ON PRODUCTS OF SYMMETRIES

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In **(2)** Halmos and Kakutani proved that any unitary operator on an infinite-dimensional Hilbert space is a product of at most four symmetries (self-adjoint unitaries). It is the purpose of this paper to show that if the unitary is an element of a properly infinite von Neumann algebra A (i.e., one with no finite non-zero central projections), then the symmetries may be chosen from A. A principal tool used in establishing this result is Theorem 1, which was proved by Murray and von Neumann **(6**, 3.2.3**)** for type II_1 factors; see also **(3**, Lemma 5**)**. The author would like to thank David Topping for raising the question, and for several stimulating conversations on the subject. He is also indebted to the referee for several helpful suggestions.

The terminology used will be that of Dixmier **(1)**, and this book will be the basic reference for standard results concerning von Neumann algebras. The proof of Theorem 1 uses the dimension function; for information concerning this topic the reader may consult **(5)**.

LEMMA. *Let A be a non-abelian von Neumann algebra and n a normal operator in A. Then there is a projection $p \in A$ such that $0 \neq p \precsim 1 - p$ and $pn = np$.*

Proof. There is by the spectral theorem a non-central projection $q \in A$ that commutes with n. By the theorem of comparability of projections **(1**, p. 228**)**, we can select a projection e in the centre Z of A such that $qe \precsim q'e$ and $q'e' \precsim qe'$ (where $p' = 1 - p$ for any projection p). Since $q \notin Z$, either $qe \neq 0$ or $q'e' \neq 0$ and we can take p to be one of these.

THEOREM 1. *Let A be a finite von Neumann algebra, and let n and q be respectively a normal operator and a projection in A. Then there is a projection $p \in A$ such that $p \sim q$ and $pn = np$.*

Proof. Assuming that $q \neq 0$, we establish first the existence of a projection $p \in A$ commuting with n such that $0 \neq p \precsim q$. Let D be the normalized dimension function on the projections of A. By passing to a suitable direct summand of A, we may suppose that there is $\epsilon > 0$ with $D(q)(x) \geqslant \epsilon$ for all points x of the spectrum of A. It follows that the central support of q is 1. This implies that $e \precsim q$ for any abelian projection e **(1**, p. 251**)**. We may therefore assume that no non-zero abelian projection commutes with n. Invoking the lemma, there exists a non-zero projection $s_1 \in A$ such that $s_1 \precsim 1 - s_1$ and s_1 commutes with n. Suppose that non-zero projections $s_1, s_2, \ldots, s_k \in A$ have been chosen, each commuting with n, and satisfying $s_i \leqslant s_{i-1}$ and $s_i \precsim s_{i-1} - s_i$. Since s_k

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commutes with n, the algebra $s_k A s_k$ is non-abelian, and so the lemma implies the existence of a non-zero projection $s_{k+1} \in s_k A s_k$ such that $s_{k+1} \precsim s_k - s_{k+1}$ and s_{k+1} commutes with $s_k n s_k$. It follows that s_{k+1} commutes with n and that $D(s_{k+1})(x) \leqslant \frac{1}{2}D(s_k)(x)$ for all x. After finitely many steps, one has $D(s_n)(x) < \epsilon$ for all x, and so $0 \neq s_n \precsim q$ as required.

To complete the proof, select by Zorn's lemma a maximal well-ordered ascending family $\{p_\alpha\}$ of projections in A, each commuting with n and satisfying $p_\alpha \precsim q$. If p is the supremum of the p_α, then $pn = np$ and $p \precsim q$ by (**4**, Lemma 6.4) since A is finite. Suppose that $p \sim p_0 \leqslant q$ and $q = p_0 + r$ with $r \neq 0$. Then $p' \sim r + q'$ (**1**, p. 243), so $r \sim s \leqslant p'$ for some projection $s \in A$. By the first paragraph, there is in $p'Ap'$ a projection t commuting with n such that $0 \neq t \precsim s$. But then $p + t$ commutes with n and satisfies

$$p + t \precsim p + s \sim p_0 + r = q,$$

contradicting the maximality of $\{p_\alpha\}$. Hence $r = 0$, $p \sim q$, and the proof is complete.

We note that the finiteness hypothesis cannot be dropped from this theorem. For example, let A be the algebra of all bounded linear transformations on $L^2[0, 1]$, let n be multiplication by the independent variable, and let q be the projection on a finite-dimensional subspace.

The following remark will be needed several times in what follows. If p is a properly infinite projection in a von Neumann algebra A, and if $p_1, p_2, \ldots$ are projections in A with $p_n \precsim p$ for all n, then $\sup p_n \precsim p$. To see this, let $e_n = \sup\{p_1, p_2, \ldots, p_n\}$, and let $f_1 = e_1$ and $f_n = e_n - e_{n-1}$ for $n > 1$. Then

$$f_n = \sup\{e_{n-1}, p_n\} - e_{n-1} = e'_{n-1} - \inf\{e'_{n-1}, p'_n\} \sim p_n - \inf\{p_n, e_{n-1}\}$$

by (**1**, cor. 1, p. 227), so that $f_n \precsim p_n \precsim p$. Since p is properly infinite, it is the sum of a sequence $g_1, g_2, \ldots$ of pairwise orthogonal projections in A with $g_n \sim p$ for all n (**1**, cor. 2, p. 319). Then $f_n \precsim g_n$, and therefore

$$\sup p_n = \sum f_n \precsim \sum g_n = p.$$

THEOREM 2. *Let A be a non-abelian von Neumann algebra and n a normal operator in A. Then there exist non-zero orthogonal equivalent projections $p, q \in A$ each commuting with n.*

Proof. Suppose first that there is in A no non-zero finite projection that commutes with n. Then every non-zero projection commuting with n is properly infinite (**1**, p. 245). By the lemma, there is a projection $p \in A$ such that $pn = np$ and $0 \neq p \sim p_0 \leqslant p'$. We are going to take for q the projection on the least reducing subspace for n which contains the range of p_0. This projection may be constructed (non-spatially) as follows. Let W be the multiplicative subsemigroup of A generated by n and n^*; for each $w \in W$ let $q(w)$ be the range projection of wp_0, and let q be the supremum of the $q(w)$. Then $q(w) \in A$ for all $w \in W$, so that $q \in A$. Moreover, $q(w)$ is equivalent to the range projection of

$(wp_0)^*$ (**1**, p. 226), so that $q(w) \precsim p_0$. Therefore $q \precsim p_0$, by the remark preceding the theorem. On the other hand, $p_0 = q(1) \leqslant q$, so that $q \sim p_0$ and $q \sim p$. The range projections of nq and n^*q are both subprojections of q, so that $qnq = nq$, $qn^*q = n^*q$, and $qn = nq$. Finally, $p'wp_0 = wp'p_0 = wp_0$, $q(w) \leqslant p'$ for all $w \in W$, $q \leqslant p'$, and q is orthogonal to p.

We can therefore suppose that A contains a finite non-zero projection s which commutes with n. Consider first the case in which s is non-abelian. By the lemma there is a projection $p \in sAs$ commuting with n such that $0 \neq p \sim p_0 \leqslant s - p$. Since $(s-p)A(s-p)$ is finite, it contains by Theorem 1 a projection $q \sim p_0$ which commutes with n. Thus p and q are as required. Suppose finally that s is abelian. We can by Zorn's lemma assume that it is a maximal projection that is abelian and commutes with n. By hypothesis $s' \neq 0$. If $s'As'$ contains a finite non-zero projection r commuting with n, then $s + r$ is finite in A, non-abelian (by the maximality of s), and commutes with n, so we may argue as above. Otherwise, we proceed with $s'As'$ as in the first paragraph.

If, in the above situation, the algebra $A \cap \{n\}'$ is non-abelian, then the theorem follows trivially. The following example is due to Topping. Let M be a maximal abelian algebra, let n be normal with $n \notin M$ and such that n generates a maximal abelian algebra, and let A be the algebra generated by M and n. Then $A \cap \{n\}' = A \cap \{n\}'' = \{n\}''$ is (maximal) abelian.

THEOREM 3. *Let A be a properly infinite von Neumann algebra and n a normal operator in A. Then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of orthogonal equivalent projections in A, each commuting with n, and whose sum is* 1.

Proof. We first locate a projection $p \in A$ such that $pn = np$ and $p \sim 1 - p$. Let F be a maximal family of pairs (p_α, q_α) such that $\{p_\alpha, q_\alpha\}_\alpha$ is an orthogonal family of projections in A each commuting with n and such that $p_\alpha \sim q_\alpha$ for all α. If p and q are respectively the sums of the p_α and the q_α, then p and q are orthogonal, equivalent, and commute with n. If $r = (p+q)'$ is non-abelian, Theorem 2 applies to rAr to contradict the maximality of F. Thus r is abelian, and therefore finite. If we show that $q \sim q + r$, it will follow that $p \sim p'$ as required. Let e be the central support of q. Then $p, q \leqslant e$, so that $e' \leqslant r$, and therefore $e' = 0$ since r is finite and A is properly infinite. Thus $e = 1$, and so $r \precsim q$ (**1**, p. 251). But q is properly infinite, for if qf is finite for some central projection f, then $f = pf + qf + rf$ is finite, and hence $f = 0$. Therefore $q + r \sim q$, by the remark preceding Theorem 2.

Let $p_1 = p$. Since p' is properly infinite, so is the algebra $p'Ap'$. As above, there exists a projection $p_2 \in p'Ap'$ commuting with n such that $p_2 \sim p' - p_2$. It follows as before that p_2 is properly infinite, and that

$$p_2 \sim p_2 + (p' - p_2) = p' \sim p_1.$$

Iteration of this procedure yields a sequence $p_1, p_2, \ldots$ of orthogonal equivalent projections each commuting with n. If s is their sum, then $s' \leqslant p'_1 \sim p_1$, and replacing p_1 by $p_1 + s'$ gives a sequence with the required properties.

Corollary. *Any unitary operator u in a properly infinite von Neumann algebra A is a product of at most four symmetries from A.*

Proof. What follows is essentially a sketch of the proof given by Halmos and Kakutani. By Theorem 3 there exists a sequence $\{p_n| -\infty < n < \infty\}$ of orthogonal equivalent projections each commuting with u and with sum 1. For each n select $x_n \in A$ with $x_n^* x_n = p_n$ and $x_n x_n^* = p_{n+1}$, and let $v = \sum x_n$ and $w = v^*u$. Then v and w are unitary operators (in fact, "shifts") in A, and $u = vw$. One now shows that each is a product of two symmetries. If

$$s = \sum v^{1-2n}p_n \text{ and } t = \sum v^{-2n}p_n,$$

then s and t are symmetries in A with $st = v$. A similar construction for w completes the sketch of the proof.

The unitary group of the complex $n \times n$ matrices is generated by the symmetries and the scalar unitaries (but not by the symmetries alone). From this and (7, Theorem 1) one may readily show that the unitary group of a type I finite algebra is generated by its centre and the symmetries. The question is unresolved for type-II finite algebras.

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