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SHARP ESTIMATES OF APPROXIMATION BY SOME POSITIVE LINEAR OPERATORS

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Recently, Varshney and Singh [*Rend. Mat.* (6) 2 (1982), 219-225] have given sharper quantitative estimates of convergence for Bernstein polynomials, Szasz and Meyer-Konig-Zeller operators. We have achieved improvement over these estimates by taking moments of higher order. For example, in case of the Meyer-Konig-Zeller operator, they gave the following estimate

$$\|L_n(f)-f\| \le \left[\frac{2}{3\sqrt{3}} + \frac{2}{27}\right]\left(\frac{1}{\sqrt{n}}\right)\omega\left(f'; \frac{1}{\sqrt{n}}\right)$$

wherein $\|\cdot\|$ stands for sup norm . We have improved this result to

$$\|L_n(f)-f\| \le \left[\frac{2}{3\sqrt{3}} + \left(\frac{105}{8}\right)\left(\frac{4}{27}\right)^4\right]\left(\frac{1}{\sqrt{n}}\right)\omega\left(f'; \frac{1}{\sqrt{n}}\right) .$$

We may remark here that for this modulus of continuity $\omega(f'; 1/\sqrt{n})$ our result cannot be sharpened further by taking higher order moments.

1. Introduction

Varshney and Singh [2] have proved the following theorem.

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THEOREM. *Let* $-\infty < a < b < \infty$, p *be a fixed positive integer. Let* K_n *be a sequence of positive numbers and* $\{L_n\}_{n=1}^{\infty}$ *be a sequence of positive linear operators, all having the same domain* D *which contains the restrictions of* $1, t, t^2, \ldots, t^{2p}$ *to* $[a, b]$. *Suppose that* $\{L_n(1)\}$ *is bounded. Let* $f' \in D$ *be continuous in* $[a, b]$ *with modulus of continuity* $\omega(f'; \cdot)$. *Then, for* $n = 1, 2, \ldots$,

$$(1.1)\quad \|L_n(f)-f\| \le \|f\| \cdot \|L_n(1)-1\| + \|f'\| \cdot \mu_n^{(1)} \cdot \|L_n(1)\|^{\frac{1}{2}} + \omega\left(f';\, K_n\mu_1^{(1)}\right)\left\{\mu_n^{(1)}\|L_n(1)\|^{\frac{1}{2}} + \left[\left(\mu_n^{(p)}\right)^2/2p\cdot\left(K_n\mu_n^{(1)}\right)^{2p-1}\right]\right\}$$

where $\mu_n^{(r)} = \left\|L_n(t-x)^{2r}(x)\right\|^{\frac{1}{2}}$ *for* $r = 1, p$ *and* $\|\cdot\|$ *norm being* sup norm *over* $[a, b]$.

However if in addition $L_n(1)(x) = 1$ *and* $L_n(t)(x) = x$, *then*

$$(1.2)\quad \|L_n(f)-f\| \le \omega\left(f';\, K_n\mu_n^{(1)}\right)\left\{\mu_n^{(1)} + \left[\left(\mu_n^{(p)}\right)^2/2p\cdot\left(K_n\mu_n^{(1)}\right)^{2p-1}\right]\right\} .$$

Using the above results the best possible quantitative estimates of convergence for Bernstein polynomials, Szász and Meyer-Konig-Zeller operators are achieved in the following sections.

2. Bernstein polynomials

For $f \in C[0, 1]$, let the Bernstein operator of order n be

$$L_n(f)(x) = \sum_{k=0}^{n} B_{n,k}(x) f(k/n)$$

where

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} .$$

We prove the following lemma.

LEMMA. *Let the function* $T_{n,m}$ *be defined on* $[0, 1]$ *for positive integers* m *and* n *by*

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$$T_{n,m}(x) = \sum_{k=0}^{n} B_{n,k}(x)(k/n-x)^m$$

then

(2.1) $$nT_{n,m+1}(x) = x \cdot (1-x) \cdot \left(mT_{n,m-1}(x)+T'_{n,m}(x)\right) .$$

Proof. We have

$$T'_{n,m}(x) = \sum_{k=0}^{n} B'_{n,k}(x)(k/n-x)^m - mT_{n,m-1}(x) .$$

Because $x \cdot (1-x) \cdot B'_{n,k}(x) = B_{n,k}(x) \cdot (k-nx)$ for all $0 < k < n$, we write

$$\begin{aligned} x \cdot (1-x) \cdot T'_{n,m}(x) & \\ &= \sum_{k=0}^{n} x \cdot (1-x)B'_{n,k}(x) \cdot (k/n-x)^m - m \cdot x \cdot (1-x) \cdot T_{n,m-1}(x) \\ &= \sum_{k=0}^{n} B_{n,k}(x) \cdot n \cdot (k/n-x)^{m+1} - m \cdot x \cdot (1-x) \cdot T_{n,m-1}(x) \\ &= nT_{n,m+1}(x) - m \cdot x \cdot (1-x)T_{n,m-1}(x) . \end{aligned}$$

This leads to (2.1).

Using the above result we have

$$\mu_n^{(1)} = 1/(2\cdot\sqrt{n}) , \quad \mu_n^{(2)} \doteq (3)^{\frac{1}{2}}/4n \quad \text{and} \quad \mu_n^{(3)} \doteq \left(15/64n^3\right)^{\frac{1}{2}} .$$

Choosing $K_n = 2n^{-\alpha+\frac{1}{2}}$ for $0 < \alpha \le \frac{1}{2}$, $p = 3$ in (1.2), we obtain, for $f \in C^1[0, 1]$ and $n \ge 1$,

(2.2) $$\|L_n(f)-f\| \le \left[1 + \frac{5}{64\cdot\left(n^{5(\frac{1}{2}-\alpha)}\right)}\right] \frac{1}{2\cdot(\sqrt{n})} \omega\left(f'; n^{-\alpha}\right) .$$

We note that (2.2) is sharper than the following estimate of Varshney and Singh [2]:

$$\|L_n(f)-f\| \le \left[1 + \frac{3}{32\cdot\left(n^{3(\frac{1}{2}-\alpha)}\right)}\right] \frac{1}{2\cdot(\sqrt{n})} \omega\left(f'; n^{-\alpha}\right) .$$

3. Szász operators

For $f \in C[0, \infty)$, let the Szász operator of order n be

$$L_n(f)(x) = \sum_{k=0}^{\infty} S_{n,k}(x) f(k/n)$$

where

$$S_{n,k}(x) = \exp \cdot (-nx) \cdot (nx)^k / k! \ .$$

We now prove the following lemma.

LEMMA. *Let the function* $T_{n,m}$ *be defined on* $[0, \infty)$ *for positive integers* m *and* n *by*

$$T_{n,m}(x) = \sum_{k=0}^{\infty} S_{n,k}(x)(k/n-x)^m \ ;$$

then

$$nT_{n,m+1}(x) = x \cdot \left(mT_{n,m-1}(x)+T'_{n,m}(x)\right) \ . \tag{3.1}$$

Proof. We have

$$T'_{n,m}(x) = \sum_{k=0}^{\infty} S'_{n,k}(x)(k/n-x)^m - mT_{n,m-1}(x) \ .$$

Now, because $x \cdot S'_{n,k}(x) = (k-nx) \cdot S_{n,k}(x)$ for all $0 < k < n$, we write

$$\begin{aligned} xT'_{n,m}(x) &= \sum_{k=0}^{\infty} x \cdot S'_{n,k}(x)(k/n-x)^m - m \cdot xT_{n,m-1}(x) \\ &= \sum_{k=0}^{\infty} n \cdot S_{n,k}(x)(k/n-x)^{m+1} - m \cdot xT_{n,m-1}(x) \\ &= nT_{n,m+1}(x) - m \cdot xT_{n,m-1}(x) \ . \end{aligned}$$

This leads to (3.1).

Using the above result we have, for $x \in [0, \lambda]$, $0 < \lambda < \infty$,

$$\mu_n^{(1)} = (\lambda/n)^{\frac{1}{2}} \ , \quad \mu_n^{(2)} \doteq (3)^{\frac{1}{2}} \cdot (\lambda/n) \quad \text{and} \quad \mu_n^{(3)} \doteq \left(15 \cdot \lambda^3/n^3\right)^{\frac{1}{2}} \ .$$

Choosing $K_n = (1/\lambda)^{\frac{1}{2}}$, $p = 2$ in (1.2), we obtain, for

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$f \in C^1[0, \lambda]$ and $n \geq 1$,

(3.2) $$\|L_n(f)-f\| \leq \left[\sqrt{\lambda} + \frac{3\lambda^2}{4}\right] \frac{1}{\sqrt{n}} \omega\left(f'; n^{-\frac{1}{2}}\right) .$$

We note that (3.2) is sharper than the following estimate of Varshney and Singh,

$$\|L_n(f)-f\| \leq [\sqrt{\lambda} + \lambda/2] \frac{1}{\sqrt{n}} \omega\left(f'; n^{-\frac{1}{2}}\right) ,$$

for all $\lambda < 2/3$.

Again choosing $K_n = (1/\lambda)^{\frac{1}{2}}$, $p = 3$ in (1.2), we obtain, for $f \in C^1[0, \lambda]$ and $n \geq 1$,

(3.3) $$\|L_n(f)-f\| \leq [\sqrt{\lambda} + 5\lambda^3/2](1/\sqrt{n})\omega\left(f'; n^{-\frac{1}{2}}\right) ,$$

which is sharper than (3.2) for all $\lambda < 3/10$.

It may be observed that if we have to approximate $f \in C^1[0, \lambda]$ when λ is very small (possibly less than $3/10$), we may have still sharper estimates of the approximation by using higher values of p in (1.2).

4. Meyer-Konig-Zeller operators

For $f \in C[0, \infty)$ let

$$L_n(f)(x) = \sum_{k=0}^{\infty} z_{n,k}(x) f\left(k/(n+k)\right)$$

be the Meyer-Konig-Zeller operator of order n , where

$$z_{n,k}(x) = \binom{n+k}{n}(1-x)^{n+1} \cdot x^k .$$

We note the following result of Rathore [1, (2.4), p. 213]:

(4.1) $$L_n\left(|t-x|^{\alpha}; x\right) \doteq \left(\Gamma(\alpha+1)/2/\Gamma(\tfrac{1}{2})\right) \cdot \left(2x\cdot(1-x)^2/n\right)^{\alpha/2} .$$

Using (4.1) we have

$$\mu_n^{(1)} \doteq (2/3\cdot\sqrt{3}\cdot\sqrt{n}) , \quad \mu_n^{(2)} \doteq (\sqrt{3}\cdot 4/27n) , \quad \mu_n^{(3)} \doteq \left(\sqrt{5}\cdot 8/81\cdot n^{3/2}\right)$$

and

$$\mu_n^{(4)} \doteq \left(\sqrt{105} \cdot 16/729n^2\right) .$$

Choosing $K_n \doteq 3\sqrt{3}/2$, $p = 4$ in (1.2), we obtain, for $f \in C^1[0, 1)$ and $n \geq 1$,

$$\|L_n(f)-f\| \leq \left[2/3\sqrt{3}+(105/8)\cdot(4/27)^4\right](1/\sqrt{n})\omega(f'; 1/\sqrt{n}) . \tag{4.2}$$

We note that this is sharper than the following estimate given by Varshney and Singh [2]:

$$\|L_n(f)-f\| \leq [2/3\sqrt{3}+2/27](1/\sqrt{n})\omega(f'; 1/\sqrt{n}) .$$

We may remark that the estimate in (4.2) cannot be bettered by taking higher value of p in (1.2) for modulus of continuity $\omega(f'; 1/\sqrt{n})$.

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