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# ON COHOMOGENEITY ONE FLAT RIEMANNIAN MANIFOLDS

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**Abstract.** We study the topological properties of cohomogeneity one flat manifolds and their orbits. Among other results we prove that principal orbits of $R^n$ are isometric to $R^{n-1}$ or $S^k(c) \times R^{n-k-1}$. We show that if $M$ has one singular orbit, it is a totally geodesic submanifold of $M$ and if $M$ is orientable then there is at most one singular orbit.

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**1. Introduction.** Cohomogeneity one Riemannian manifolds have been studied from different points of view (see [**1**], [**2**], [**3**], [**4**], [**6**] and [**9**]). C. Searle [**9**] provided a complete classification of such manifolds when they are simply connected compact and of positive curvature of dimension less than or equal to six. F. Podesta and A. Spiro [**4**] studied the topological properties of cohomogeneity one negatively curved Riemannian manifolds. The aim of this paper is to study the topological properties of a flat cohomogeneity one manifold $M^n$. In Theorem 3.1 we take $M^n$ to be $R^n$ and prove that the orbits of $R^n$ are isometric to $R^{n-1}$ or $S^k(c) \times R^{n-k-1}$. Then we distinguish two cases. In Theorem 3.3 we prove that if $M$ is orientable then $M$ can admit at most one singular orbit. We show in Theorem 3.5 that if there is a unique singular orbit $B$ then $B$ is a totally geodesic submanifold of $M$ and is isometric to $R^k \times T^m$ and $\pi_1(M) = Z^m$. Also we prove that if there is not any singular orbit then each principal orbit is isometric to $R^k \times T^m$.

**2. Preliminaries.** Let $M$ be a complete Riemannian manifold of dimension $n$ and $G$ be a Lie group of isometries of $M$, which is closed in the full isometry group of $M$. We say that $M$ is of *cohomogeneity one* under the action of $G$, if $G$ has an orbit of codimension one.

For a general theory of cohomogeneity one manifolds we refer to [**1**], [**2**], [**3**], [**4**] and [**6**]. Here we briefly mention some facts about cohomogeneity one manifolds which will be needed in the sequel.

It is known that the orbit space $\Omega = M/G$ is a topological Hausdorff space homeomorphic to one of the following spaces: $R, S^1, R^+ = [0, \infty)$ and $[0, 1]$. We indicate by $k : M \to \Omega$ the projection to the orbit space.

Given a point $x \in M$, we say that the orbit $Gx$ is *principal* (resp. *singular*) if the corresponding image in the orbit space $\Omega$ is an internal (resp. boundary) point. A point $x$ whose orbit is principal (resp. singular) will be called *regular* (resp. *singular*) point. The subset of all regular points turns out to be an open and dense subset of $M$ denoted by $M_{\text{reg}}$ and the subset of singular points is denoted by $M_s$.

If $\Omega^o(\subset \Omega)$ is homeomorphic to an open interval of $R$ and $D$ is a principal orbit then $\Omega^o \times D$ is diffeomorphic to $k^{-1}(\Omega^o)$.

All principal orbits are diffeomorphic to each other and if $M/G = R$ then $M$ is diffeomorphic to $R \times D$, where $D$ is a principal orbit.

Each singular orbit is of dimension less than or equal to $n-1$. A singular orbit of dimension $n-1$ is called an *exceptional* orbit. Note that no exceptional orbit is simply connected, and if $M$ is simply connected no exceptional orbit may exist. If $M$ is orientable and all principal orbits, are connected, then any exceptional orbit is non-orientable.

If $B$ is the unique singular orbit of $M$ ($M/G = R^+$) then $\pi_1(M) = \pi_1(B)$.

Definition 2.1. A (complete) geodesic $\gamma : R \longrightarrow M$ on a Riemannian manifold of cohomogeneity one is called a *normal geodesic* if it crosses each orbit orthogonally.

It is known (see [**1**], [**2**] and [**4**]) that a geodesic $\gamma$ is normal if and only if it is orthogonal to the orbit $Gx$ at one point $x = \gamma(t)$. If $M/G = S^1$ or [0,1] then a normal geodesic $\gamma : R \longrightarrow M$ intersects each principal orbit $D$ infinitely many times (for infinitely many $t \in R$ we have $\gamma(t) \in D$), while if $M/G = R^+$, $\gamma$ intersects each principal orbit in two distinct points and if $M/G = R$ then $\gamma$ intersects a principal orbit exactly once.

For the sake of completeness, we quote the following theorems which we use in the proofs.

Theorem 2.2 [**7**, p. 374]. *Let $\tilde{M}$ be a space form of constant curvature $c \leq 0$ and let $M$ be a hypersurface in $\tilde{M}$ whose principal curvatures are constant. Then at most two of them are distinct.*

Theorem 2.3 [**8**, Theorem 1]. *Suppose $\tilde{M}$ is a real space form and $M$ a hypersurface in $\tilde{M}$. Suppose the principal curvatures of $M$ are constant and at most two are distinct. Then $M$ is congruent to an open subset of one of the standard examples.*

Remark 2.4. When $\tilde{M} = \mathbb{R}^{n+1}$, the standard examples are hyperplanes, spheres and cylinders over spheres. (See [**8**, Section 1]).

Theorem 2.5 [**6**, Theorem 6.1]. *Suppose $M$ is a complete Riemannian $G$-manifold that admits sections, and $N$ is a principal orbit of $M$. Then*

(a) *$exp(\nu(N_x))$ is a properly embedded totally geodesic submanifold of $M$ for all $x \in N$, ($\nu(N)$ is the normal bundle of $N$ in $M$).*

(b) *$\nu(N)$ is flat and has trivial holonomy; in fact if $v_1, \ldots, v_k$ is a basis for $\nu(N)_x$ then the $G$-invariant normal fields $\tilde{v}_i(gx) = dg_x(v_i)$ form a global parallel frame for $\nu(N)$.*

(c) *The principal curvatures of $N$ with respect to any parallel normal field are constant.*

Theorem 2.6. *Let $M$ be a complete hypersurface of the Euclidean space $R^n$, whose principal curvatures are constant. Then $M$ is isometric to one of the following spaces:*

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$$(1)\quad R^{n-1}; \qquad (2)\quad S^k(c) \times R^{n-1-k},\ 1 \leq k \leq n-1,\ c > 0.$$

*Proof*. Since $M$ is complete, the theorem is a simple consequence of Theorems 2.2 and 2.3.

THEOREM 2.7 [**10**, pp. 88, 89]. *Let $M^n$ be a connected homogeneous Riemannian flat manifold. Then $M^n$ is isometric to the product $R^m \times T^{n-m}$ of a Euclidean space with a flat Riemannian torus.*

THEOREM 2.8. *If $M$ is a simply connected cohomogeneity one Riemannian manifold of non-positive curvature, then there is at most one singular orbit.*

*Proof*. The proof is similar to the proof of Proposition 3.3 of [**4**]. To facilitate the reader we mention it briefly. Suppose that there exist two singular orbits $B_i = G/H_i$, $i = 1, 2$, $B_1 \neq B_2$, where $H_1, H_2$ are maximal compact in $G$. The two subgroups $H_1$ and $H_2$ are conjugate to each other, so there exist points $z_1 \in B_1$, $z_2 \in B_2$ with the same isotropy subgroup, say $H_1$. The unique geodesic joining $z_1$ to $z_2$ would be left pointwise fixed by $H_1$, so that $H_1$ should be a subgroup of a regular isotropy subgroup, which is a contradiction.

**3. Main results.** Throughout the following $M$ will denote a complete Riemannian manifold of dimension $n$ which is flat and of cohomogeneity one under the action of a connected Lie group $G$. If $M$ is not simply connected then $\tilde{M}$ the universal covering manifold of $M$ is of cohomogeneity one under the action of a Lie group $\tilde{G}$, a connected covering manifold of $G$ (see [**3**, p. 63]).

If $\tilde{\pi} : \tilde{G} \longrightarrow G, \pi : \tilde{M} \longrightarrow M$ are the covering maps then for each orbit $\tilde{D}$ in $\tilde{M}, \pi(\tilde{D})$ is an orbit of $M$, and for each orbit $D$ in $M$, we have $D = \pi(\tilde{D})$, for some orbit $\tilde{D}$ of $\tilde{M}$.

Each deck transformation $\varphi$ maps orbits to orbits. Thus if $\tilde{M}/\tilde{G} = R^+$ or $R$, then $\varphi$ induces an isometry $\varphi^*$ on $\tilde{M}/\tilde{G}$ such that $k\varphi(\tilde{D}) = \varphi^* k(\tilde{D})$, where $k : \tilde{M} \longrightarrow \tilde{M}/\tilde{G}$ is the projection onto the orbit space and $\tilde{D}$ is an orbit of $\tilde{M}$.

THEOREM 3.1. *Let $M = R^n$ be of cohomogeneity one under the action of a connected Lie group $G \subset Iso(R^n)$. Then either each principal orbit is isometric to $R^{n-1}$ and there is not any singular orbit or each principal orbit is isometric to $S^k(c) \times R^{n-k-1}$, $1 \leq k \leq n-1$, $k$ is fixed for all orbits, and the unique singular orbit is isometric to $R^{n-k-1}$.*

*Proof*. Let $D$ be a principal orbit. By using Theorems 2.5 (c) and 2.6 we get that $D$ is isometric to $R^{n-1}$ or $S^k(c) \times R^{n-k-1}$, for some $k$, $1 \leq k \leq n-1$ ($c$ depends on the orbit $D$). Since principal orbits are diffeomorphic to each other we get that each principal orbit of $M$ is isometric to $R^{n-1}$ or each principal orbit is isometric to $S^k(c) \times R^{n-k-1}$, $1 \leq k \leq n-1$. Now we consider two cases.

*Case* 1. Each principal orbit is isometric to $R^{n-1}$. By the fact that a line in $R^n$ which is normal to a hyperplane $R^{n-1}$, intersects it exactly once, we get that a normal geodesic $\gamma$ intersects each principal orbit exactly in one point, therefore we have $M/G = R$ and there is not any singular orbit.

*Case* 2. Each principal orbit is isometric to $S^k(c) \times R^{n-k-1}$, for some $k, 1 \leq k \leq n-1$. By the fact that $M = M_{\text{reg}} \cup M_s$, and $M_{\text{reg}}$ is open and dense in $M$, we get that for each $c \in (0, \infty)$ there exists a principal orbit $S^k(c) \times R^{n-k-1}$, so we conclude that $M_{\text{reg}} = \bigcup_{c \in (0,\infty)} S^k(c) \times R^{n-k-1} = (R^{k+1} - \{0\}) \times R^{n-k-1}$; hence $\{0\} \times R^{n-k-1}$ is the singular orbit.

From the proof of this theorem we have the following corollary.

COROLLARY 3.2. *If $R^n$ is of cohomogeneity one under the action of a connected Lie group $G \subset Iso(R^n)$, then the singular orbit (if there is any) is non-exceptional.*

The corollary is in accordance with the fact that if $M$ is simply connected, then no exceptional orbit may exist (see [**3**]).

THEOREM 3.3. *If $M$ is an orientable cohomogeneity one flat Riemannian manifold, then there is at most one singular orbit.*

*Proof*. By Theorem 2.8 we need only consider the case of $M$ not simply connected. Let the Lie group $G$ act by cohomogeneity one on $M$ and $\tilde{G}$ be the corresponding covering Lie group of $G$ which acts by cohomogeneity one on $\tilde{M} = R^n$ (the universal covering manifold of $M$). By Theorem 3.1 we have two cases.

*Case* 1. Each orbit of $\tilde{M}$ is isometric to $R^{n-1}$.

In this case each orbit $D$ of $M$ would be a totally geodesic hypersurface of $M$ (because $D = \pi(\tilde{D})$ for some orbit $\tilde{D}$ of $\tilde{M}$); therefore each orbit $D$ of $M$ is a homogeneous flat hypersurface of $M$. By Theorem 2.7 we get that $D$ is isometric to $R^k \times T^m, k + m = n - 1$, $D$ cannot be singular since no exceptional singular orbit is orientable; therefore in this case there is not any singular orbit.

*Case* 2. $\tilde{M}$ has a unique non-exceptional singular orbit (i.e $\tilde{M}/\tilde{G} = R^+$).

We show that $M$ cannot admit two singular orbits. Let $M$ admits two singular orbits.

Since $\tilde{M}/\tilde{G} = R^+$, a normal geodesic $\tilde{\gamma}$ in $\tilde{M}$ intersects each principal orbit in two points, while since $M/G = [0, 1]$ the normal geodesic $\gamma = \pi \circ \tilde{\gamma}$ intersects each principal orbit $D$ infinitely many times. So we conclude that $\pi^{-1}(D)$ has more than one connected component. Hence for a principal orbit $\tilde{D} \subset \pi^{-1}(D)$ there exists a deck transformation $\varphi$ such that $\varphi(\tilde{D}) \neq \tilde{D}$. Now let $\tilde{B}$ be the singular orbit of $\tilde{M}$. For dimension reasons we get that $\varphi(\tilde{B}) = \tilde{B}$. Let $\varphi^*$ be the induced isometry on the orbit space $\tilde{M}/\tilde{G} = R^+$. We have $\varphi^*(0) = \varphi^*(k(\tilde{B})) = k\varphi(\tilde{B}) = k(\tilde{B}) = 0$, therefore for each $t \in R^+, \varphi^*(t) = t$; so for each orbit $\tilde{D}$ in $\tilde{M}$ we would have $\varphi(\tilde{D}) = \tilde{D}$, which is a contradiction. Therefore $M$ cannot admit two singular orbits.

REMARK 3.4. In fact we proved that the singular orbit of $M$ (if there is any) is non-exceptional.

THEOREM 3.5. *Let $M$ be a flat non-simply connected cohomogeneity one Riemannian manifold under the action of a Lie group $G \subset Iso(M)$,*

(a) *If there is a unique singular orbit $B$, then $B$ is a totally geodesic submanifold of $M$ and is isometric to $R^k \times T^m$ for some non-negative integers $m, k$ and $\pi_1(M) = Z^m$.*

(b) *If there is no singular orbit, then each principal orbit is isometric to $R^k \times T^m$ for some non-negative integers $m,k$.*

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(c) *In the case* (b)*, if $M/G = R$ then $M$ is diffeomorphic to $R^r \times T^t$ for some non-negative integers $r, t$, $r + t = n$.*

*Proof.* (a) Let $\tilde{M} = R^n$ be the universal covering manifold of $M$ and let $\tilde{G}$ be the corresponding covering Lie group of $G$, which acts by cohomogeneity one on $\tilde{M}$. By Theorem 3.1 we have two cases.

*Case* 1. $\tilde{M}$ has a unique singular orbit $\tilde{B}$ isometric to $R^\ell$, $\ell < n - 1$.

In this case the orbit $\pi(\tilde{B})$ in $M$ has dimension less than $n - 1$. Therefore it is a singular orbit. Since by assumption $M$ has only one singular orbit $B$, we have $B = \pi(\tilde{B})$. As $\tilde{B}$ is totally geodesic in $\tilde{M}$, $B$ is totally geodesic in $M$ and hence is flat. Therefore by Theorem 2.7 we get that $B$ is isometric to $R^k \times T^m$ for some non-negative integers $m, k, m + k = l$. Also $\pi_1(M) \cong \pi_1(B) \cong Z^m$ by the preliminaries.

*Case* 2. Each orbit of $\tilde{M}$ is isometric to $R^{n-1}$. From the fact that $B = \pi(\tilde{B})$ (for some orbit $\tilde{B}$ of $\tilde{M}$) we get that $B$ is a totally geodesic submanifold of $M$. So $B$ is flat and homogeneous. Hence, by Theorem 2.7, $B$ is isometric to $R^k \times T^m$ for some non-negative integers $m, k$, $m + k = n - 1$. So $\pi_1(M) \cong \pi_1(B) \cong Z^m$.

(b),(c). In this case $\tilde{M}$ does not have any singular orbit (because if $\tilde{M}$ admitted a singular orbit, by case 1 we get that $M$ admits a singular orbit). Therefore by Theorem 3.1 we get that each orbit $\tilde{D}$ of $\tilde{M}$ is isometric to $R^{n-1}$. Thus each orbit $D(= \pi(\tilde{D}))$ of $M$ is a totally geodesic submanifold of $M$. So $D$ is flat and homogeneous; thus, by Theorem 2.7, $D$ is isometric to $R^m \times T^t$, $m + t = n - 1$. If $M/G = R$, from the fact that $M$ is diffeomorphic to $R \times D$ we get that $M$ is diffeomorphic to $R^r \times T^t$, $r + t = n$.

Example 3.6. Suppose that $M = T^l \times R$, where $T^l$ is a flat $l$-torus, and $G = T^l$ acts on $T^l$ by translation and on $R$ trivially. Then $M$ is a cohomogeneity one flat manifold, $M/G = R$, each orbit is isometric to $T^l$ and $\pi_1(M) = Z^l$.

Example 3.7. Suppose that $M = S^1 \times R^{n-1}$, $n \geq 2$, and $G = R^{n-1}$ acts on $M$ by translation on $R^{n-1}$ and trivially on $S^1$. Then $M/G = S^1$, each orbit is isometric to $R^{n-1}$, $\pi_1(M) = Z$.

Example 3.8. Suppose that $M = S^1 \times R^{n-1}$, $n \geq 3$, and $G = S^1 \times O(n-1)$ acts on $M$ componentwise. Then $M$ is a cohomogeneity one flat manifold, each principal orbit is diffeomorphic to $S^1 \times S^{n-2}$, the unique singular orbit is $S^1$, and $M/G = R^+$.

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