\limsup_{t\to\infty} y'(t) = 0.$$

To establish (c), return to the nonlinear Riccati equation

$$z'/z = z + p(t)f(1/z, -1)$$

and integrate over $[a, t]$. If (c) fails, then

$$\int_a^t \frac{z'}{z} \geqq \frac{1}{2}\int_a^t z$$

for large t, and one reaches a contradiction as in the proof of 3.1. Therefore, (c) is proved.

Proof of Theorem 4.2. Let $f \in \mathscr{F}$ be a rational function. Write $f(1, -z) = P(z)/Q(z)$ where P and Q are polynomials. Since $f(1, -z) > 0$ we may assume that $P(z) > 0$ and $Q(z) > 0$. If P has degree N, Q has degree M, and $N > M$, then for $x \neq 0$,

$$f(x, y) = xf(1, y/x) = xP(y/x)/Q(y/x)$$

for all y. Choose $y = |x|^r$, where $r = (N - M - 1)/(N - M)$. Then

$$\begin{aligned} f(x, y) &= x(a(y/x)^{N-M} + \ldots) \\ &= a(|x|/x)^{N-M-1} + v(x, |x|) \end{aligned}$$

with $a \neq 0$ and $v \to 0$ as $x \to 0$. This violates $f(0, 0) = 0$. Therefore $N \leqq M$. Put $l = M - N \geqq 0$. Then $z^l P(z)/Q(z)$ is bounded, and there is a constant $c > 0$ such that

$$1/c \leqq z^l P(z)/Q(z) \leqq c$$

for all large $z > 0$. Theorem 4.3 applies, and $y'' + p(t)f(y, y') = 0$ is strongly oscillatory.

Proof of Theorem 4.3. Suppose not and let $y(t)$ be a solution of $y'' + \lambda p(t)f(y, y') = 0$ $\lambda > 0$, $y(t) > 0$ for $t \geqq a$. By Lemma 4.1, $z(t) = -y'(t)/y(t) \to \infty$ as $t \to \infty$, and therefore

$$1/c \leqq z^m f(1, -z) \leqq c, \qquad t \geqq T \geqq a.$$

From Lemma 1.1 one obtains $z' = z^2 + \lambda p(t)f(1, -z)$, and division by $f(1, -z)$, integration over $[T, t]$, yields

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} = \int_T^t \frac{z^2 dz}{f(1, -z)} + \lambda \int_T^t p(s)ds.$$

Since $1/f(1, -z) \geqq z^m/c$, one has

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} \geqq \frac{1}{c} \int_T^t z^{2+m} + \lambda \int_T^t p(s)ds.$$

On the other hand, $1/f(1, -u) \leqq cu^m$ for u large, which gives

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} \leqq \frac{c[z(t)]^{m+1}}{m+1}$$

for $m \neq -1$, and

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} \leqq c \log z(t)$$

for $m = -1$. Of course, it is assumed that $z(t) \geqq \min\{u_0, 1\}$ for $t \geqq T$. For the case $m > -1$ we obtain, for large t,

$$\frac{c}{m+1} [z(t)]^{m+1} \geqq \frac{1}{2c} \int_T^t [z(s)]^{2+m} ds.$$

Define

$$W(t) = \int_T^t [z(s)]^{2+m} ds.$$

Then

$$W'(t) = ([z(t)]^{m+1})^{(m+2)/(m+1)}.$$

Therefore,

$$W'(t) \geqq \left(\frac{m+1}{2c^2} W(t)\right)^{(m+2)/(m+1)}.$$

Then $W' \geqq kW^r$ where k is a positive constant, $r = (m+2)/(m+1)$. Therefore, for $t \geqq b$ and b large enough,

$$(1/(1-r))[W^{1-r}(t) - W^{1-r}(b)] \geqq k(t-b).$$

But $1 - r = (m + 1 - m - 2)/(m+1) = -1/(m+1) < 0$ gives a contradiction, because $W > 0$ and $k > 0$.

The case $m = -1$ leads to

$$c \log z(t) \geqq \frac{1}{2c} \int_T^t [z(s)]^{2+m} ds,$$

and therefore

$$z(t) \geqq \exp\left(\frac{1}{2c^2} \int_T^t [z(s)]^{m+2} ds\right).$$

Again one lets

$$W(t) = \int_T^t [z(s)]^{m+2} ds,$$

and there results the inequality

$$W'(t) = [z(t)]^{m+2} \geqq \exp\left(\frac{m+2}{2c^2} W(t)\right);$$

as before, a contradiction is reached.

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Proof of Corollary 4.4. Choose $c > 0$ large enough to satisfy

$$1/c < \liminf_{u\to\infty} u^m f(1, -u) \leqq \limsup_{u\to\infty} u^m f(1, -u) < c$$

and apply the Theorem.

Proof of Theorem 4.5. By Lemma 4.1, the failure of the Theorem implies the existence of a solution $y(t)$ of $y'' + \lambda p(t)f(y, y') = 0$ positive on a half-line $t \geqq T$ with $z(t) = -y'(t)/y(t) \to \infty$ as $t \to \infty$. The function $z(t)$ satisfies, by Lemma 1.1,

$$z'/z = z + \lambda p_+(t)f(1/z, -1) - \lambda p_-(t)f(1/z, -1),$$

where $p = p_+ - p_-$, $\lambda > 0$. Furthermore, division of $z' = z^2 + p(t)f(1, -z)$ by $f(1, -z)$ and integration over $[T, t]$ gives

$$\int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} = \int_T^t \frac{z^2(s)ds}{f(1, -z(s))} + \lambda \int_T^t p(s)ds$$

whereby the sign conditions on f and z yield

$$G(z(t)) - G(z(T)) = \int_{z(T)}^{z(t)} \frac{du}{f(1, -u)} \geqq \lambda \int_T^t p(s)ds.$$

By choosing T larger, if necessary, we can assume that $z(T) > 0$, so that $G(z(T)) > 0$. Hence,

$$G(z(t)) \geqq \lambda \int_T^t p(s)ds.$$

However, $G(u)$ is strictly increasing, and one obtains

$$z(t) \geqq G^{-1}\left(\lambda \int_T^t p(s)ds\right).$$

Therefore,

$$1/z(t) \in \left\{u : uG^{-1}\left(\lambda \int_T^t p(s)ds\right) \leqq 1, u > 0\right\},$$

which forces $p_-(t)f(1/z(t), -1) \leqq Q(t, \lambda, T)$. Furthermore,

$$z'/z \geqq z - Q(t, \lambda, T)$$

from a preceding equation. We can proceed now as in the proof of Theorem 3.1 to obtain a contradiction to any one of the conditions (i), (ii), (iii).

5. An example. It will be shown that there exists an equation $y'' + p(t)f(y, y') = 0$ with $\int^\infty p = \infty$ and $f \in \mathscr{F}$ having a positive solution on a half-line. Therefore, the equivalence of strong oscillation for $y'' + p(t)f(y, y') = 0$ and $y'' + p(t)y = 0$ in the case $p(t) \geqq 0$ cannot be carried over to the case when $p(t)$ changes sign.

In view of the preceding theorems, an involved construction can be expected. The first obstacle is the virtual exhaustion of possible candidates $f \in \mathscr{F}$ by results of section 4. Secondly, the nonlinearity of (1.1) forces one to abandon linear techniques.

The attack on the construction problem is made through the Riccati integral equation

$$\int_1^{z(t)} \Psi(u)du - \int_1^t z^2(s)\Psi(z(s))ds = \int_1^t p(s)ds. \tag{5.1}$$

If $z(t)$ is a C^1 solution of (5.1), then

$$y = \exp\left(-\int_1^t z\right)$$

satisfies

$$y'' + p(t)f(y, y') = 0$$

where $f(y, y') = y/\Psi(-y'/y)$. The idea is to construct $z(t)$ and $\Psi(u) > 0$ so that the left hand side of (5.1) approaches ∞ as $t \to \infty$; then let $p(t)$ be defined via (5.1) to obtain the example.

It takes some development to insure that the function f constructed in this way is locally Lipschitz. Accordingly, the following definition is summoned.

Definition 5.2. Define $\mu(s) = s$, $0 \leqq s < c$, $\mu(s) = c(1 - s)/(1 - c)$ for $c \leqq s \leqq 1$, $\mu(s) = 0$ elsewhere. Put

$$\lambda(t) = \left[\int_0^t \mu(s)ds\right] \Big/ \left[\int_0^1 \mu(s)ds\right].$$

The *blending function* $\lambda(t)$ has the following properties:

$$\int_0^1 s\mu(s)ds = \frac{c(c+1)}{6}, \quad \int_0^1 \mu(s)ds = c/2,$$

$\lambda(0) = \lambda'(0) = \lambda'(1) = 0$, $\lambda(1) = 1$, $\lambda'(t) > 0$ on $(0, 1)$,

$$2\int_0^1 \lambda(s)ds - 1 = (1 - 2c)/3.$$

We take $c = 3/4$.

Let f_1 and f_2 be two functions defined on $[a, b]$ and let $\lambda(t)$ be the blending function defined above. Define the *convex connection* $r(t) \equiv r([a, b], f_1, f_2; t)$ of f_1 and f_2 over $[a, b]$ by the formula

$$r(t) = \begin{cases} f_1(t) & t \leqq a, \\ \lambda\left(\dfrac{t-a}{b-a}\right) f_2(t) + \left[1 - \lambda\left(\dfrac{t-a}{b-a}\right)\right] f_2(t) & a \leqq t \leqq b, \\ f_2(t) & t \geqq b. \end{cases}$$

In particular, if f_1 and f_2 are of class C^1, then so is $r(t)$, and if $f_1 \leqq f_2$, then $f_1 \leqq r \leqq f_2$. Further, if the derivatives obey $f_2' \geqq f_1'$, then $f_2 \geqq f_1$ implies $r(t)$ is increasing on $[a, b]$.

For future use, the following identity is recorded here:

$$\lim_{b\to a^+} (b-a)^{-1}\int_a^b r([a,b], f_1, f_2; t)dt = \frac{5}{12}f_2(a) + \frac{7}{12}f_1(a). \tag{5.2}$$

In this identity, f_1 and f_2 are continuously differentiable on $[c, d]$ and $a \in [c, d)$. To prove the identity, first integrate by parts to get the integral equal to

$$[f_2(b) - f_1(b)]\int_a^b \lambda\left(\frac{t-a}{b-a}\right)dt - \int_a^b \left(\int_a^t \lambda\right)(f_2'(t) - f_1'(t))dt + \int_a^b f_1(t)dt.$$

One easily verifies that

$$\int_a^b \lambda\left(\frac{t-a}{b-a}\right)dt = (b-a)\int_0^1 \lambda(s)ds, \quad \int_0^1 \lambda(s)ds = 5/12.$$

It is now routine to verify the limit identity (5.2).

The construction will make use of (5.2) by choosing $f_1(t) = -t^{-1}$, $f_2(t) = t^{-1}$. If we apply the identity (5.2), then the convex connection $r(t)$ of Definition 5.2 will satisfy

$$\int_a^b r(t)dt < 0$$

for $b - a$ sufficiently small, $1 \leqq a < b < \infty$.

The function $\Psi(u)$ which appears in (5.1) will be constructed in terms of an auxiliary function $g(t)$ by the formula

$$\Psi(u) = \left(1 - \int_1^u g(t)dt\right)^{-1}. \tag{5.3}$$

The function $z(t)$ appearing in (5.1) will be constructed by the formula

$$z(t) = \sum_{n=1}^{\infty} c_n\chi_{A_n}(t) + r(B_n, c_{n-1}, c_n; t)\chi_{B_n}(t). \tag{5.4}$$

In this formula, $\{c_n\}$ is a sequence of real numbers, and $\{A_n\}$, $\{B_n\}$ are sequences of intervals on the real axis.

Let E_n, F_n, G_n, H_n be pairwise disjoint intervals with

$$\bigcup_{k=1}^{\infty} [E_k \cup F_k \cup G_k \cup H_k] = [1, \infty),$$

to be defined by induction. The function $g(t)$ will be constructed by the identity

$$\begin{aligned} g(t) = \sum_{n=1}^{\infty} \left(\frac{1}{t}\right)\chi_{E_n}(t) + r\left(F_n, \frac{1}{t}, \frac{-1}{t}; t\right)\chi_{F_n}(t) \\ + \left(-\frac{1}{t}\right)\chi_{G_n}(t) + \sum_{n=1}^{\infty} r\left(H_n, -\frac{1}{t}, \frac{1}{t}; t\right)\chi_{H_n}(t.) \end{aligned} \tag{5.5}$$

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The sets E_n, F_n, G_n, H_n are defined inductively to satisfy the following requirements:

(1) $1 - \int_1^u g > 0$, $\quad u \in V_k \equiv E_k \cup F_k \cup G_k \cup H_k, k \geqq 1$;

(2) $1 - \int_1^u g < 1/(k+1)$, $\quad u \in F_k, k \geqq 1$;

(3) $1 - \int_1^u g = 1/k$, $\quad$ at the right endpoint c_k of G_k, $k \geqq 1$;

(4) $1 - \int_1^u g \geqq 1/(k+1)$, $\quad u \in H_k, k \geqq 1$;

(5) $1 - \int_1^u g < 1/k$, $\quad$ at the right endpoint d_k of H_k, $k \geqq 1$;

(6) $\bigcup_{k=1}^{n} V_k \subseteq \{u : 1 \leqq u \leqq e^{2n}\}$;

(7) $\int_{V_k} \left(1 - \int_1^u g\right)^{-1} du \geqq a_k$, $\quad$ the sequence $\{a_k\}$ being entirely arbitrary, but we will select

$$a_n = \int_{n+2}^{n+3} 2xe^{4x}dx, n \geqq 1.$$

The function $g(t)$ is further required to satisfy the following:

(8) $\int_1^\infty g = 1$,

(9) $|g(t)| \leqq 1/t, t \geqq 1$.

Once $g(t)$ is defined on $[1, \infty)$, it is easy to extend its definition to $[0, \infty)$: simply extend $g(t)$ to $[0, 1]$ so that g is continuous, $g(t) \equiv 0$ near 0, and

$$1 - \int_1^u g > 0$$

for $u \geqq 0$. Since $\Psi(u)$ is now defined on $[0, \infty)$ and constant near 0, it follows that $\Psi(u)$ can be extended to $(-\infty, \infty)$ by the relation $\Psi(-u) = \Psi(u)$.

The inductive definition of g is tedious to write down, and all computations will be left to the reader. The case $k = 1$ is similar to the induction step, and only the latter will be given here.

Suppose the pairwise disjoint intervals E_k, F_k, G_k, H_k have been constructed for $k = 1, \ldots, n-1$, and relations (1)–(7) are valid. Let $c = d_{n-1}$ be the right endpoint of H_{n-1}, and put $E_n = [c, t)$, where t is to be determined. One has

$$1 - \int_1^s g = \left(1 - \int_1^c g\right) - \int_c^s g = \log\left(\frac{mc}{s}\right)$$

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where

$$m = \exp\left(1 - \int_1^c g\right), c \leqq s < t.$$

One calculates

$$\int_c^t \left(1 - \int_1^s g\right)^{-1} ds = \int_{\log(mc/t)}^{\log m} \frac{mcdu}{ue^u} \geqq c \int_{\log(mc/t)}^{\log m} \frac{du}{u}.$$

The integral on the right diverges if $t = mc$, so by (5) one sees that

$$t < mc \leqq e^{2n-1}.$$

Select t to make

$$\int_c^t \left(1 - \int_1^s g\right)^{-1} ds > a_n, \quad \text{and} \quad 1 - \int_1^t g < \frac{1}{n+1}.$$

If $F_n = [t, t + \epsilon)$ and $\epsilon > 0$, and F_n is small enough, then $t + \epsilon < e^{2n-1}$ and

$$1 - \int_1^u g < \frac{1}{n+1}$$

for $u \in F_n$. Put $t_1 = t + \epsilon$ and $G_n = (t_1, t_2)$, where t_2 is to be determined. Since

$$1 - \int_1^{t_2} g = \log\left(\frac{kt_2}{t_1}\right)$$

with $\log k < 1/(n + 1)$, t_2 can be chosen less than $t_1 e$ to satisfy relation (3). Hence

$$t_2 < t_1 e \leqq e^{2n}.$$

Let H_k be a small interval $[t_2, t_2 + \delta]$; for $\delta > 0$ small enough, the remark concerning $r([a, b], f_1, f_2; t)$ shows that (5) can be satisfied, and by choosing δ even smaller, if necessary, (4) can be satisfied, as well as $t_2 + \delta < e^{2n}$. By the induction hypothesis and the above construction, (6) holds, completing the induction.

It should be mentioned that (2), (3), (4) and the construction imply (1), whereas (8) follows by construction, and (9) by the definition of g. To make

$$\int_{H_n} \Psi = O(1) \text{ as } n \to \infty,$$

it suffices to make H_n smaller, as this has no effect on relations (4), (5), (6).

Let E_n, F_n, G_n, H_n be the sets constructed to satisfy relations (1)–(9), and let $\{c_n\}$ be the sequence in relation (3). Select a sequence $\{\alpha_n\}$ of numbers in (0, 1) satisfying the relation

$$(10) \quad \sum_{n=2}^{\infty} \alpha_n c_n^2 \sup\{\Psi(u) : c_{n-1} \leqq u \leqq c_n\} \leqq 1.$$

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Put

$$A_n = [n + \alpha_n, n + 1], \qquad B_n = [n, n + \alpha_n). \tag{5.6}$$

Let $z(t)$ be defined by (5.4), suppose n is a large integer and $t \in [n, n + 1]$. Then $c_{n-1} \leqq z(t) \leqq c_n$ by (5.4), and relation (7) yields

$$\begin{aligned}\int_1^{z(t)} \Psi(u)du &\geqq \int_1^{c_{n-1}} \Psi(u)du \\ &\geqq \sum_{j=1}^{n-1} \int_{V_j} \Psi(u)du - \int_{H_{n-1}} \Psi(u)du \\ &\geqq \sum_{j=1}^{n-1} a_j + O(1).\end{aligned}$$

By relations (9) and (10) one computes the estimate

$$\begin{aligned}\int_1^t z^2(s)\Psi(z(s))ds &\leqq \sum_{k=1}^{n+1} \alpha_k c_k^2 \sup\{\Psi(u) : c_{k-1} \leqq u \leqq c_k\} \\ &\qquad + \sum_{k=1}^{n+1} \int_{k+\alpha_k}^{k+1} z^2(s)\Psi(z(s))ds \\ &\leqq 1 + \sum_{k=1}^{n+1} kc_k^2 \\ &\leqq (a_{n-1} + a_{n-2} + \ldots + a_1)/2 + O(1).\end{aligned}$$

Combining these two calculations gives

$$\int_1^{z(t)} \Psi(u)du - \int_1^t z^2(s)\Psi(z(s))ds \geqq \int_3^t xe^{4x}dx + O(1) \tag{5.7}$$

as $t \to \infty$.

Let $P(t)$ be the left side of inequality (5.7) and define $p(t) = P'(t)$. Then put

$$f(x, y) = \begin{cases} x/\Psi(y/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $\int^\infty p = \infty$ by virtue of inequality (5.7), and f is continuous, because $1/\Psi$ is bounded. One easily verifies the relations $xf(x, y) > 0$ $(x \neq 0)$ and $f(\lambda x, \lambda y) = \lambda f(x, y)$.

To prove that f is locally Lipschitzian, it suffices to prove that f has bounded partials in $\mathbf{R}^2$. Now

$$f(x, y) = x\left(1 - \int_1^{|y/x|} g\right) \text{ for } x \neq 0,$$

and

$$\int_1^\infty g = 1,$$

so $f_x(0, b) = 0$ for $b \neq 0$,

$$f_x(0, 0) = 1 + \int_0^1 g.$$

If $a \neq 0$, then b/x and b/a have the same sign for x near a, and one calculates

$$f_x(a, b) = \left(1 - \int_1^{|b/a|} g\right) + \left|\frac{b}{a}\right| g\left(\left|\frac{b}{a}\right|\right).$$

Surely $f_y(0, b) = 0$, and

$$f_y(a, 0) = \lim_{y \to 0} \frac{1}{y}\left(a \int_1^0 g - a \int_1^{|y/a|} g\right) = \lim_{y \to 0} -\frac{a}{y} \int_0^{|y/a|} g = \lim_{t \to 0} -\frac{1}{t} \int_0^{|t|} g = 0,$$

because $g \equiv 0$ near 0. Finally, if $b \neq 0$, $a \neq 0$, then

$$f_y(a, b) = -\operatorname{sgn}(b/a) g(|b/a|),$$

so the partials are bounded by (8) and (9).

The function

$$y(t) = \exp\left(-\int_1^t z(s) ds\right)$$

is of class C^2 and satisfies

$$y'' + p(t) f(y, y') = 0.$$

Therefore, $\int^\infty p = \infty$ and $f \in \mathscr{F}$ is not strong enough to insure strong oscillation. In particular, Bihari's theorem [**3**] cannot be extended to arbitrary continuous $p(t)$ without additional assumptions on $p(t)$ and/or $f \in \mathscr{F}$.

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University of Utah,
Salt Lake City, Utah