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LETTERS TO THE EDITOR

ON ARNOLD'S TREATMENT OF MORAN'S BOUNDS

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Abstract

We prove a conjecture of Arnold (1968) which simplifies the determination of an optimal bound on absorption probability originally due to Moran (1960).

ABSORPTION PROBABILITY; MARKOV-CHAIN MODEL WITH SELECTION

In a problem concerning calculation of an optimal bound on absorption probability in Wright's Markov-chain model with selection (for the context refer to Moran (1960), Arnold (1968)), it is required to find

$$\underline{\theta}_N = \sup \{\theta \mid \pi_i(\theta) \leqq p_i,\ i = 0, 1, 2, \cdots, 2N\},$$

where for $\theta \in (0, \infty)$,

$$\pi_i(\theta) = \frac{1 - \exp(-\theta i/N)}{1 - \exp(-2\theta)}, \qquad p_i = \frac{(1+\sigma)i}{2N + \sigma i}, \qquad i = 0, \cdots, 2N.$$

Here N (integer, $\geqq 1$), and σ (>0) are assumed known. Since for $i = 1, 2, 3, \cdots, 2N-1$, $\pi_i(\theta)$ is an increasing function of $\theta > 0$, and $\lim_{\theta \downarrow 0^+} \pi_i(\theta) = i/2N < p_i$, $\lim_{\theta \uparrow \infty} \pi_i(\theta) = 1 > p_i$, it follows that there exists a unique $\theta_i^* > 0$ such that $\pi_i(\theta_i^*) = p_i$, with $\pi_i(\theta) < p_i$ for $\theta < \theta_i^*$, $\pi_i(\theta) > p_i$, $\theta > \theta_i^*$. Clearly

$$\underline{\theta}_N = \min \{\theta_1^*, \theta_2^*, \cdots, \theta_{2N-1}^*\}$$

where for each i, θ_i^* is the unique root in $(0, \infty)$ of the equation $f_\theta(i/2N) = 0$, where for $0 \leqq x \leqq 1$,

$$f_\theta(x) = (1+\sigma)x(1 - \exp(-2\theta)) - (1 + \sigma x)(1 - \exp(-2\theta x)).$$

Thus for fixed $i = 1, \cdots, 2N-1$, $f_\theta(i/2N) > 0$ for $\theta < \theta_i^*$; <0 for $\theta > \theta_i^*$. Using a different notation, Arnold (1968) arrived at this result; and conjectured that

$$\theta_1^* > \theta_2^* > \cdots > \theta_{2N-1}^*. \tag{1}$$

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We prove this conjecture, whence $\underline{\theta}_N = \theta^*_{2N-1}$, resulting in considerable saving in computational labour as perceived by Arnold, in determining $\underline{\theta}_N$.

Suppose the conjecture is false: then there exist i_1, i_2, $1 \leqq i_1 < i_2 \leqq 2N-1$, such that $\theta^*_{i_1} \leqq \theta^*_{i_2}$. Taking any henceforth fixed θ satisfying $\theta^*_{i_1} \leqq \theta \leqq \theta^*_{i_2}$, it follows that $f_\theta(i_2/2N) \geqq f_{\theta_{i_2}}(i_2/2N) = 0 = f_{\theta_{i_1}}(i_1/2N) \geqq f_\theta(i_1/2N)$. By the mean-value theorem, since $f(0) = 0$ there is a ξ_1, $0 < \xi_1 < i_1/2N$, such that $f'_\theta(\xi_1) \leqq 0$; and, since $f_\theta(1) = 0$, a ξ_2, $i_2/2N < \xi_2 < 1$, such that $f'_\theta(\xi_2) \leqq 0$. By applying the mean-value theorem again, there is a ξ_3, $i_1/2N < \xi_3 < i_2/2N$, such that $f'_\theta(\xi_3) \geqq 0$. Now applying the mean-value theorem to the function $f'_\theta(x)$, there exist numbers ζ_1, $(\xi_1 < \zeta_1 < \xi_3)$ and $\zeta_2(\xi_3 < \zeta_2 < \xi_2)$ such that $f''_\theta(\zeta_1) \geqq 0$, $f''_\theta(\zeta_2) \leqq 0$. Since $f''_\theta(x) = 4\theta \exp(-2\theta x)(\theta - \sigma + \theta\sigma x)$, there is a unique $x = x_0$ such that $f''_\theta(x_0) = 0$, and for $x > x_0$, $f''_\theta(x) > 0$, while for $x < x_0$, $f''_\theta(x) < 0$. Since we have $\zeta_1 < \zeta_2$ with $f''_\theta(\zeta_1) \geqq 0$, $f''_\theta(\zeta_2) \leqq 0$ a contradiction results, completing the proof.

It follows from (1) that

$$\begin{aligned}\bar{\theta}_N &= \inf\{\theta \mid \pi_i(\theta) \geqq p_i,\ i = 0, 1, 2, \cdots, 2N\} \\ &= \max\{\theta^*_1, \theta^*_2, \cdots, \theta^*_{2N-1}\} = \theta^*_1.\end{aligned}$$

It is of interest to find quantities such as $\underline{\theta} = \sup\{\theta; f_\theta(x) \geqq 0, 0 \leqq x \leqq 1\}$, so $\underline{\theta} \leqq \underline{\theta}_N$, and $\bar{\theta}$ defined analogously (so $\bar{\theta} \geqq \bar{\theta}_N$), which will lead to bounds at least as tight as those of Moran (1960) and likewise valid for *all* N. It is readily seen by a contradiction argument similar to the above that $f_\theta(x) \geqq 0$ for all $0 \leqq x \leqq 1$ if and only if $f'_{\underline{\theta}}(1) \leqq 0$, which leads to $\underline{\theta}$ as the unique root in $(0, \infty)$ of $\exp 2\theta - 1 = (1+\sigma)2\theta$, while $\bar{\theta}$ is the unique root of $1 - \exp - 2\theta = 2\theta/(1+\sigma)$, being the smallest θ in $(0, \infty)$ for which $f'_\theta(0) \leqq 0$. Note (without digression as to causes) that $\{2\bar{\theta}/(1+\sigma)\}$ is the *survival probability* of a Galton–Watson process with offspring p.g.f. $f(s) = \exp(1+\sigma)(s-1)$ and $e^{-2\bar{\theta}}$ is the extinction probability. The argument used to prove (1) can again be used to prove e.g. that $\underline{\theta}_N > \underline{\theta}_{N+1}$, whence as $N \to \infty$ $\underline{\theta}_N \downarrow \underline{\theta}$; and similarly $\bar{\theta}_N \uparrow \bar{\theta}$. Note also that Moran's (1960) explicit simple bounding interval, $[\sigma/(1+\sigma), \sigma]$, containing $[\underline{\theta}, \bar{\theta}]$, in particular leads to simple explicit bounds on the above survival probability.

References

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