THE FIELD STRUCTURE OF THE REAL NUMBER SYSTEM

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1

It is well-known that the real number system can be characterised as a topological space [1], [3], as an ordered set [2], and as an ordered field [4]. It is the aim of this note to give two characterisations of the system purely as a field (see Theorems 4 and 9) without any extra notion of order, topology, *et cetera.*

2

We shall say that two sub-fields F_1, F_2 of a field F are square-isomorphic if there exists an isomorphism between the two sub-fields such that whenever $x \in F_1$ is the square of some element of F, its image in F_2 is also the square of some element of F and vice versa. It is to be noted that our definition is relative to the containing field F.

THEOREM 1. *No two distinct subfields F_1, F_2 of the real number field R are square-isomorphic.*

PROOF. We may without loss of generality suppose that $F_1 \not\subset F_2$. Let α (which can be supposed to be greater than zero) be an element belonging to F_1 but not to F_2. If there exists a square isomorphism f between F_1 and F_2, let $\beta = f(\alpha)$. We notice first that any non-zero element of F_1 or F_2 is a square element (in R) if and only if it is positive. Let θ be a rational number between α and β. Then either $\alpha < \theta < \beta$ or $\alpha > \theta > \beta$. Also $f(\theta) = \theta$ since f must induce an automorphism in its prime field and the only automorphism of the rational number field is the identity. We find in the first case that the square element $\theta - \alpha$ of F_1 goes over into the non-square element $\theta - \beta$ and that in the second case the square element $\alpha - \theta$ of F_1 goes over into the non-square element $\beta - \theta$ of F_2 — which contradict the square isomorphism of f, and thus establishes our result.

REMARKS. It is to be noted that every real-closed subfield of the real number field has the property (say P) stated in Theorem 1, since the positive elements and the square elements coincide in every real-closed field [4].

We now wish to introduce the notion of a real closed field — as a field having any one of the following three equivalent properties:

1. It has an algebraic closure which is a proper finite extension.

2. Any sum of non-zero squares is non-zero and this fails in every algebraic extension.

3. It can be ordered but no algebraic extension thereof can be ordered. (In this case the order is unique.)

THEOREM 2. *A real closed (and therefore ordered) field F has the property P if and only if it is archimedian ordered.*

PROOF. The 'if' part follows from the above remarks and the well-known fact that every archimedean ordered field is isomorphic to a sub-field of the real number field.

If it were not archimedean ordered, let x be an element greater than all the integers $n = 1, 2, 3, \cdots$. It is easy to see that x is transcendental over Γ. ($x^n + \cdots + a_n > 0$ whenever the a_i's are all rational.) We proceed to show that the sub-fields $\Gamma(x)$ and $\Gamma(x^2)$ are square isomorphic. We notice also that the typical element

$$\frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m}$$

of $\Gamma(x)$ is a positive element (and hence a square element) if and only if $a_n b_m > 0$. In this case the element

$$\frac{a_0 + a_1 x^2 + \cdots + a_n x^{2n}}{b_0 + b_1 x^2 + \cdots + b_m x^{2m}}$$

is also > 0. The mapping $f(x) \to f(x^2)$ is seen to be a square isomorphism between the sub-fields $\Gamma(x)$ and $\Gamma(x^2)$ of F.

THEOREM 3. *If a real closed field F has the real number field R as a sub-field it must be non-archimedean ordered* [1] *(and hence cannot have property P).*

PROOF. If it were archimedean ordered (being real closed, it must be ordered) it must be isomorphic to a subfield of the real number field. Since the order in a real closed field is unique, it must also be order isomorphic to a subfield S of the real number field. In this isomorphism of F with S, R will be order isomorphic to a subfield Σ of S and hence of R. But an order

[1] This is perhaps not an immediate consequence of the fact that every archimedean ordered field is isomorphic to a sub-field of the real numbers.

isomorphism between R and Σ is also a square isomorphism in (real closed) R which contradicts Theorem 1.

THEOREM 4. *A field F is isomorphic to the field of real numbers if and only if it is a maximal real closed field with property P.*

PROOF. The necessary part follows from the well-known fact [4] that the real number field is real closed and Theorems 1 and 3.

The sufficiency part follows from the fact that such a field F is by Theorem 2 archimedean and so a subfield of the real number field. By its maximality it coincides with the real number field.

We shall merely mention the following characterization, the proof of which is fairly obvious now.

THEOREM 5. *A field F is isomorphic to the real number field R if and only if*

1) *F has transcendence degree c over its prime field.*

2) *F is real closed.*

3) *There exist transcendence bases B and C of F and R such that $\Gamma(B)$ and $\Gamma(C)$ are square isomorphic.*

3

We start by characterising the real closed subfields of dimension $\leqq 1$ of the real number field.

THEOREM 6. *The real number field and any other real closed subfield thereof has only one automorphism — the identity.*

PROOF. Any automorphism of this field is order (square) preserving since it is a real closed field. If α is an element such that $\alpha \to \beta \neq \alpha$ then there exists an element k between α and β, where k is rational. Then either $\alpha < k < \beta$ or $\alpha > k > \beta$. Then the positive element $k-\alpha$ $(\alpha-k)$ goes over into the negative element $k-\beta$ $(\beta-k)$ which yields the desired contradiction.

THEOREM 7. *Any non-archimedean ordered real closed field F of transcendence degree less than or equal to one, over its prime field, has a non-trivial automorphism.*

PROOF. The case < 1 is void. Let Γ be the prime field and let x be an element $>$ all integers n and hence also all elements of Γ. Then it is easily seen that $x^n+a_1x^{n-1}+\cdots+a_n > 0$ for all $a_1, \cdots a_n$ in Γ; and thus x is transcendental. Any element

$$\frac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m}$$

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of $\Gamma(x)$ is positive if and only if $a_0 b_0 > 0$. This shows that the mapping $x \rightarrow x+1$ induces an order preserving automorphism of $\Gamma(x)$. Since F is real closed and algebraic over $\Gamma(x)$, a standard procedure shows that this automorphism can be extended to F.

Since every archimedean ordered field (and only these) are imbeddable in the real number field, we now get the following

THEOREM 8. *A real closed field of transcendence dimension $\leqq 1$ allows of a unique automorphism if and only if it is archimedean ordered.*

As another purely field-characterisation of the real number field we can now prove

THEOREM 9. *A field F is isomorphic to the real number field R if and only if*

(i) *it is real closed,*

(ii) *every real closed subfield of F of transcendence dimension $\leqq 1$ allows of only the identity automorphism,*

(iii) *every real closed field of transcendence dimension $\leqq 1$ allowing only the identity automorphism is isomorphic to some subfield of F.*

PROOF. *Necessity.* (i) is a classical property of the real numbers; (ii) has been established in Theorem 6; (iii) by Theorem 8 every such field is archimedean ordered and hence isomorphic to a subfield of the real numbers.

Sufficiency. First we note that F must be archimedean ordered: otherwise, let $x \in F$ be greater than all integers n, and hence all the elements of the prime (rational) field Γ. Since the relative algebraic closure of a subfield of a real closed field is also real closed [4], the relative algebraic closure of $\Gamma(x)$ is a real closed subfield of transcendence dimension one, which being non-archimedean allows a non-trivial automorphism, contradicting our hypothesis (ii). Thus F being archimedean must be a subfield of R. If $F \neq R$ and x is an element of R not in F, let S be the relative algebraic closure in R of $\Gamma(x)$. This is a real closed subfield of R of transcendence dimension $\leqq 1$ which (being a subfield of R and hence archimedean) allows no automorphism. So by hypothesis (iii) there exists a subfield Σ of F isomorphic to S. Since S is real closed so is Σ, and hence the isomorphism is square-preserving in the real closed field R. That is, the two subfields S and Σ of R are square-isomorphic, which is impossible. Thus $F = R$ and our proposition is proved.

References

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