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Perverse, Hodge and motivic realizations of étale motives

Florian Ivorra

Abstract

Let $k = \mathbb{C}$ be the field of complex numbers. In this article we construct Hodge realization functors defined on the triangulated categories of étale motives with rational coefficients. Our construction extends to every smooth quasi-projective k-scheme, the construction done by Nori over a field, and relies on the original version of the basic lemma proved by Beĭlinson. As in the case considered by Nori, the realization functor factors through the bounded derived category of a perverse version of the Abelian category of Nori motives.

Contents

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers and k be a field of characteristic zero with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$.

1.1 In the present article we consider the triangulated categories of étale motives $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$ over quasi-projective k-schemes. These categories were introduced by Ayoub in [Ayo07a, Ayo07b] and are the $\mathbb{Q}$-linear étale counterpart of the stable homotopy category of schemes of Morel and Voevodsky. The theory developed in [Ayo07b] provides these categories with a formalism of six operations. As shown in [MVW06, Ayo14], the category $\mathbf{DA}^{\text{ét}}(k, \mathbb{Q})$ is equivalent to the triangulated category of motives $\mathrm{DM}(k, \mathbb{Q})$ considered by Voevodsky. Hence the category $\mathrm{DM}_{\mathrm{gm}}(k, \mathbb{Q})$ of *geometric motives* of [Voe00] can be seen as a full subcategory of $\mathbf{DA}^{\text{ét}}(k, \mathbb{Q})$.

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1.2 As part of the vision of Grothendieck, these categories should have realization functors. For Betti cohomology (see [Ayo10]) or ℓ-adic cohomology (see [Hub00, Hub04, Ivo07, Ivo10, Ayo14]) such functors have been constructed.

On the Hodge-theoretic side, however, the picture is far from complete as the only realization functor available is defined over $\mathrm{Spec}(k)$. Let $\mathrm{MHS}^p_{\mathbb{Q}}$ be the Abelian category of polarizable mixed $\mathbb{Q}$-Hodge structures. Three different constructions of such a realization functor

$$\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{MHS}^p_{\mathbb{Q}})$$

have been given in the literature: one due to Levine [Lev98], one due to Huber [Hub00, Hub04] (a different approach is sketched in [DG05]) and one due to Nori (though unpublished, Nori's construction has been sketched in [Lev05, Nor]). The first two constructions do not use directly the category of polarizable mixed Hodge structures; they use instead as target the more flexible category of polarizable mixed Hodge complexes $\mathrm{D}^{\mathrm{b}}_{\mathscr{H}^p}$. This category was defined by Beĭlinson in [Beĭ86] where he also constructs an equivalence of categories

$$\mathrm{D}^{\mathrm{b}}(\mathrm{MHS}^p_{\mathbb{Q}}) \to \mathrm{D}^{\mathrm{b}}_{\mathscr{H}^p}.$$

However, such an equivalence is not available in higher dimensions, though partial results have been obtained in [Ivo15]. They are not sufficient to get a realization except perhaps on the triangulated category of smooth motives. Let us also mention that Levine's construction is also indirect as the source category is rather is own category of motives $\mathcal{DM}(k,\mathbb{Q})$ (known to be equivalent to $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ by [Lev98]).

1.3 The approach we generalize to higher dimensions in this work is the construction due to Nori. Recall that, using a Tannakian approach, he defined an Abelian category of mixed motives $\mathrm{NMM}(k)$ over k. Roughly speaking, being a motive in $\mathrm{NMM}(k)$ is the best structure that one can put on the relative homology of a pair of k-varieties. In particular, as the relative homology of pairs carries a (polarizable) mixed Hodge structure, one has a faithful exact functor

$$\mathrm{NMM}(k) \to \mathrm{MHS}^p_{\mathbb{Q}}.$$

Using the so-called 'basic lemma', a special case of a more general result on perverse sheaves due to Beĭlinson, Nori constructs a finer realization functor

$$\mathrm{DM}_{\mathrm{gm}}(k) \to \mathrm{D}^{\mathrm{b}}(\mathrm{NMM}(k)). \tag{1}$$

1.4 In this work we use the original version of the basic lemma proved by Beĭlinson to extend the construction of Nori to all smooth quasi-projective k-schemes. Recall that if $\mathscr{A}$ is an essentially small $\mathbb{Q}$-linear Abelian category, then the Yoneda functor

$$\mathsf{i} : \mathscr{A} \to \mathbf{Sha}(\mathscr{A},\mathbb{Q}),$$

where $\mathbf{Sha}(\mathscr{A},\mathbb{Q})$ is the Abelian category of additive sheaves of $\mathbb{Q}$-vector spaces on $\mathscr{A}$ for the topology defined by epimorphisms, is exact and fully faithful (this is the Gabriel–Quillen embedding). Moreover, it induces a fully faithful functor

$$\mathrm{D}^{\mathrm{b}}(\mathscr{A}) \to \mathrm{D}(\mathbf{Sha}(\mathscr{A},\mathbb{Q})).$$

Let us state our main results.

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Main results. *Let X be a smooth quasi-projective k-scheme and $\mathscr{M}(X)$ be either the category $\mathscr{N}(X)$ of perverse Nori motives (see [Ivo16]), the category $\mathscr{H}(X) := \mathrm{MHM}(X, \mathbb{Q})$ of mixed Hodge modules [Sai88, Sai90], or the category $\mathscr{P}(X)$ of perverse sheaves.*

(i) *We construct two triangulated functors defining an adjunction*

$$\mathsf{RL}_X^{\mathscr{M}} : \mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) : \mathsf{RR}_X^{\mathscr{M}},$$

the right-hand side being the unbounded derived category of $\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})$.

(ii) *Let $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q})$ be the full triangulated category of constructible étale motives. The left adjoint $\mathsf{RL}_X^{\mathscr{M}}$ then induces a triangulated functor*

$$\mathsf{RL}_X^{\mathscr{M}} : \mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$$

which returns (1) when $X = \mathrm{Spec}(\mathbb{C})$.

(iii) *If $a : Y \to X$ is a smooth quasi-projective morphism of k-schemes and Y is affine, then the image of the homological motive $\mathrm{M}_X(Y)$ under $\mathsf{RL}_X^{\mathscr{H}}$ is isomorphic to the Hodge homology complex $a_!^{\mathscr{H}} a^!_{\mathscr{H}}(\mathbb{Q}_X^{\mathscr{H}})$ where*

$$a_!^{\mathscr{H}} : \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(Y, \mathbb{Q})) \rightleftarrows \mathrm{D}^{\mathrm{b}}(\mathrm{MHM}(X, \mathbb{Q})) : a^!_{\mathscr{H}}$$

are the extraordinary adjoint functors part of the formalism of the six operations developed by Saito.

By construction there are $\mathbb{Q}$-linear faithful exact functors $\mathsf{R}_X^{\mathscr{H}} : \mathscr{N}(X) \to \mathrm{MHM}(X, \mathbb{Q})$ and $\mathsf{rat}_X^{\mathscr{H}} : \mathrm{MHM}(X, \mathbb{Q}) \to \mathscr{P}(X)$ (the latter associates its underlying perverse sheaf with a mixed Hodge module). The functors $\mathsf{RL}_X^{\mathscr{H}}$ and $\mathsf{RL}_X^{\mathscr{P}}$ are obtained from $\mathsf{RL}_X^{\mathscr{N}}$ via these functors.

However, for readers interested only in the Hodge realization, let us note that the present work is completely independent of [Ivo16]. The construction does not need the categories of perverse motives of that work and can be done directly using mixed Hodge modules. Note that for the usual Nori motives (i.e., Nori motives over $\mathrm{Spec}(k)$), a similar realization has been constructed independently in [CDS14].

1.5 This article is the first part of an ongoing project devoted to the construction of Hodge realization functors and their compatibility with the six operations. Such results are needed, for example, in [Wil12] (see [Wil12, Remark 3.12]) to compare the motivic intersection complex defined by Wildeshaus unconditionally in certain situations with its Hodge-theoretic analog.

More generally, they would allow us to deduce results in Hodge theory from results proved in the motivic context. As a motivating example, we sketch an application of this principle to the Hodge theory of symmetric spaces. Various compactifications of arithmetic quotients of Hermitian symmetric spaces have been introduced. Though such a quotient $\mathcal{X}$ is algebraic (by [BB66] it is the analytic space associated with the $\mathbb{C}$-points of a quasi-projective $\mathbb{C}$-scheme X), some of these compactifications, such as the Borel–Serre compactification introduced in [BS73] or its reductive variant $\overline{\mathcal{X}}^{\mathrm{rbs}}$ (see [Zuc01, Zuc82]), are purely topological. The Baily–Borel compactification $\overline{\mathcal{X}}^{\mathrm{bb}}$, introduced in [BB66], is the analytic space associated with the $\mathbb{C}$-points of a normal $\mathbb{C}$-scheme $\overline{X}^{\mathrm{bb}}$ and is therefore algebraic. Although the reductive Borel–Serre compactification is neither an algebraic variety nor even an analytic space, a procedure to construct a mixed Hodge structure on its cohomology has been given in [Zuc04]. This suggests that the reductive Borel–Serre compactification, in some sense, is not that far from being algebraic and might even be motivic.

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This has been shown in [AZ12]. More precisely, the two compactifications are related by a morphism $p : \overline{\mathcal{X}}^{\mathrm{rbs}} \to \overline{\mathcal{X}}^{\mathrm{bb}}$, and the main theorem of [AZ12] shows that the complex of sheaves $\mathrm{R}p_*\mathbb{Q}_{\overline{\mathcal{X}}^{\mathrm{rbs}}}$ on the Baily–Borel compactification is the Betti realization of an étale motive on $\overline{X}^{\mathrm{bb}}$. Hodge realization functors may then be used to produce a mixed Hodge structure of the cohomology of the Borel–Serre compactification, presumably the same as in [Zuc04] (see [AZ12, Remark 5.9]), or even in the relative context to see that $\mathrm{R}p_*\mathbb{Q}_{\overline{\mathcal{X}}^{\mathrm{rbs}}}$ itself underlies a complex of mixed Hodge modules on $\overline{X}^{\mathrm{bb}}$.

As another application, note that, using [IS13, AIS13], it should be possible to relate the nearby cycle functors in the mixed Hodge modules context to tubes in rigid analytic geometry or motivic integration theory. Hodge realization functors could also have applications to periods and motivic Galois groups (see, for example, [CDS14]).

2. Background on étale motives

In this section we briefly recall the construction of the categories $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ of étale motives over a quasi-projective k-scheme X and some of their properties. For model categories introduced by Quillen in [Qui67] we refer, for example, to [Hir03, Hov99].

2.1 The triangulated categories $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ were introduced in [Ayo07a, Ayo07b], where they are particular cases of the categories $\mathbf{SH}_{\mathfrak{M}}(X)$ obtained by choosing the topology to be the étale topology and the model category $\mathfrak{M}$ of coefficients to be the model category $\mathsf{Ch}(\mathbb{Q})$ of chain complexes of $\mathbb{Q}$-vector spaces. They are the $\mathbb{Q}$-linear étale counterpart of the stable homotopy category of X-schemes of Morel and Voevodsky (see [Jar00, MV01, Voe98]) and have been studied in further detail in [Ayo14].

They are part of a stable homotopic 2-functor $\mathbf{DA}^{\text{ét}}(-, \mathbb{Q})$ on the category of quasi-projective k-schemes as defined in [Ayo07a, Définition 2.4.13]. The theory developed by Ayoub in [Ayo07a, Ayo07b] provides, for these triangulated categories, a formalism of six operations as envisioned by Grothendieck.

We consider ultimately the full triangulated category $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q})$ of constructible motives, defined as the smallest triangulated subcategory of $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ stable by direct factors and containing the homological motives of smooth quasi-projective X-schemes. As shown in [Ayo07a, Scholie 2.2.34], these categories of constructible motives are stable under the six operations.

2.2 If $\mathscr{A}$ is an additive category, we denote by $\mathsf{Ch}(\mathscr{A})$ the category of chain complexes of objects in $\mathscr{A}$. Let Λ be a commutative ring. We denote simply by $\mathsf{Ch}(\Lambda) := \mathsf{Ch}(\mathsf{Mod}(\Lambda))$ the category of chain complexes of Λ-modules. We consider on $\mathsf{Ch}(\Lambda)$ the projective model category structure for which the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms.

2.3 Let X be a quasi-projective k-scheme. Let Sm/X be the category of smooth quasi-projective X-schemes. The construction of the category $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ starts with the category of presheaves of $\mathbb{Q}$-vector spaces $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ endowed with its projective model structure: the fibrations (respectively, equivalences) are the maps of presheaves of complexes $\mathscr{X} \to \mathscr{Y}$ such that $\mathscr{X}(Y) \to \mathscr{Y}(Y)$ is a fibration (respectively, an equivalence) in $\mathsf{Ch}(\mathbb{Q})$ for every $Y \in \mathsf{Sm}/X$.

A left Bousfield localization of this projective model structure provides the ét-local model structure. For the ét-local structure, the weak equivalences are the morphisms of complexes of

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presheaves that induce isomorphisms on the étale sheafification of the homology presheaves. Note that the étale sheafification functor then induces an equivalence of triangulated categories

$$a_{\text{ét}} : \mathsf{Ho}_{\text{ét}}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))) \xrightarrow{\sim} \mathrm{D}(\mathbf{Sh}_{\text{ét}}(\mathsf{Sm}/X, \mathbb{Q}))$$

where the left-hand side is the homotopy category for the ét-local projective model structure and the right-hand side is the unbounded derived category of the Abelian category of étale sheaves of $\mathbb{Q}$-vector spaces on Sm/X (see [Ayo07b, Corollaire 4.4.42] for a proof).

The ét-local model structure is then further localized with respect to the class of maps

$$\mathbf{A}^1_Y \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$$

where $Y \in \mathsf{Sm}/X$. The left Bousfield localization of the ét-local model structure with respect to the above maps is called the $(\mathbf{A}^1, \text{ét})$-local projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{eff},\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{\mathbf{A}^1,\text{ét}}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$$

is called the category of effective étale motives (with rational coefficients).

The last step of the construction is the stabilization. Let T_X be the presheaf

$$T_X := \frac{\mathbf{G}_{m,X} \otimes \mathbb{Q}}{X \otimes \mathbb{Q}}.$$

Consider the category $\mathsf{Spt}_{T_X}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$ of T_X-spectra of presheaves of complexes of $\mathbb{Q}$-vector spaces (see [Ayo07b, Définition 4.3.6]). The $(\mathbf{A}^1, \text{ét})$-local projective model structure induces on it a model structure (see [Ayo07b, Définition 4.3.29]): the so-called $(\mathbf{A}^1, \text{ét})$-local stable projective model structure. Its homotopy category

$$\mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) := \mathsf{Ho}_{(\mathbf{A}^1,\text{ét})-\text{st}}(\mathsf{Spt}_{T_X}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))))$$

is the triangulated category of étale motives with rational coefficients.

With a scheme $Y \in \mathsf{Sm}/X$ is associated a homological motive $\mathrm{M}_X(Y)$ given by the T_X-spectrum $\mathsf{Sus}^0_{T_X}(X \otimes \mathbb{Q})$.

2.4 It follows from [DHI04] that the fibrant objects for the ét-local projective model structure are the fibrant objects for the projective model structure that satisfy étale descent (see [DHI04, Définition 4.3] or [CD13, Définition 3.2.5, § 3.2.9] for the definition). Working with rational coefficients substantially simplifies the description of these ét-local fibrant objects.

It follows from [Voe10, Proposition 3.8] and [CD13, Theorem 3.3.23] that an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ is fibrant for the ét-local projective model structure if and only if it is fibrant for the projective model structure, satisfies elementary Galois descent (in the sense of Definition A.2), the B.G. property in the Nisnevich topology, and is such that $\mathscr{X}(\emptyset)$ is acyclic.

As a consequence the fibrant objects for the $(\mathbf{A}^1$, ét)-local projective model structure are the presheaves $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ that are fibrant for the projective model structure, satisfy Galois descent, the $\mathbf{A}^1$-B.G. property in the Nisnevich topology and such that $\mathscr{X}(\emptyset)$ is acyclic.

By [Mor12, Theorem A.14], if an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ satisfies the $\mathbf{A}^1$-B.G. property in the Zariski topology and the affine B.G. property in the Nisnevich topology, then it satisfies the B.G. property in the Nisnevich topology.

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This description may be reinterpreted as follows:

Proposition 2.1. *The* $(\mathbf{A}^1, \text{ét})$*-local projective model structure on the category* $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ *is the left Bousfield localization of the projective model structure with respect to the following classes of maps:*

$$
\begin{tikzcd}
\emptyset \otimes \mathbb{Q} \arrow[d] & (Y' \otimes \mathbb{Q})^G \arrow[d] & \mathbf{A}^1_Y \otimes \mathbb{Q} \arrow[d] & (V \otimes \mathbb{Q}) \arrow[r] & (U \otimes \mathbb{Q}) \oplus (E \otimes \mathbb{Q}) \arrow[d] \\
0 & Y \otimes \mathbb{Q} & Y \otimes \mathbb{Q} & & (Y \otimes \mathbb{Q})
\end{tikzcd}
\tag{2}
$$

where $r : Y' \to Y$ *is a Galois cover with Galois group* G *and*

$$
\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}
$$

is either an elementary Zariski square or an elementary affine Nisnevich square.

Remark 2.2. Let G be a finite group of order m. The functor $M \mapsto M^G$ is an exact functor from the category of left $\mathbb{Q}[G]$-modules to the category of $\mathbb{Q}$-vector spaces. Indeed, using restriction and corestriction, it is easy to see that the cohomology group $H^i(G, M)$ has m-torsion for every integer $i > 0$ and every left $\mathbb{Z}[G]$-module M. In particular, if M is a left $\mathbb{Q}[G]$-module, then $H^i(G, M) = 0$ for $i > 0$ which implies the exactness.

Proof of Proposition 2.1. It is enough to check that both model structures have the same fibrant objects. Let $\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))$ be the homotopy category associated with the projective model structure and $\mathscr{L}$ be a class of maps in $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$. Note that the fibrant objects for the left Bousfield localization with respect to $\mathscr{L}$ are the $\mathscr{L}$-local objects, that is, the presheaves $\mathscr{X}$ that are fibrant for the projective model structure and satisfy the following property: for every map $w : \mathscr{V} \to \mathscr{W}$ in $\mathscr{L}$ and every integer $n \in \mathbb{Z}$, the induced morphism of $\mathbb{Q}$-vector spaces

$$\mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))}(\mathscr{W}, \mathscr{X}[n]) \to \mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))}(\mathscr{V}, \mathscr{X}[n])$$

is an isomorphism.

Let us first observe that a fibrant presheaf $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ has elementary Galois descent (in the sense of Definition A.2) if and only it is $\mathscr{L}$-local with respect to the class $\mathscr{L}$ consisting of the second type of maps. Indeed, let $r : Y' \to Y$ be a Galois cover with Galois group G. Then one has a commutative square

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))}(Y \otimes \mathbb{Q}, \mathscr{X}[n]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))}((Y' \otimes \mathbb{Q})^G, \mathscr{X}[n]) \arrow[d, equal] \\
\mathsf{H}^n(\mathscr{X}(Y)) \arrow[r] & \mathsf{H}^n(\mathscr{X}(Y'))^G
\end{tikzcd}
$$

by the additivity of the functor $\mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})))}(-, \mathscr{X}[n])$ and the definition of the G-invariants in Appendix A. Our observation follows from the fact that the right-hand side is isomorphic to $\mathsf{H}^n(\mathscr{X}(Y')^G)$ by Remark 2.2.

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Similarly, the $\mathscr{L}$-local presheaves, when $\mathscr{L}$ consists of the third and the fourth type of maps, are exactly the fibrant presheaves that satisfy the $\mathbf{A}^1$-B.G. property in the Zariski topology and the affine B.G. property in the Nisnevich topology. Since $\mathscr{X}(\emptyset)$ is acyclic if and only if the map

$$\mathsf{H}^n(\mathscr{X}(\emptyset)) = \mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q})))}(\emptyset\otimes\mathbb{Q},\mathscr{X}[n]) \to \mathrm{Hom}_{\mathsf{Ho}(\mathbf{PSh}(\mathsf{Sm}/X,\mathsf{Ch}(\mathbb{Q})))}(0,\mathscr{X}[n]) = 0$$

is an isomorphism for every integer $n\in\mathbb{Z}$, we see that a presheaf $\mathscr{X}$ is $\mathscr{L}$-local when $\mathscr{L}$ consists of all four types of maps if and only if $\mathscr{X}$ is fibrant for the $(\mathbf{A}^1,\text{ét})$-local projective model structure. This concludes the proof. $\square$

2.5 The category $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$ is equivalent to the triangulated category of motives $\mathrm{DM}(k,\mathbb{Q})$ considered by Voevodsky. This result is a particular case of [CD13, Corollary 16.2.22] (see also [Ayo14, Théorème B.1]). Hence the category $\mathrm{DM}_{\mathrm{gm}}(k,\mathbb{Q})$ of *geometric motives* of [Voe00] can be seen as a full subcategory of $\mathbf{DA}^{\text{ét}}(k,\mathbb{Q})$, that contains the additive category $\mathsf{M}_{\mathsf{rat}}(k,\mathbb{Q})$ of Chow motives (over k with rational coefficients).

In [CD13, Définition 14.2.1] Cisinski and Déglise introduced the category $\mathrm{DM}_{\mathrm{B}}(X)$ of Beĭlinson motives. As shown in [CD13, Theorem 15.2.16] this category turns out to be equivalent to the previously defined $\mathbf{DA}^{\text{ét}}(X,\mathbb{Q})$. Note that the category of Beĭlinson motives is $\mathbb{Q}$-linear and was defined only after Ayoub introduced and constructed the formalism of six operations on the category of étale motives.

3. Perverse homology of pairs

Let k be a field of characteristic zero with a fixed embedding of fields $\sigma : k \hookrightarrow \mathbb{C}$.

3.1 Let X be a quasi-projective k-scheme. For the sake of brevity of notation, we denote by $\mathscr{P}(X)$ the category of perverse sheaves with rational coefficients (or the full subcategory $\mathscr{P}(X)^{\mathrm{go}}$ of perverse sheaves of geometric origin [BBD82, 6.2.4]) and by $\mathscr{H}(X)$ the category of mixed Hodge modules $\mathrm{MHM}(X,\mathbb{Q})$ introduced by Saito in [Sai88, Sai90] (or the full subcategory $\mathrm{MHM}(X,\mathbb{Q})^{\mathrm{go}}$ of mixed Hodge modules of geometric origin [Sai91, (2.6) Définition]).

Let $\mathscr{M}\in\{\mathscr{H},\mathscr{P}\}$. Recall that the derived categories of the Abelian categories $\mathscr{M}(X)$ are endowed with a six-functor formalism. More precisely, every morphism $f: Y\to X$ of quasi-projective k-schemes induces two pairs of adjoint functors

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)) \arrow[r, shift left, leftarrow, "f^*_{\mathscr{M}}"] \arrow[r, shift right, "f_*^{\mathscr{M}}"'] & \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y)) \arrow[r, shift left, "f_!^{\mathscr{M}}"] \arrow[r, shift right, leftarrow, "f^!_{\mathscr{M}}"'] & \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)). \end{tikzcd}$$

We denote by

$$\mathsf{H}^i_{\mathscr{M}} : \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)) \to \mathscr{M}(X), \quad i\in\mathbb{Z},$$

the cohomological functor associated with the usual t-structure. Recall that by definition $\mathbb{Q}_Y^{\mathscr{M}} = \pi^*\mathbb{Q}_k^{\mathscr{M}}$, where $\pi: Y\to\mathrm{Spec}(k)$ is the structural morphism and $\mathbb{Q}_k^{\mathscr{M}}$ is either the trivial Hodge structure of weight 0 or $\mathbb{Q}$. We set $\mathsf{H}_i^{\mathscr{M}} = \mathsf{H}^{-i}_{\mathscr{M}}$. In this section we fix an integer $d\in\mathbb{N}$ (later taken to be the dimension of X).

Remark 3.1. Let $\mathscr{A},\mathscr{B}$ be $\mathbb{Q}$-linear Abelian categories and $\mathscr{A}\to\mathscr{B}$ be a $\mathbb{Q}$-linear faithful exact functor. Then the induced functor $\mathrm{D}^{\mathrm{b}}(\mathscr{A})\to\mathrm{D}^{\mathrm{b}}(\mathscr{B})$ is conservative (i.e., a morphism in $\mathrm{D}^{\mathrm{b}}(\mathscr{A})$ is an isomorphism if and only if its image in $\mathrm{D}^{\mathrm{b}}(\mathscr{B})$ is an isomorphism). In particular, the canonical functor $\mathrm{D}^{\mathrm{b}}(\mathscr{H}(X))\to\mathrm{D}^{\mathrm{b}}(\mathscr{P}(X))$ is conservative.

3.2 A relative X-triple is a triple $(Y \xrightarrow{a} X, Z, i)$ where Y is a quasi-projective k-scheme, $a : Y \to X$ is a morphism of k-schemes, Z is a closed subset of Y, and $i \in \mathbb{Z}$ is an integer.

Definition 3.2. Let $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$ and (Y, Z, i) be a relative X-triple. We set

$$\mathsf{TH}_X^{\mathscr{M}}(Y \xrightarrow{a} X, Z, i) := \mathsf{H}_{\mathscr{M}}^{2d-i}(a_!^{\mathscr{M}}(u_*^{\mathscr{M}} u_{\mathscr{M}}^! a_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}})))$$

where $u : U \hookrightarrow Y$ is the open immersion of the complement of Z in Y.

Note that by definition $\mathsf{TH}_X^{\mathscr{M}}(Y \xrightarrow{a} X, Z, i)$ is an object in $\mathscr{M}(X)$ which depends only on the reduced structure of Y. If there is no possibility of confusion, we will also use the notation (Y, Z, i) to denote a relative X-triple and write $\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i)$ instead of $\mathsf{TH}_X^{\mathscr{M}}(Y \xrightarrow{a} X, Z, i)$. Recall that $\mathbb{Q}_Y^{\mathscr{M}}$ is not in general an object in $\mathscr{M}(Y)$. If Y is smooth over k of pure dimension n, then $\mathbb{Q}_Y^{\mathscr{M}}[n]$ belongs to $\mathscr{M}(Y)$.

Remark 3.3. With the notation of [Ivo16], one has

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) = \mathsf{T}_X^{\mathscr{M}}(Y, Z, i - 2d).$$

3.3 Let (Y_1, Z_1, i) and (Y_2, Z_2, i) be relative X-triples. Assume that $f : Y_2 \to Y_1$ is a morphism of X-schemes, such that $f(Z_2) \subseteq Z_1$. Then there are morphisms in $\mathscr{M}(X)$,

$$f_{\mathsf{TH}}^{\mathscr{M}} : \mathsf{TH}_X^{\mathscr{M}}(Y_2, Z_2, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y_1, Z_1, i), \tag{3}$$

such that if (Y_3, Z_3, i) is a relative X-triple, and $g : Y_3 \to Y_2$ is a morphism of X-schemes such that $g(Z_3) \subseteq Z_2$, then

$$f_{\mathsf{TH}}^{\mathscr{M}} \circ g_{\mathsf{TH}}^{\mathscr{M}} = (fg)_{\mathsf{TH}}^{\mathscr{M}}.$$

Recall that the morphism (3) is obtained as follows. Consider the commutative diagram

$$\begin{tikzcd}
f^{-1}(U_1) \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & U_2 \arrow[r, "u_2"] & Y_2 \arrow[d, "f"] \arrow[dr, "a_2"] & \\
U_1 \arrow[rr, "u_1"] & & Y_1 \arrow[r, "a_1"'] & X
\end{tikzcd}$$

in which U_1 (respectively, U_2) is the open complement of Z_1 (respectively, Z_2) and all arrows are the canonical morphisms. Using smooth base change and adjunction, we have a morphism in $\mathrm{D^b}(\mathscr{M}(Y_1))$,

$$\begin{aligned}
f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}}(u_2)_{\mathscr{M}}^!(a_2)_{\mathscr{M}}^! \to {} & f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}} u_*^{\mathscr{M}} u_{\mathscr{M}}^!(u_2)_{\mathscr{M}}^!(a_2)_{\mathscr{M}}^! \\
= {} & f_!^{\mathscr{M}}(u_2)_*^{\mathscr{M}} u_*^{\mathscr{M}} f_{\mathscr{M}}^!(u_1)_{\mathscr{M}}^!(a_1)_{\mathscr{M}}^! \\
= {} & f_!^{\mathscr{M}} f_{\mathscr{M}}^!(u_1)_*^{\mathscr{M}}(u_1)_{\mathscr{M}}^!(a_1)_{\mathscr{M}}^! \longrightarrow (u_1)_*^{\mathscr{M}}(u_1)_{\mathscr{M}}^!(a_1)_{\mathscr{M}}^!.
\end{aligned}$$

Successively applying $(a_1)_!^{\mathscr{M}}$ and the cohomological functor $\mathsf{H}_{\mathscr{M}}^{2d-i}$ to this morphism, we obtain the morphism (3) in $\mathscr{M}(X)$.

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3.4 Now let (Y, Z, i) be a relative X-triple, and $W \subseteq Z$ be a closed subset. Then we have a boundary morphism

$$\partial^{\mathscr{M}}_{\mathsf{TH}} : \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i) \to \mathsf{TH}^{\mathscr{M}}_X(Z, W, i-1) \tag{4}$$

defined as follows. Consider the commutative diagram

$$\begin{tikzcd}
U := Y\backslash Z \arrow[r, "j"] \arrow[rr, bend left, "u"] & Y\backslash W \arrow[r, "v_Y"] & Y \arrow[r, "a"] & X \\
& V := Z\backslash W \arrow[u, "z_V"] \arrow[r, "v"] \arrow[ur, phantom, "\square"] & Z \arrow[u, "z"'] \arrow[ur, "b"'] &
\end{tikzcd}$$

where v, v_Y, j are the open immersions, z the closed immersion and a, b the structural morphisms. Since j is an open immersion, $j^* = j^!$ and the localization triangle in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y\backslash W))$

$$(z_V)^{\mathscr{M}}_! (z_V)^!_{\mathscr{M}} \to \mathrm{id} \to j^{\mathscr{M}}_* j^*_{\mathscr{M}} \xrightarrow{+1},$$

applied to $(v_Y)^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$, provides a morphism

$$j^{\mathscr{M}}_* u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to (z_V)^{\mathscr{M}}_! v^!_{\mathscr{M}} b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1].$$

As z and z_V are closed immersions, applying $(v_Y)^{\mathscr{M}}_*$ yields a morphism

$$u^{\mathscr{M}}_* u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to z^{\mathscr{M}}_! v_* v^!_{\mathscr{M}} b^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[1].$$

By applying $a^{\mathscr{M}}_!$ and the cohomological functor H^{2d-i} one gets the boundary map (4).

3.5 Recall that in [Ivo16] we have constructed a $\mathbb{Q}$-linear Abelian category $\mathscr{N}(X)$ with a faithful exact functor $\mathscr{N}(X) \to \mathscr{P}(X)$ that factors through $\mathrm{MHM}(X, \mathbb{Q})$. By construction, to every relative X-triple (Y, Z, i) is attached an object $\mathsf{TH}^{\mathscr{N}}_X(Y, Z, i)$ in $\mathscr{N}(X)$. These objects enjoy the same functorialities as previously described. More precisely, if (Y_1, Z_1, i) and (Y_2, Z_2, i) are relative X-triples and $f : Y_2 \to Y_1$ is a morphism of X-schemes, such that $f(Z_2) \subseteq Z_1$, then the category $\mathscr{N}(X)$ contains a morphism

$$f^{\mathscr{N}}_{\mathsf{TH}} : \mathsf{TH}^{\mathscr{N}}_X(Y_2, Z_2, i) \to \mathsf{TH}^{\mathscr{N}}_X(Y_1, Z_1, i) \tag{5}$$

which maps to (3) via the functor $\mathscr{N}(X) \to \mathscr{M}(X)$. Similarly, if (Y, Z, i) is a relative X-triple, and $W \subseteq Z$ is a closed subset, then $\mathscr{N}(X)$ contains a morphism. Then we have a boundary morphism

$$\partial^{\mathscr{N}}_{\mathsf{TH}} : \mathsf{TH}^{\mathscr{N}}_X(Y, Z, i) \to \mathsf{TH}^{\mathscr{N}}_X(Z, W, i-1) \tag{6}$$

compatible again with (4).

3.6 The next lemma is elementary but will be useful in what follows.

LEMMA 3.4. *Let (Y, Z, i) be a relative triple. Then*

$$\cdots \to \mathsf{TH}^{\mathscr{M}}_X(Z, \emptyset, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y, \emptyset, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i) \to \mathsf{TH}^{\mathscr{M}}_X(Z, \emptyset, i-1) \to \cdots \tag{7}$$

is a long exact sequence in $\mathscr{M}(X)$.

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Proof. Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Apply the distinguished triangle $z_!^{\mathscr{M}} z^!_{\mathscr{M}} \to \mathrm{Id} \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} \xrightarrow{+1}$ to $a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})$ and take its image by $a_!^{\mathscr{M}}$ to get the distinguished triangle

$$(a \circ z)_!^{\mathscr{M}} (a \circ z)^!_{\mathscr{M}} (\mathbb{Q}_X^{\mathscr{M}}) \to a_!^{\mathscr{M}} a^!_{\mathscr{M}} (\mathbb{Q}_X^{\mathscr{M}}) \to a_!^{\mathscr{M}} u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}} (\mathbb{Q}_X^{\mathscr{M}}) \xrightarrow{+1}.$$

The associated long exact sequence yields the desired long exact sequence. $\square$

The morphisms (3) and (4) (or (5) and (6) as well) are compatible. More precisely, we have the following lemma.

Lemma 3.5. *Let $f : Y_2 \to Y_1$ be a X-morphism of quasi-projective k-varieties. Let $W_2 \subseteq Z_2$ and $W_1 \subseteq Z_1$ such that $f(Z_2) \subseteq Z_1$ and $f(W_2) \subseteq W_1$. Then the square of morphisms in $\mathscr{M}(X)$*

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_2, Z_2, i) \arrow[r, "\partial_{\mathsf{TH}}^{\mathscr{M}}"] \arrow[d, "f_{\mathsf{TH}}^{\mathscr{M}}"] & \mathsf{TH}_X^{\mathscr{M}}(Z_2, W_2, i-1) \arrow[d, "f_{\mathsf{TH}}^{\mathscr{M}}"] \\
\mathsf{TH}_X^{\mathscr{M}}(Y_1, Z_1, i) \arrow[r, "\partial_{\mathsf{TH}}^{\mathscr{M}}"] & \mathsf{TH}_X^{\mathscr{M}}(Z_1, W_1, i-1)
\end{tikzcd}$$

is commutative.

Proof. To prove the lemma, it is enough to use the definition of the different maps and the three properties, listed below, of the boundary morphisms appearing in localization triangles (the details are left to the reader).

(a) Let $f : Y' \to Y$ be a morphism of quasi-projective k-schemes, Z be a closed subset in Y, and U be its open complement. Consider the commutative square of morphisms of quasi-projective k-schemes

$$\begin{tikzcd}
U' \arrow[r, "u'"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Z' \arrow[l, "z'"'] \arrow[d, "f"] \\
U \arrow[r, "u"] & Y & Z \arrow[l, "z"']
\end{tikzcd}$$

The canonical morphisms

$$(z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} f^!_{\mathscr{M}} = (z')_!^{\mathscr{M}} f^!_{\mathscr{M}} z^!_{\mathscr{M}} \to f^!_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}}, \quad (u')_*^{\mathscr{M}} (u')^*_{\mathscr{M}} f^!_{\mathscr{M}} = (u')_*^{\mathscr{M}} f^!_{\mathscr{M}} u^*_{\mathscr{M}} \to f^!_{\mathscr{M}} u_*^{\mathscr{M}} u^*_{\mathscr{M}}$$

are isomorphisms by smooth base change and fit into a morphism of localization distinguished triangles

$$\begin{tikzcd}
(z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} f^!_{\mathscr{M}} \arrow[r] \arrow[d] & f^!_{\mathscr{M}} \arrow[r] \arrow[d, equal] & (u')_*^{\mathscr{M}} (u')^*_{\mathscr{M}} \arrow[r] \arrow[d] & (z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} f^!_{\mathscr{M}}[1] \arrow[d] \\
f^!_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}} \arrow[r] & f^!_{\mathscr{M}} \arrow[r] & f^!_{\mathscr{M}} u_*^{\mathscr{M}} u^*_{\mathscr{M}} \arrow[r] & f^!_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}}[1]
\end{tikzcd}$$

(b) Let Y be a quasi-projective k-scheme, and Z (respectively, W) be a closed subset of Y with open complement U (respectively, V). Assume $W \subseteq Z$ (so that $U \subseteq V$). We then have the open and closed immersions

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[rr, bend left, "u"] & V \arrow[r, "v"] & Y & Z \arrow[l, "z"'] & W \arrow[l, "i"'] \arrow[ll, bend right, "w"']
\end{tikzcd}$$

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and the canonical morphisms

$$v_*^{\mathscr{M}} v^*_{\mathscr{M}} \to v_*^{\mathscr{M}} j_*^{\mathscr{M}} j^*_{\mathscr{M}} v^*_{\mathscr{M}} = u_*^{\mathscr{M}} u^*_{\mathscr{M}}, \quad w_!^{\mathscr{M}} w^!_{\mathscr{M}} = z_!^{\mathscr{M}} i_!^{\mathscr{M}} i^!_{\mathscr{M}} z^!_{\mathscr{M}} \to z_!^{\mathscr{M}} z^!_{\mathscr{M}}$$

fit into the morphism of localization distinguished triangles

$$\begin{tikzcd}
w_!^{\mathscr{M}} w^!_{\mathscr{M}} \arrow[r] \arrow[d] & \mathrm{Id} \arrow[r] \arrow[d, equal] & v_*^{\mathscr{M}} v^*_{\mathscr{M}} \arrow[r] \arrow[d] & w_!^{\mathscr{M}} w^!_{\mathscr{M}}[1] \arrow[d] \\
z_!^{\mathscr{M}} z^!_{\mathscr{M}} \arrow[r] & \mathrm{Id} \arrow[r] & u_*^{\mathscr{M}} u^*_{\mathscr{M}} \arrow[r] & z_!^{\mathscr{M}} z^!_{\mathscr{M}}[1]
\end{tikzcd}$$

(c) Let $j : Y' \to Y$ be an open immersion of quasi-projective k-schemes, Z be a closed subset in Y, and U be its open complement. Consider the commutative square of morphisms of quasi-projective k-schemes

$$\begin{tikzcd}
U' \arrow[r, "u'"] \arrow[d, "j"] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "j"] \arrow[dr, phantom, "\square"] & Z' \arrow[l, "z'"'] \arrow[d, "j"] \\
U \arrow[r, "u"] & Y & Z \arrow[l, "z"']
\end{tikzcd}$$

The canonical morphisms

$$j^*_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}} \to (z')_!^{\mathscr{M}} j^*_{\mathscr{M}} z^!_{\mathscr{M}} = (z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} j^*_{\mathscr{M}}, \quad j^*_{\mathscr{M}} u_*^{\mathscr{M}} u^*_{\mathscr{M}} f^!_{\mathscr{M}} \to (u')_*^{\mathscr{M}} j^*_{\mathscr{M}} u^*_{\mathscr{M}} \to (u')_*^{\mathscr{M}} (u')^*_{\mathscr{M}} j^*_{\mathscr{M}}$$

are isomorphisms by smooth base change and fit into a morphism of localization distinguished triangles

$$\begin{tikzcd}
j^*_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}} \arrow[r] \arrow[d] & j^*_{\mathscr{M}} \arrow[r] \arrow[d, equal] & j^*_{\mathscr{M}} u_*^{\mathscr{M}} u^*_{\mathscr{M}} \arrow[r] \arrow[d] & j^*_{\mathscr{M}} z_!^{\mathscr{M}} z^!_{\mathscr{M}}[1] \arrow[d] \\
(z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} j^*_{\mathscr{M}} \arrow[r] & j^*_{\mathscr{M}} \arrow[r] & (u')_*^{\mathscr{M}} (u')^*_{\mathscr{M}} j^*_{\mathscr{M}} \arrow[r] & (z')_!^{\mathscr{M}} (z')^!_{\mathscr{M}} j^*_{\mathscr{M}}[1]
\end{tikzcd}$$

□

3.7 We now give some properties of relative $\mathscr{M}$-homology objects needed later to construct the realization functors.

LEMMA 3.6. *Let $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let (Y, Z, i) be a relative X-triple and*

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

be a Nisnevich square. Then there is a long exact sequence in $\mathscr{M}(X)$:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(Y, Z, i+1) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(V, Z_V, i) \arrow[d] & & \\
& & \mathsf{TH}_X^{\mathscr{M}}(U, Z_U, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E, Z_E, i) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) \arrow[r] & \cdots
\end{tikzcd} \tag{8}$$

where $Z_V := Z \times_X V$, $Z_U := Z \times_X U$ and $Z_E := Z \times_X E$.

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Proof. Let $w : W \hookrightarrow Y$ be an open immersion of the complement of Z in Y. Consider the diagram obtained by base change

$$\begin{tikzcd}
V_W \arrow[r, "v_W"] \arrow[d, "e'_W"'] \arrow[drr, "w_V"] \arrow[dr, phantom, "\square"] & E_W \arrow[d, dotted, "e_W"] \arrow[drr, "w_E"] & & \\
U_W \arrow[r, dotted, "u_W"] \arrow[drr, "w_U"'] & W \arrow[drr, dotted, "w"] & V \arrow[r, "v"] \arrow[d, "e'"] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
 & & U \arrow[r, "u"] & Y
\end{tikzcd}$$

Let $h = e \circ v = u \circ e'$ and $h_W = e_W \circ v_W = u_W \circ e'_W$. We have a distinguished triangle

$$h_!^{\mathscr{M}} h^!_{\mathscr{M}} \to u_!^{\mathscr{M}} u^!_{\mathscr{M}} \oplus e_!^{\mathscr{M}} e^!_{\mathscr{M}} \to \mathrm{Id} \xrightarrow{+1} .$$

Applying this triangle to $w_*^{\mathscr{M}} w^*_{\mathscr{M}}$ yields the distinguished triangle

$$\begin{tikzcd}
h_!^{\mathscr{M}} (w_V)_*^{\mathscr{M}} (w_V)^*_{\mathscr{M}} h^!_{\mathscr{M}} \arrow[r] & u_!^{\mathscr{M}} (w_U)_*^{\mathscr{M}} (w_U)^*_{\mathscr{M}} u^!_{\mathscr{M}} \oplus e_!^{\mathscr{M}} (w_E)_*^{\mathscr{M}} (w_E)^*_{\mathscr{M}} e^!_{\mathscr{M}} \arrow[d] \\
 & w_*^{\mathscr{M}} w^*_{\mathscr{M}} \xrightarrow{+1}
\end{tikzcd} \tag{9}$$

since using smooth base change, we get

$$e_!^{\mathscr{M}} e^!_{\mathscr{M}} w_*^{\mathscr{M}} w^*_{\mathscr{M}} = e_!^{\mathscr{M}} (w_E)_*^{\mathscr{M}} (w_E)^*_{\mathscr{M}} e^!_{\mathscr{M}}, \quad u_!^{\mathscr{M}} u^!_{\mathscr{M}} w_*^{\mathscr{M}} w^*_{\mathscr{M}} = u_!^{\mathscr{M}} (w_U)_*^{\mathscr{M}} (w_U)^*_{\mathscr{M}} u^!_{\mathscr{M}},$$
$$h_!^{\mathscr{M}} h^!_{\mathscr{M}} w_*^{\mathscr{M}} w^*_{\mathscr{M}} = h_!^{\mathscr{M}} (w_V)_*^{\mathscr{M}} (w_V)^*_{\mathscr{M}} h^!_{\mathscr{M}}.$$

Applying the triangle (9) to $a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})$ and taking the image under $a_!^{\mathscr{M}}$ yields a new distinguished triangle. Then the long exact sequence (8) is the long exact sequence associated with this triangle. □

COROLLARY 3.7. *Let (Y, Z, i) be a relative X-triple and*

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

be a Nisnevich square. Then there is an exact sequence in $\mathscr{N}(X)$:

$$\mathsf{TH}_X^{\mathscr{N}}(V, Z_V, i) \to \mathsf{TH}_X^{\mathscr{N}}(U, Z_U, i) \oplus \mathsf{TH}_X^{\mathscr{N}}(E, Z_E, i) \to \mathsf{TH}_X^{\mathscr{N}}(Y, Z, i) \tag{10}$$

where $Z_V := Z \times_X V$, $Z_U := Z \times_X U$ and $Z_E := Z \times_X E$.

Proof. This follows from Lemma 3.6 since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful. (Note that it is not clear a priori that the boundary morphism in the long exact sequence (8) exists in the category of perverse Nori motives $\mathscr{N}(X)$.) □

LEMMA 3.8. *Let (Y, Z, i) be a relative triple and $p : Y' \to Y$ be a Galois covering with Galois group G. Then the morphism*

$$\mathsf{TH}_X^{\mathscr{M}}(Y', Z', i)^G \to \mathsf{TH}_X^{\mathscr{M}}(Y, Z, i)$$

is an isomorphism.

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Proof. Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let $z : Z \hookrightarrow Y$ be a closed immersion and $u : U \hookrightarrow Y$ be the open immersion of the complement. Consider their pullbacks $u' : U' \hookrightarrow Y'$ and $z' : Z' \hookrightarrow Y'$ along p. Let $A \in \mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y))$. Then we have the commutative diagram

$$\begin{tikzcd}
{[p^!_{\mathscr{M}}(z')^{\mathscr{M}}_!(z')^!_{\mathscr{M}}p^!_{\mathscr{M}}(A)]^G} \arrow[r] \arrow[d] & {[p^{\mathscr{M}}_! p^!_{\mathscr{M}}(A)]^G} \arrow[r] \arrow[d] & {[p^{\mathscr{M}}_!(u')^{\mathscr{M}}_*(u')^*_{\mathscr{M}}p^!_{\mathscr{M}}(A)]^G} \arrow[r, "+1"] \arrow[d] & {} \\
z^{\mathscr{M}}_! z^!_{\mathscr{M}}(A) \arrow[r] & A \arrow[r] & u^{\mathscr{M}}_* u^!_{\mathscr{M}}(A) \arrow[r, "+1"] & {}
\end{tikzcd}$$

in which the lines are distinguished triangles in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(Y))$. The first two vertical arrows are isomorphisms by étale descent for Betti cohomology (the result in Hodge theory follows from the case of perverse sheaves by Remark 3.1), hence so is the map

$$[p^{\mathscr{M}}_!(u')^{\mathscr{M}}_*(u')^*_{\mathscr{M}}p^!_{\mathscr{M}}(A)]^G \to u^{\mathscr{M}}_* u^!_{\mathscr{M}}(A).$$

This implies that the maps $\mathsf{TH}^{\mathscr{M}}_X(Y', Z', i)^G \to \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i)$ are isomorphisms for every integer $i \in \mathbb{Z}$. $\square$

Lemma 3.9. *Let Y be a quasi-projective k-scheme and $T \to Y$ be a finite-rank vector bundle. Then for every integer $i \in \mathbb{Z}$,*

$$\mathsf{TH}^{\mathscr{M}}_X(T, Y, i) = 0$$

where Y is embedded in T via the zero section.

Proof. Since the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Now consider the zero section $\sigma : Y \to T$ and denote the open immersion of the complement by u. Let $p : T \to Y$ be the projection. By homotopy invariance

$$p^{\mathscr{M}}_! p^!_{\mathscr{M}} \to \mathrm{Id}$$

is an isomorphism. We have the distinguished triangle $\sigma^{\mathscr{M}}_! \sigma^!_{\mathscr{M}} \to \mathrm{Id} \to u^{\mathscr{M}}_* u^!_{\mathscr{M}} \xrightarrow{+1}$. But $p \circ \sigma = \mathrm{Id}$, hence the canonical morphism

$$a^{\mathscr{M}}_! p^{\mathscr{M}}_! \sigma^{\mathscr{M}}_! \sigma^!_{\mathscr{M}} p^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) \to a^{\mathscr{M}}_! p^{\mathscr{M}}_! p^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)$$

is an isomorphism, and thus

$$a^{\mathscr{M}}_! p^{\mathscr{M}}_! u^{\mathscr{M}}_* u^!_{\mathscr{M}} p^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X) = 0$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$. In particular, for all integers $i \in \mathbb{Z}$, we have the vanishing $\mathsf{TH}^{\mathscr{M}}_X(T, Y, i) = 0$. $\square$

Lemma 3.10. *Let (Y, Z, i) be a relative X-triple. We have a decomposition into direct summands*

$$\mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z}, i) = \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i-1)(1). \tag{11}$$

If $W \subseteq Z$ is a closed subset, then the decomposition (11) is compatible with boundary maps, that is, the square

$$\begin{tikzcd}
\mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Y}, \mathbf{G}_{m,Z}, i) \arrow[r] \arrow[d, equal] & \mathsf{TH}^{\mathscr{M}}_X(\mathbf{G}_{m,Z}, \mathbf{G}_{m,W}, i-1) \arrow[d, equal] \\
\mathsf{TH}^{\mathscr{M}}_X(Y, Z, i) \oplus \mathsf{TH}^{\mathscr{M}}_X(Y, Z, i-1)(1) \arrow[r] & \mathsf{TH}^{\mathscr{M}}_X(Z, W, i-1) \oplus \mathsf{TH}^{\mathscr{M}}_X(Z, W, i-2)(1)
\end{tikzcd}$$

is commutative.

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Proof. Again we may assume $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let $z : Z \hookrightarrow Y$ be a closed immersion, and $u : U \hookrightarrow Y$ be its open complement. We denote by $\pi : \mathbf{G}_{m,k} \to \mathrm{Spec}(k)$ the projection. Recall that there is an isomorphism

$$\pi_!^{\mathscr{M}} \pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}) = \mathbb{Q}_k^{\mathscr{M}} \oplus \mathbb{Q}_k^{\mathscr{M}}(1)[1]$$

in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(\mathrm{Spec}(k)))$. We have an isomorphism

$$(u \times_k \mathrm{Id})^!_{\mathscr{M}}(a \times_k \pi)^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) = u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \boxtimes \pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}).$$

The object $(a \times_k \pi)_!^{\mathscr{M}}(u \times_k \mathrm{Id})_*^{\mathscr{M}}(u \times_k \mathrm{Id})^!_{\mathscr{M}}(a \times_k \pi)^!(\mathbb{Q}_X^{\mathscr{M}})$ of $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ is therefore isomorphic to

$$(a_!^{\mathscr{M}} u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})) \boxtimes (\pi_!^{\mathscr{M}} \pi^!_{\mathscr{M}}(\mathbb{Q}_k^{\mathscr{M}}) = a_!^{\mathscr{M}} u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) \oplus (a_!^{\mathscr{M}} u_*^{\mathscr{M}} u^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))(1)[1].$$

This yields the decomposition into direct summands in (11). The commutativity of the square is easy to verify from the definition of boundary maps. □

4. Perverse cellular complexes

We assume that X is a *smooth* quasi-projective k-variety. We may assume that X is connected of dimension d. We denote by SmAff/X the category of smooth quasi-projective X-schemes that are affine.

In this section, given a scheme $Y \in \mathsf{SmAff}/X$, we use the basic lemma [Beĭ87, Lemma 3.3] proved by Beĭlinson to associate with certain stratifications of Y an explicit complex of mixed Hodge modules, perverse sheaves or perverse motives that computes its relative homology. This construction is the crucial step towards the realization functor. Note that the assumption on the smoothness of X is used in the proof of Proposition 4.15.

4.1 Let Y be a quasi-projective k-scheme. A stratification $Y_\bullet$ of Y is a sequence of closed subsets of Y,

$$Y_\bullet : \cdots \subseteq Y_i \subseteq Y_{i+1} \subseteq \cdots, \quad i \in \mathbb{Z},$$

such that $\dim(Y_i) \leqslant i$ for every integer $i \in \mathbb{Z}$. Note that the condition on dimensions implies that $Y_{-1} = \emptyset$.

Let $Y_\bullet$ and $Y'_\bullet$ be two stratifications of Y. We say that $Y'_\bullet$ is finer than $Y_\bullet$, and write $Y_\bullet \leqslant Y'_\bullet$, if $Y_i \subseteq Y'_i$ for every integer $i \in \mathbb{Z}$. This defines an order relation on the set Strat_Y of all stratifications of Y. The ordered set Strat_Y is filtered. Indeed, since

$$\dim(Y_i \cup Y'_i) \leqslant i,$$

there is a stratification $Y''_\bullet$ given by $Y''_i := Y_i \cup Y'_i$ and it is finer than $Y_\bullet$ and $Y'_\bullet$.

Let $f : Y \to Y'$ be a morphism of schemes of quasi-projective X-schemes and $Y_\bullet$ be a stratification of Y. Let

$$Y'_i := \overline{f(Y_i)}$$

be the closure of the image of Y_i in Y'. Then $Y'_\bullet$ is a stratification of Y'. Indeed, by [EGAIVa, Théorème (4.1.2)], for every integer $i \in \mathbb{Z}$,

$$\dim(Y'_i) \leqslant \dim(Y_i) \leqslant i.$$

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We call this stratification the image of $Y_\bullet$ by f and write $f_\sharp(Y_\bullet) := Y'_\bullet$. This defines a functor $f_\sharp : \mathsf{Strat}_Y \to \mathsf{Strat}_{Y'}$. Let $g : Y' \to Y''$ be another morphism of quasi-projective X-schemes. Then, for every integer $i \in \mathbb{Z}$,

$$\overline{g(\overline{f(Y_i)})} = \overline{g(f(Y_i))}.$$

This means that the two stratifications $g_\sharp(f_\sharp(Y_\bullet))$ and $(gf)_\sharp(Y_\bullet)$ are the same. In other words, $g_\sharp \circ f_\sharp = (g \circ f)_\sharp$ as functors.

Remark 4.1. We do not require a stratification to be exhaustive, that is, we do not require the existence of an integer n such that $Y_n = Y$. If f is a closed immersion of codimension greater than zero, the image of an exhaustive stratification by f is not an exhaustive stratification. For functoriality reasons, it is therefore essential to work with all stratifications, not only the exhaustive ones. Note that every stratification admits a finer stratification that is exhaustive.

4.2 The following definition is essential in what follows.

Definition 4.2. Let Y be a quasiprojective X-scheme. A stratification

$$Y_\bullet : \emptyset = Y_{-1} \subseteq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_{n-1} \subseteq Y_n = Y$$

of Y is said to be cellular if and only if there exists an integer n such that $Y_n = Y$ and for every $i \in \mathbb{Z}$ the following conditions are satisfied:

- if $\dim(Y_i) = i$, then for every $k \in \mathbb{Z}$, $k \neq i$, one has

  $$\mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, k) = 0$$

  in $\mathscr{M}(X)$;
- if $\dim(Y_i) \leqslant i - 1$, then $Y_i = Y_{i-1}$.

Note that in the second case $\mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, k) = 0$ for every $k \in \mathbb{Z}$. Assume $Y \neq \emptyset$ and let $n \in \mathbb{N}$ be the smallest integer such that $Y_n = Y$. Then we must have $n \leqslant \dim(Y)$ by the second condition. For a stratification, $Y_\bullet$ being cellular is not a property with respect to Y but with respect to the morphism $Y \to X$. This will cause no confusion in what follows as our scheme X is fixed once and for all.

If $f : Y \to Y'$ is a morphism of quasi-projective X-schemes, the image of a cellular stratification under the functor $f_\sharp$ may not be a cellular stratification. It may not even be exhaustive (see Remark 4.1). So as far as functoriality is concerned, it is essential to consider all stratifications and not only the cellular ones.

Remark 4.3. The long exact sequence (7) provides the exact sequences

$$\mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, k+1) \to \mathsf{TH}_X^{\mathscr{M}}(Y_{i-1}, \emptyset, k) \to \mathsf{TH}_X^{\mathscr{M}}(Y_i, \emptyset, k) \to \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, k).$$

In particular, if $Y_\bullet$ is a cellular stratification of Y, then for $k < i - 1$ or $k > i$ the canonical morphism

$$\mathsf{TH}_X^{\mathscr{M}}(Y_{i-1}, \emptyset, k) \to \mathsf{TH}_X^{\mathscr{M}}(Y_i, \emptyset, k)$$

is an isomorphism in $\mathscr{M}(X)$. This implies that, for $k < i$ or $k > n$, the morphism

$$\mathsf{TH}_X^{\mathscr{M}}(Y_i, \emptyset, k) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, k)$$

is an isomorphism in $\mathscr{M}(X)$.

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Remark 4.4. Let $Y_\bullet$ be a cellular stratification of Y and n be an integer such that $Y = Y_n$. It is easy to see by induction that $\mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) = 0$ for every integer $i \in \mathbb{Z}$ such that $i < 0$ or $i > n$. Indeed, for $n = 0$, this follows from Definition 4.2. For $n > 0$, Remark 4.3 implies that

$$\mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, i) \xrightarrow{\simeq} \mathsf{TH}_X^{\mathscr{M}}(Y_n, \emptyset, i) = \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i)$$

for $i < 0$ or $i > n$ and vanishing follows by induction.

The following result is an immediate application of [Beĭ87, Lemma 3.3].

Lemma 4.5. *Let $a : Y \to X$ be an affine morphism of finite type. Assume that $\dim(Y) = n$. For a closed subset W such that $\dim(W) \leqslant n - 1$, there exists a closed subset Z of Y containing W and such that $\dim(Z) \leqslant n - 1$ and for every integer $i \neq n$,*

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) = 0.$$

If Y is integral then we may choose Z such that its open complement is smooth over k.

Note that we do not have to assume here that the scheme Y is affine, only that the morphism $a : Y \to X$ is affine.

Proof. As the functor $\mathscr{N}(X) \to \mathscr{P}(X)$ is exact and faithful, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. We may assume that Y is reduced. By replacing W by the union of W and the irreducible components of Y of dimension less than or equal to $n - 1$, we may assume that W contains all the irreducible components of Y of dimension less than or equal to $n - 1$. Then $Y \backslash W$ is open in Y and of pure dimension n (i.e., all its irreducible components are of dimension n). As k is of characteristic zero, by [EGAIVb, Proposition (17.15.12)], there is an affine dense open subset V in $Y \backslash W$ which is smooth over k. Since V is smooth of pure dimension n, $\mathbb{Q}_V^{\mathscr{M}}[n]$ is an object in $\mathscr{M}(V)$ and

$$A := v_*^{\mathscr{M}} v_{\mathscr{M}}^! a_{\mathscr{M}}^! (\mathbb{Q}_X^{\mathscr{M}})[2d - n]$$

belongs to $\mathscr{M}(Y)$. Apply [Beĭ87, Lemma 3.3] to this object $A \in \mathscr{M}(Y)$. This yields an affine open U' in Y such that $\dim(Y \backslash U') \leqslant n - 1$ and such that

$$\mathsf{H}_{\mathscr{M}}^i (a_!^{\mathscr{M}} (u')_*^{\mathscr{M}} (u')_{\mathscr{M}}^* A) = 0$$

in $\mathscr{M}(X)$ for every integer $i \in \mathbb{Z} \backslash \{0\}$. Let U be the intersection of the two dense open subsets U' and V, and Z its complement in Y. We have $W \subseteq Z$, $\dim(Z) \leqslant n - 1$ and the square of open immersions

$$\begin{tikzcd} U \arrow[r, "j'"] \arrow[d, "j"] \arrow[dr, phantom, "\square"] & U' \arrow[d, "u'"] \\ V \arrow[r, "v"] & Y \end{tikzcd}$$

is cartesian. By smooth base change

$$\begin{aligned} (u')_*^{\mathscr{M}} (u')_{\mathscr{M}}^* A &= (u')_*^{\mathscr{M}} (u')_{\mathscr{M}}^* v_*^{\mathscr{M}} v_{\mathscr{M}}^! a_{\mathscr{M}}^! (\mathbb{Q}_X^{\mathscr{M}})[2d - n] \\ &= (u')_*^{\mathscr{M}} (j')_*^{\mathscr{M}} j_{\mathscr{M}}^* v_{\mathscr{M}}^! a_{\mathscr{M}}^! (\mathbb{Q}_X^{\mathscr{M}})[2d - n] \\ &= (u)_*^{\mathscr{M}} (u)_{\mathscr{M}}^! a_{\mathscr{M}}^! (\mathbb{Q}_X^{\mathscr{M}})[2d - n]. \end{aligned}$$

Hence $\mathsf{TH}_X^{\mathscr{M}}(Y, Z, i) = 0$ for $i \neq n$ and the proof is complete. ∎

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Remark 4.6. In the above proof, the application of [Beĭ87, Lemma 3.3] allows us to choose the open immersion $u' : U' \hookrightarrow Y$ so that the canonical morphism $A \to (u')^{\mathscr{M}}_*(u')^*_{\mathscr{M}}A$ is a monomorphism in $\mathscr{M}(Y)$. This property is not used in the proof of Lemma 4.5 but plays an essential role in the proof of Proposition 4.15.

Note that for $X = \mathrm{Spec}(k)$, Lemma 4.5 is nothing more than the so-called basic lemma of Nori [Nor02, Basic Lemma – first form]. As a consequence, the subset Cell_Y of cellular stratifications is cofinal in Strat_Y. That is to say, we have the following lemma.

Lemma 4.7. *Let $a : Y \to X$ be an affine morphism of finite type and $Y_\bullet$ be a stratification of Y. Then there exists a cellular stratification of Y finer than $Y_\bullet$.*

Proof. We construct the stratification by induction. Let us set $Y'_n = Y_n$. Assume that we have constructed a sequence of closed subsets $Y'_r \subseteq Y'_{r+1} \subseteq \cdots \subseteq Y'_n = Y$ such that $Y_i \subseteq Y'_i$ and $\dim(Y'_i) \leqslant i$ for every $r \leqslant i \leqslant n$ and such that for $r + 1 \leqslant i \leqslant n$ the following conditions are satisfied:

- if $\dim(Y'_i) = i$, then for every $k \in \mathbb{Z}$, $k \neq i$, one has

$$\mathsf{TH}^{\mathscr{M}}_X(Y'_i, Y'_{i-1}, k) = 0$$

  in $\mathscr{M}(X)$;
- if $\dim(Y'_i) \leqslant i - 1$, then$Y'_i = Y'_{i-1}$.

If $\dim(Y'_r) \leqslant r - 1$, then we set $Y'_{r-1} = Y'_r$. Otherwise $\dim(Y'_r) = r$, and since $Y_{r-1} \subseteq Y'_r$ and $\dim(Y_{r-1}) \leqslant r - 1$, we may apply Lemma 4.5 to obtain a closed subset Y'_{r-1}, such that $Y_{r-1} \subseteq Y'_{r-1} \subseteq Y'_r$, $\dim(Y'_{r-1}) \leqslant r - 1$ and

$$\mathsf{TH}^{\mathscr{M}}_X(Y'_r, Y'_{r-1}, i) = 0$$

for every integer $i \neq r$. □

Corollary 4.8. *Let $a : Y \to X$ be an affine morphism of finite type. There exists a cellular stratification $Y_\bullet$ on Y.*

Proof. Let n be the dimension of Y. It suffices to apply Lemma 4.7 to the stratification $Y_\bullet$ such that $Y_i = \emptyset$ for $i < n$ and $Y_i = Y$ for $i \geqslant n$. □

Corollary 4.9. *The ordered subset Cell_Y of Strat_Y formed by the cellular stratifications is filtered.*

Proof. Since Strat_Y is filtered, this follows immediately from Lemma 4.7. □

4.3 The starting point of the main construction is to consider the following complexes associated with stratifications of (affine) quasi-projective X-schemes.

Definition 4.10. Let Y be a quasi-projective X-scheme. Let $Y_\bullet$ be a stratification of Y. We denote by $\mathsf{TH}^{\mathscr{M}}_X(Y, Y_\bullet)$ the complex in $\mathsf{Ch}(\mathscr{M}(X))$ given by

$$\cdots \to \mathsf{TH}^{\mathscr{M}}_X(Y_i, Y_{i-1}, i) \to \mathsf{TH}^{\mathscr{M}}_X(Y_{i-1}, Y_{i-2}, i-1) \to \cdots \to \mathsf{TH}^{\mathscr{M}}_X(Y_0, Y_{-1}, 0) \to 0$$

where $\mathsf{TH}^{\mathscr{M}}_X(Y_0, Y_{-1}, 0)$ is placed in degree 0 and all differentials are given by the corresponding boundary morphisms $\partial^{\mathscr{M}}_{\mathsf{TH}}$.

These complexes are functorial. Indeed, let $f : Y \to Y'$ be a X-morphism of quasi-projective k-varieties. Let $Y_\bullet$ be a stratification of Y, and $Y'_\bullet$ be a stratification of Y' such that $f(Y_i) \subseteq Y'_i$ for every integer $i \in \mathbb{Z}$ (i.e., $Y'_\bullet$ is finer than the image $f_\sharp(Y_\bullet)$ of $Y_\bullet$ by f). Then by Lemma 3.5, the morphisms

$$f_{\mathsf{TH}}^{\mathscr{M}} : \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y'_i, Y'_{i-1}, i)$$

define a morphism of complexes

$$f_{\mathsf{TH}}^{\mathscr{M}} : \mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet) \to \mathsf{TH}_X^{\mathscr{M}}(Y', Y'_\bullet).$$

In particular, we have a morphism of complexes

$$f_{\mathsf{TH}}^{\mathscr{M}} : \mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet) \to \mathsf{TH}_X^{\mathscr{M}}(Y', f_\sharp(Y_\bullet))$$

and for every morphism $f' : Y' \to Y''$ of quasi-projective k-schemes, the diagram

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet) \arrow[r, "f_{\mathsf{TH}}^{\mathscr{M}}"'] \arrow[rr, bend left, "(f'f)_{\mathsf{TH}}^{\mathscr{M}}"] & \mathsf{TH}_X^{\mathscr{M}}(Y', f_\sharp(Y_\bullet)) \arrow[r, "(f')_{\mathsf{TH}}^{\mathscr{M}}"'] & \mathsf{TH}_X^{\mathscr{M}}(Y'', (f'f)_\sharp(Y_\bullet))
\end{tikzcd}$$

is commutative.

DEFINITION 4.11. We define $\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)$ by

$$\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet) := \mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)[-2d].$$

The next proposition shows that complexes associated with cellular stratifications do compute the $\mathscr{M}$-homology of quasi-projective k-schemes.

PROPOSITION 4.12. *Assume that $Y_\bullet$ is a cellular stratification of Y. For every integer $i \in \mathbb{Z}$, there is an isomorphism*

$$\phi(Y, Y_\bullet, i) : \mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \xrightarrow{\sim} \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i)$$

such that for every cellular stratification $Y'_\bullet$ finer that $Y_\bullet$ the diagram is commutative:

$$\begin{tikzcd}
\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \arrow[dd] \arrow[dr, "\phi(Y,Y_\bullet,i)"] & \\
& \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) \\
\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y'_\bullet)) \arrow[ur, "\phi(Y,Y'_\bullet,i)"'] &
\end{tikzcd}$$

where the vertical morphism is the functoriality morphism.

Proof. Let n be an integer such that $Y_n = Y$. Let us construct the isomorphisms $\phi(Y, Y_\bullet, i)$ by induction on n. If $n = 0$, then $\mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) = 0$ for every integer $i \neq 0$ and the lemma is obvious. Assume $n = 1$. Using the long exact sequence from Lemma 3.4, Definition 4.2 and Remark 4.4, we obtain the exact sequence

$$0 \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, 1) \to \mathsf{TH}_X^{\mathscr{M}}(Y, Y_0, 1) \to \mathsf{TH}_X^{\mathscr{M}}(Y_0, \emptyset, 0) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, 0) \to 0$$

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which proves the lemma in that case. Assume $n \geqslant 2$. Let $Z = Y_{n-1}$ and

$$Z_\bullet : \emptyset = Z_{-1} \subseteq Z_0 = Y_0 \subseteq Z_1 = Y_1 \subseteq \cdots \subseteq Z_{n-1} = Y_{n-1} = Z$$

be the induced stratification. If $i < 0$ or $i > n$ we set $\phi(Y, Y_\bullet, i) = 0$, which is an isomorphism since $\mathsf{H}_i^{\mathscr{M}}(\mathsf{R}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i) = 0$ by Remark 4.4. Let $0 \leqslant i \leqslant n-2$. We have by induction an isomorphism

$$\mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{H}_i^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Z, Z_\bullet)) \xrightarrow{\phi(Z, Z_\bullet, i)} \mathsf{TH}_X^{\mathscr{M}}(Z, \emptyset, i) = \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, i)$$

and we let $\phi(Y, Y_\bullet, i)$ be the composition of this isomorphism and the canonical morphism $\mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, i)$, which is an isomorphism by Remark 4.3.

Now we have a commutative diagram

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_n, Y_{n-1}, n) \arrow[r, "\partial_n"] \arrow[dr] \arrow[drr, bend right, "(12)"'] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, Y_{n-2}, n-1) \arrow[r, "\partial_{n-1}"] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-2}, Y_{n-3}, n-2) \\
& \operatorname{Ker}(\partial_{n-1}) = \mathsf{H}_{n-1}^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Z, Z^\bullet)) \arrow[u] \arrow[r, "\phi(Z, Z_\bullet, n-1)"] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, n-1)
\end{tikzcd}$$

where the morphism (12) is the morphism in the long exact sequence

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, n) \arrow[d] & & \\
\mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, n) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(Y_n, Y_{n-1}, n) \arrow[r, "(12)"] & \mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, n-1) \arrow[d] \\
& & \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, n-1) \arrow[d] \\
& & \mathsf{TH}_X^{\mathscr{M}}(Y_n, Y_{n-1}, n-1)
\end{tikzcd}$$

obtained by Lemma 3.4. However, $\mathsf{TH}_X^{\mathscr{M}}(Y_n, Y_{n-1}, n-1) = 0$ by Definition 4.2, and $\mathsf{TH}_X^{\mathscr{M}}(Y_{n-1}, \emptyset, n) = 0$ by Remark 4.4. We obtain therefore an isomorphism

$$\phi(Y, Y_\bullet, n) : \operatorname{Ker}(\partial_n) = \mathsf{H}_n^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, n)$$

and an isomorphism

$$\phi(Y, Y_\bullet, n-1) : \mathsf{H}_{n-1}^{\mathscr{M}}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \to \mathsf{TH}_X^{\mathscr{M}}(Y, \emptyset, n-1).$$

Hence the statement. $\square$

Remark 4.13. By Definition 3.2, one may view the isomorphisms constructed in Lemma 4.12 as isomorphisms

$$\mathsf{H}_{\mathscr{M}}^{2d-i}(\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{H}_{\mathscr{M}}^{-i}(\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)) \xrightarrow{\phi(Y, Y_\bullet, i)} \mathsf{H}_{\mathscr{M}}^{2d-i}(a_!^{\mathscr{M}} a_{\mathscr{M}}^!(\mathbb{Q}_X^{\mathscr{M}})).$$

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Hence $\psi(Y, Y_\bullet, i) := \phi(Y, Y_\bullet, 2d - i)$ are isomorphisms

$$\psi(Y, Y_\bullet, i) : \mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet)) \xrightarrow{\sim} \mathsf{H}^i_{\mathscr{M}}(a^{\mathscr{M}}_! a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X))$$

such that for every cellular stratification $Y'_\bullet$ finer that $Y_\bullet$ the diagram

$$\begin{tikzcd}
\mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet)) \arrow[d] \arrow[dr, "\psi(Y,Y_\bullet,i)"] & \\
& \mathsf{H}^i_{\mathscr{M}}(a^{\mathscr{M}}_! a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)) \\
\mathsf{H}^i_{\mathscr{M}}(\mathsf{r}^{\mathscr{M}}_X(Y, Y'_\bullet)) \arrow[ur, "\psi(Y,Y'_\bullet,i)"'] &
\end{tikzcd}$$

commutes, where the vertical morphism is the functoriality morphism.

Corollary 4.14. *Let $Y_\bullet$ and $Y'_\bullet$ be cellular stratifications. If $Y'_\bullet$ is finer that $Y_\bullet$, then the canonical map*

$$\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet) \to \mathsf{r}^{\mathscr{M}}_X(Y, Y'_\bullet)$$

is a quasi-isomorphism in $\mathsf{Ch}^{\mathrm{b}}(\mathscr{M}(X))$.

In the Hodge or perverse case, the result can be strengthened.

Proposition 4.15. *Assume $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. Let $a : Y \to X$ be an affine morphism. Assume that Y is smooth of pure dimension n. Then there exists a cellular stratification $Y_\bullet$ of Y such that $\mathsf{r}^{\mathscr{M}}_X(Y, Y_\bullet)$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ to*

$$a^{\mathscr{M}}_! a^!_{\mathscr{M}} \mathbb{Q}^{\mathscr{M}}_X.$$

Proof. Let r be an integer $0 \leqslant r \leqslant n$. Assume that $Z \subseteq Y$ is a closed subset such that $\dim(Z) \leqslant r$ and

$$\mathsf{H}^i_{\mathscr{M}}(z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)) = 0$$

for every integer $i \neq 2d - r$. Let $z : Z \hookrightarrow Y$ be the closed immersion. Consider the object

$$A := \mathsf{H}^{2d-r}_{\mathscr{M}}(z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)) \simeq z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[2d - r]$$

in $\mathscr{M}(Z)$. By [Beĭ87, Lemma 3.3], there exists a dense affine open subscheme U in Z such that the open immersion $u : U \hookrightarrow Z$ satisfies the following conditions.

- The canonical morphism $A \to u^{\mathscr{M}}_* u^*_{\mathscr{M}} A$ is a monomorphism.
- For every $i \in \mathbb{Z}\backslash\{0\}$, one has

$$\mathsf{H}^i_{\mathscr{M}}(a^{\mathscr{M}}_! z^{\mathscr{M}}_! (u^{\mathscr{M}}_* u^*_{\mathscr{M}} A)) = 0.$$

Consider the distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$,

$$w^{\mathscr{M}}_! w^!_{\mathscr{M}}(A) \to A \to u^{\mathscr{M}}_* u^*_{\mathscr{M}}(A) \xrightarrow{+1} \tag{12}$$

where $w : W \hookrightarrow Z$ is the closed immersion of the complement of U in Z. Note that $\dim(W) \leqslant r-1$ and

$$w^{\mathscr{M}}_! w^!_{\mathscr{M}}(A) = w^{\mathscr{M}}_! w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}^{\mathscr{M}}_X)[2d - r].$$

Since $A \to u_*^{\mathscr{M}} u_{\mathscr{M}}^* A$ is a monomorphism in the category $\mathscr{M}(Z)$, using the long exact sequence obtained by applying the cohomological functor $\mathsf{H}^0_{\mathscr{M}}$ to the distinguished triangle (12), we get

$$w_!^{\mathscr{M}} \mathsf{H}^i_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = \mathsf{H}^i_{\mathscr{M}}(w_!^{\mathscr{M}} w^!_{\mathscr{M}}(A)) = 0$$

for $i \neq 1$. This implies that $\mathsf{H}^1_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d+1-r]$ belongs to $\mathscr{M}(Z)$ and

$$\mathrm{Coker}[A \to u_*^{\mathscr{M}} u^*_{\mathscr{M}} A] = \mathsf{H}^1_{\mathscr{M}}(w^!_{\mathscr{M}}(A)) = w^!_{\mathscr{M}} z^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d+1-r].$$

Since a is a smooth morphism, one has $a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}) = \mathbb{Q}_Y^{\mathscr{M}}(n-d)[2n-2d]$. Hence, in the case $z = \mathrm{Id}_Y$, the object

$$A := a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n]$$

belongs to $\mathscr{M}(X)$, since Y is smooth over k of dimension n. Using the above considerations, we construct simultaneously by induction an acyclic resolution $A^\bullet$ of A for the left exact functor $\mathsf{H}^0_{\mathscr{M}} \circ a_!^{\mathscr{M}}$ and a cellular stratification $Y_\bullet$ of Y such that

$$\mathsf{H}^j_{\mathscr{M}}((y_i)^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})) = 0$$

for every integer $j \in \mathbb{Z}\backslash\{2d-i\}$. The resolution is given in terms of the stratification by

$$A^i = (y_{n-i})_!^{\mathscr{M}} (u_{n-i})_*^{\mathscr{M}} (u_{n-i})^*_{\mathscr{M}}((y_{n-i})^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))[2d+i-n]$$

and

$$\mathrm{Coker}[A^i \to A^{i+1}] = (y_{n-i-1})_!^{\mathscr{M}}((y_{n-i-1})^!_{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}}))[2d+i+1-n]$$

where $u_i : Y_i \backslash Y_{i-1} \hookrightarrow Y_i$ is the open immersion and $y_i : Y_i \hookrightarrow Y$ the closed immersion. Since the resolution is acyclic for the left exact functor $\mathsf{H}^0_{\mathscr{M}} \circ a_!^{\mathscr{M}}$ there is an isomorphism in $\mathrm{D^b}(\mathscr{M}(X))$ between $a_!^{\mathscr{M}} A = a_!^{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})[2d-n]$ and the complex

$$\cdots \to 0 \to \mathsf{H}^0_{\mathscr{M}}(a_!^{\mathscr{M}}(A^0)) \to \mathsf{H}^0_{\mathscr{M}}(a_!^{\mathscr{M}}(A^1)) \to \cdots \to \mathsf{H}^0_{\mathscr{M}}(a_!^{\mathscr{M}}(A^n)) \to 0 \to \cdots \tag{13}$$

where $\mathsf{H}^0_{\mathscr{M}}(a_!^{\mathscr{M}}(A^0))$ is in degree zero. By construction

$$\mathsf{H}^0_{\mathscr{M}}(a_!^{\mathscr{M}}(A^i)) = \mathsf{TH}_X^{\mathscr{M}}(Y_{n-i}, Y_{n-i-1}, n-i)$$

and the complex (13) is nothing more than $\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)[-n]$. Hence $a_!^{\mathscr{M}} a^!_{\mathscr{M}}(\mathbb{Q}_X^{\mathscr{M}})$ is isomorphic to $\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)$. This concludes the proof of the proposition. $\square$

5. The Gabriel–Quillen embedding theorem and homotopical lifting

Let X be a smooth quasi-projective k-scheme. To construct a realization functor from the category of étale constructible motives, it is handy to have it first defined on the 'big category' $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ (it may also be useful in some instances to have such a 'big realization'). However, for this, the bounded derived category $\mathrm{D^b}(\mathscr{M}(X))$ of mixed Hodge modules is too small.

In this section we elaborate on the Gabriel–Quillen embedding theorem (see [TT90, Appendix A] for a very detailed treatment), to explain how one can remedy this problem and embed the bounded derived category into the homotopy category of some stable model category that does the job.

We also prove the results from homotopical algebra needed to construct the realization functors, in particular Proposition 5.7, which allows us to lift certain functors defined on Sm/X to a Quillen adjunction on the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$.

In this section $\mathscr{A}$ is an essentially small $\mathbb{Q}$-linear Abelian category. We denote by $0_{\mathscr{A}}$ the zero object in $\mathscr{A}$.

5.1 Let $\mathbf{PSh}(\mathscr{A},\mathbb{Q})$ be the category of presheaves of $\mathbb{Q}$-vector spaces on $\mathscr{A}$. Since $\mathscr{A}$ is $\mathbb{Q}$-linear, we have the Yoneda functor

$$\begin{aligned} \mathsf{i} : \mathscr{A} &\rightarrow \mathbf{PSh}(\mathscr{A},\mathbb{Q}) \\ A &\mapsto \mathrm{Hom}_{\mathscr{A}}(-,A). \end{aligned}$$

Denote by $\mathbf{PSha}(\mathscr{A},\mathbb{Q})$ the full subcategory of $\mathbf{PSh}(\mathscr{A},\mathbb{Q})$ with objects the additive presheaves of $\mathbb{Q}$-vector spaces (or equivalently the $\mathbb{Q}$-linear presheaves). The forgetful functor admits a left adjoint

$$a_{\mathrm{ad}} : \mathbf{PSh}(\mathscr{A},\mathbb{Q}) \to \mathbf{PSha}(\mathscr{A},\mathbb{Q})$$

given, for some $\mathscr{F} \in \mathbf{PSh}(\mathscr{A},\mathbb{Q})$, by the colimit $a_{\mathrm{ad}}(\mathscr{F}) := \mathrm{colim}_{(A,\mathsf{i}(A)\to\mathscr{F})\in\mathsf{i}\downarrow\mathscr{F}}\, \mathsf{i}(A)$ where $\mathsf{i}\downarrow\mathscr{F}$ is the over category in the category of presheaves of sets (in other words, $\mathscr{F}$ is viewed as a presheaf of sets and i as functor from $\mathscr{A}$ to presheaves of sets on $\mathscr{A}$).

Consider the Grothendieck pretopology on $\mathscr{A}$ (see [SGA4, Exposé II, Définition 1.3]) such that covering families of an object $A \in \mathscr{A}$ are families with one element $\{a : B \twoheadrightarrow A\}$ where a is an epimorphism, and let $\mathbf{Sh}(\mathscr{A},\mathbb{Q})$ be the category of sheaves of $\mathbb{Q}$-vector spaces for this topology. A presheaf $\mathscr{F} \in \mathbf{PSh}(\mathscr{A},\mathbb{Q})$ is a sheaf if and only if for every epimorphism $a : B \twoheadrightarrow A$ the sequence

$$0 \to \mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(B \times_A B)$$

is exact and the objects in

$$\mathbf{Sha}(\mathscr{A},\mathbb{Q}) := \mathbf{PSha}(\mathscr{A},\mathbb{Q}) \cap \mathbf{Sh}(\mathscr{A},\mathbb{Q})$$

are precisely the left exact $\mathbb{Q}$-linear contravariant functors from $\mathscr{A}$ to the category of $\mathbb{Q}$-vector spaces.

We have the sheafification functor

$$a_{\mathrm{epi}} : \mathbf{PSh}(\mathscr{A},\mathbb{Q}) \to \mathbf{Sh}(\mathscr{A},\mathbb{Q}).$$

Let us recall briefly its construction (see, for example, [TT90, § A.7.8]). For $A \in \mathscr{A}$, let $\mathcal{C}_A$ be the following filtered category. The objects in $\mathcal{C}_A$ are epimorphisms $B \twoheadrightarrow A$. Between two objects there is at most one map. There exists a map $(b : B \twoheadrightarrow A) \to (b' : B' \twoheadrightarrow A)$ if and only if there is a map $b'' : B' \to B$ such that $b \circ b'' = b'$. Given a presheaf $\mathscr{F} \in \mathbf{PSh}(\mathscr{A},\mathbb{Q})$, sending an object $B \twoheadrightarrow A$ to $\mathrm{Ker}(\mathscr{F}(B) \to \mathscr{F}(B \times_A B))$ is a functor from the filtered category $\mathcal{C}_A$ to the category of $\mathbb{Q}$-vector spaces. One then defines

$$\mathrm{L}\mathscr{F}(A) := \operatorname*{colim}_{(B\twoheadrightarrow A)\in\mathcal{C}_A} \mathrm{Ker}(\mathscr{F}(B) \to \mathscr{F}(B \times_A B))$$

and $a_{\mathrm{epi}}\mathscr{F} = \mathrm{LL}\mathscr{F}$.

Remark 5.1. Given $\mathscr{F} \in \mathbf{PSh}(\mathscr{A},\mathbb{Q})$, recall that $\mathrm{L}\mathscr{F} = 0$ if and only if, for every $A \in \mathscr{A}$ and every $\alpha \in \mathscr{F}(A)$, there exists an epimorphism $b : B \twoheadrightarrow A$ in $\mathscr{A}$ such that $b^*\alpha = 0$ in $\mathscr{F}(B)$ (see [TT90, A.7.11. Lemma]).

In particular, given a sequence $\mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}''$ in $\mathbf{PSh}(\mathscr{A},\mathbb{Q})$, the sequence

$$a_{\mathrm{epi}}\mathscr{F}' \xrightarrow{a_{\mathrm{epi}}\phi} a_{\mathrm{epi}}\mathscr{F} \xrightarrow{a_{\mathrm{epi}}\psi} a_{\mathrm{epi}}\mathscr{F}''$$

is exact in $\mathbf{Sh}(\mathscr{A},\mathbb{Q})$ if and only if for every $A \in \mathscr{A}$ and every $\alpha \in \mathscr{F}(A)$ such that $\psi(\alpha) = 0$ in $\mathscr{F}''(A)$ there exist an epimorphism $b : B \to A$ in $\mathscr{A}$ and an element $\beta \in \mathscr{F}'(B)$ such that $\phi(\beta) = b^*\alpha$.

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If $\mathscr{F}$ is additive then $a_{\mathrm{epi}}\mathscr{F}$ is also additive (see [TT90, § A.7.8]), hence the functor a_{epi} induces a functor

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \tag{14}$$

which is left adjoint to the forgetful functor. Note that $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is a Grothendieck Abelian category.

Lemma 5.2. *The category $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is a $\mathbb{Q}$-linear Abelian category. The Yoneda functor*

$$\begin{aligned} \mathsf{i} : \mathscr{A} &\to \mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \\ A &\mapsto \mathrm{Hom}_{\mathscr{A}}(-, A) \end{aligned}$$

is a fully faithful exact functor and $\mathscr{A}$ is stable by extension in $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$. Moreover, the induced functors

$$\mathrm{D}^{\mathrm{b}}(\mathscr{A}) \to \mathrm{D}^{\mathrm{b}}_{\mathscr{A}}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q})), \quad \mathrm{D}^{-}(\mathscr{A}) \to \mathrm{D}^{-}_{\mathscr{A}}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q}))$$

are equivalences of categories.

Remark 5.3. Let $\mathbf{Sha}(\mathscr{A}, \mathbb{Z})$ be the category of additive sheaves of Abelian groups on $\mathscr{A}$ for the topology of epimorphisms (i.e., the category of additive left exact functors from $\mathscr{A}$ to the category of Abelian groups as considered in [Gab62, II § 2] and [Qui73]). Since $\mathscr{A}$ is $\mathbb{Q}$-linear, the canonical functor $\mathbf{Sha}(\mathscr{A}, \mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathbb{Z})$ is an exact equivalence of categories. In particular, the statement of Lemma 5.2 is simply the embedding theorem proved by Gabriel in [Gab62] and generalized to exact categories by Quillen in [Qui73].

5.2 We endow the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its τ-local projective model structure where τ is the topology of epimorphisms (i.e., we consider the left Bousfield localization of the projective model structure of Lemma B.2 with respect to the maps that induce quasi-isomorphims on the associated complexes of sheaves). Let us consider the full subcategory $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ formed by the additive sheaves of complexes of $\mathbb{Q}$-vector spaces. The functor (14) induces a functor

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \to \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

left adjoint to the forgetful functor.

Consider the classes W, Fib of maps in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ defined as follows. A map $\mathscr{F} \to \mathscr{G}$ belongs to W (respectively, Fib) if and only if it is a τ-local weak equivalence (respectively, a τ-local fibration) in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Let Cof be the class of maps in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ that have the left lifting property with respect to maps in $\mathsf{W} \cap \mathsf{Fib}$.

By [Ayo07b, Lemme 4.4.41], the triple $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$ is a model structure (called the projective model structure) on the category $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) = \mathsf{Ch}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q}))$ and we have a Quillen adjunction

$$a_{\mathrm{epi}} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

for the projective model structures. Note that since (14) is an exact functor, the left adjoint preserves equivalences (i.e., quasi-isomorphisms).

Remark 5.4. Note that $\mathbf{Sha}(\mathscr{A}, \mathbb{Q})$ is an Abelian category, and the weak equivalences for the above model structure are the quasi-isomorphisms. In particular,

$$\mathsf{Ho}(\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))) = \mathrm{D}(\mathbf{Sha}(\mathscr{A}, \mathbb{Q})).$$

5.3 Let $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ be the category of simplicial objects in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. It is tensored and cotensored over the category of simplicial sets $\Delta^{\mathrm{op}}\mathsf{Sets}$. For $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $S \in \Delta^{\mathrm{op}}\mathsf{Sets}$, the tensor product $\mathscr{F} \odot S$ is defined as the simplicial object

$$n \mapsto \mathscr{F}_n \odot S_n := \coprod_{s \in S_n} \mathscr{F}_n.$$

Since the model category $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is proper, cofibrantly generated and stable (see [Ayo07b, Lemme 4.4.35] and [CD09, Theorem 2.1] for the right properness), by [RSS01, Proposition 4.5] the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ has a simplicial model structure, that is, such that

$$- \odot - : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \Delta^{\mathrm{op}}\mathsf{Sets} \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor. This model structure is called the canonical model structure and is obtained from the Reedy model structure. Recall that the Reedy weak equivalences are the level weak equivalences and that a map $\mathscr{F} \to \mathscr{G}$ is called a Reedy cofibration if for every integer n the map

$$\mathscr{F}_n \coprod_{\mathrm{L}_n(\mathscr{F})} \mathrm{L}_n(\mathscr{G}_n) \to \mathscr{G}_n$$

is a cofibration in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ where $\mathrm{L}_n(-)$ is the nth latching space functor. The left derived functor of the colimit functor provides a functor

$$\mathbb{L} \operatorname*{colim}_{\Delta^{\mathrm{op}}} : \mathsf{Ho}_{\mathrm{Reedy}}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))) \to \mathsf{Ho}(\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))),$$

and a map $\mathscr{F} \to \mathscr{G}$ is a canonical equivalence if its image under this functor is an isomorphism. The canonical cofibrations are the Reedy cofibrations, and fibrations are defined as maps having the right lifting property with respect to the class of trivial cofibrations.

Let $\mathsf{cc}(\mathscr{F})$ be the constant simplicial object and $\mathsf{Ev}(\mathscr{G}) = \mathscr{G}_0$. Then the adjoint functors cc and Ev provide a Quillen equivalence

$$\mathsf{cc} : \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \rightleftarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \mathsf{Ev} \tag{15}$$

(see [RSS01, Theorem 3.6]).

5.4 Given a complex of $\mathbb{Q}$-vector spaces K, let K_{cst} be the constant presheaf of $\mathbb{Q}$-vector spaces on $\mathscr{A}$. We denote by $\mathscr{F} \otimes \mathscr{G}$ the tensor product of two presheaves $\mathscr{F}, \mathscr{G} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})$. Note that if $\mathscr{F}$ is an object in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ (respectively, $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})))$, then $\mathscr{F} \otimes K_{\mathrm{cst}}$ belongs also to $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ (respectively, $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})))$. In particular, we have a functor

$$-\otimes (-)_{\mathrm{cst}} : \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})).$$

In the following proposition, we consider its extension to simplicial objects (a proof is given in Appendix B).

PROPOSITION 5.5. *The bifunctor*

$$-\otimes (-)_{\mathrm{cst}} : \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor where $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ *is endowed with the canonical model structure and* $\mathsf{Ch}(\mathbb{Q})$ *with the projective model structure.*

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Let $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. One then has an adjunction

$$\mathscr{F} \otimes (-)_{\mathrm{cst}} : \mathsf{Ch}(\mathbb{Q}) \rightleftarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \underline{\mathrm{Hom}}(\mathscr{F}, -) \tag{16}$$

where, for every $\mathscr{G}$ in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, the complex $\underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{G})$ is the equalizer of the maps

$$\prod_{n\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}}(\mathscr{F}_n, \mathscr{G}_n) \rightrightarrows \prod_{n\to m\in\Delta^{\mathrm{op}}} \underline{\mathrm{Hom}}(\mathscr{F}_n, \mathscr{G}_m),$$

and $\underline{\mathrm{Hom}}(\mathscr{F}_n, \mathscr{G}_n)$ and $\underline{\mathrm{Hom}}(\mathscr{F}_n, \mathscr{G}_m)$ are the usual complexes of graded morphisms. As a consequence one immediately gets the following corollary.

Corollary 5.6. *Let $\mathscr{F}$ be a cofibrant object in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Then (16) is a Quillen adjunction.*

5.5 Let $\mathcal{S}$ be a category and

$$r : \mathcal{S} \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

be a functor. With this functor are associated two functors

$$r^* : \mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : r_*$$

defined as follows. Given an object $\mathscr{F} \in \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, $r_*(\mathscr{F})$ is the presheaf on $\mathcal{S}$ with values in $\mathsf{Ch}(\mathbb{Q})$ defined by

$$r_*(\mathscr{F})(X) := \underline{\mathrm{Hom}}(r(X), \mathscr{F})$$

for $X \in \mathcal{S}$. Given a presheaf $\mathscr{X} \in \mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q}))$, the object $r^*(\mathscr{X})$ is defined as the coequalizer in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$,

$$r^*(\mathscr{X}) = \mathrm{Coeq}\Big[\bigoplus_{X\to Y\in\mathsf{Fl}(\mathcal{S})} r(X) \otimes \mathscr{X}(Y)_{\mathrm{cst}} \rightrightarrows \bigoplus_{X\in\mathcal{S}} r(X) \otimes \mathscr{X}(X)_{\mathrm{cst}}\Big]. \tag{17}$$

Recall that with an object $X \in \mathcal{S}$ and a presheaf $\mathscr{X} \in \mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q}))$ is associated an object $X \otimes \mathscr{X} \in \mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q}))$ (see, for example, [Ayo07b, § 4.4]). Given an object $K \in \mathsf{Ch}(\mathbb{Q})$, we denote by K_{cst} the constant presheaf on $\mathcal{S}$ with value K.

Proposition 5.7. *The functors*

$$r^* : \mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : r_*$$

are adjoint and the functors r and $r^*(- \otimes \mathbb{Q})$, are canonically isomorphic. Moreover, if $r(X)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ for every $X \in \mathcal{S}$, then they form a Quillen adjunction for the projective model structure on $\mathbf{PSh}(\mathcal{S}, \mathsf{Ch}(\mathbb{Q}))$.

Proof. We simply denote by Hom the set of morphisms in the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Then $\mathrm{Hom}_{\mathbf{PSha}}(r^*(\mathscr{X}), \mathscr{F})$ is by definition the equalizer of

$$\prod_{X\in\mathcal{S}} \mathrm{Hom}_{\mathbf{PSha}}(r(X) \otimes \mathscr{X}(X)_{\mathrm{cst}}, \mathscr{F}) \rightrightarrows \prod_{X\to Y\in\mathsf{Fl}(\mathcal{S})} \mathrm{Hom}_{\mathbf{PSha}}(r(X) \otimes \mathscr{X}(Y)_{\mathrm{cst}}, \mathscr{F}).$$

But for objects $U, V \in \mathcal{S}$,

$$\begin{aligned}\mathrm{Hom}(r(U) \otimes \mathscr{X}(V)_{\mathrm{cst}}, A) &= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathscr{X}(V), \underline{\mathrm{Hom}}(r(U), \mathscr{F})) \\ &= \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathscr{X}(V), r_*(\mathscr{F})(U)).\end{aligned}$$

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This means that $\mathrm{Hom}(r^*(\mathscr{X}),\mathscr{F})$ is equal to the set $\mathrm{Hom}_{\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))}(\mathscr{X},r_*(\mathscr{F}))$ of morphisms in $\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))$.

Assume that $r(X)$ is cofibrant for every $X\in\mathcal{S}$. If $a:\mathscr{F}\to\mathscr{G}$ is fibration (respectively, a trivial fibration), then by Corollary 5.6, for every object $X\in\mathcal{S}$, the induced map

$$\underline{\mathrm{Hom}}(r(X),\mathscr{F})\to\underline{\mathrm{Hom}}(r(X),\mathscr{G})$$

is a fibration (respectively, a trivial fibration). Hence the map $r_*(a)$ is a projective fibration (respectively, projective trivial fibration). This implies that the pair (r^*,r_*) is a Quillen adjunction.

It remains to construct an isomorphism $r^*(X\otimes\mathbb{Q})\simeq r(X)$ in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ functorial in X. Let $\mathscr{F}$ be an object in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$. Then there are isomorphisms functorial in $\mathscr{F}$ and X,

$$\begin{aligned}\mathrm{Hom}(r^*(X\otimes\mathbb{Q}),\mathscr{F})&\simeq\mathrm{Hom}_{\mathbf{PSh}(\mathcal{S},\mathsf{Ch}(\mathbb{Q}))}(X\otimes\mathbb{Q},r_*(\mathscr{F}))\\&\simeq\mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathbb{Q},r_*(\mathscr{F})(X))\\&=\mathrm{Hom}_{\mathsf{Ch}(\mathbb{Q})}(\mathbb{Q},\underline{\mathrm{Hom}}(r(X),\mathscr{F}))\\&\simeq\mathrm{Hom}(r(X),\mathscr{F})\end{aligned}$$

(see, for example, [Ayo07b, Proposition § 4.4]). The result then follows by the Yoneda lemma. □

5.6 Now let $\mathscr{M}\in\{\mathscr{N},\mathscr{H},\mathscr{P}\}$ and consider the category $\mathscr{M}(X)$. The functor $A\mapsto A(1)$ is a $\mathbb{Q}$-linear autoequivalence of the category $\mathscr{M}(X)$. It induces a $\mathbb{Q}$-linear exact equivalence of categories

$$\mathsf{T}^{\mathscr{M}}_X:\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))\to\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})).$$

For every simplicial sheaf $\mathscr{F}\in\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$, the object $\mathsf{T}^{\mathscr{M}}_X(\mathscr{F})$ is the simplicial sheaf such that for every $n\in\mathbb{N}$ and $A\in\mathscr{M}(X)$,

$$\mathsf{T}^{\mathscr{M}}_X(\mathscr{F})_n(A)=\mathscr{F}_n(A(-1))[1].$$

Note that for every $A\in\mathscr{M}(X)$ we have an isomorphism (functorial in A)

$$\mathsf{cc}(\mathsf{i}(A(1)[1]))=\mathsf{T}^{\mathscr{M}}_X(\mathsf{cc}(\mathsf{i}(A))).$$

Remark 5.8. Since the functor $\mathsf{T}^{\mathscr{M}}_X$ commutes with colimits, for every $\mathscr{F}\in\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$ and every $S\in\Delta^{\mathrm{op}}\mathsf{Sets}$ there is a canonical isomorphism

$$\mathsf{T}^{\mathscr{M}}_X(\mathscr{F})\odot S=\mathsf{T}^{\mathscr{M}}_X(\mathscr{F}\odot S).$$

Note that $\mathsf{T}^{\mathscr{M}}_X$ is a Quillen equivalence for the canonical model structure on $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$. Let

$$\mathfrak{M}\mathscr{M}(X):=\mathsf{Sp}_{\mathsf{T}^{\mathscr{M}}_X}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q})))$$

be the category of $\mathsf{T}^{\mathscr{M}}_X$-spectra in the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))$ as defined in [Hov01, Definition 1.1]. Then, by [Hov01, Theorem 5.1], the canonical functors

$$\mathsf{Sus}^0_{\mathsf{T}^{\mathscr{M}}_X}:\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X),\mathsf{Ch}(\mathbb{Q}))\leftrightarrows\mathfrak{M}\mathscr{M}(X):\mathsf{Ev}_0\tag{18}$$

are a Quillen equivalence where the right-hand side is endowed with its stable model structure.

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Lemma 5.9. *Let $\mathcal{C}$ be an essentially small category and*

$$F : \mathcal{C} \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$$

be a functor. Then there is a natural isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\operatorname*{hocolim}_{\mathcal{C}} F) \simeq \operatorname*{hocolim}_{\mathcal{C}}(\mathsf{T}_X^{\mathscr{M}} \circ F).$$

Proof. By [RSS01, Theorem 3.6, Proposition 4.5], the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its canonical model structure is a simplicial model category. The homotopy colimit functor is thus given by the Bousfield–Kan formula (see [Hir03, Definition 19.1.2]). In other words, if G is $\mathcal{C}$-diagram, then $\operatorname{hocolim}_{\mathcal{C}} G$ is the coequalizer

$$\coprod_{\sigma:\alpha\to\alpha'} G(\alpha) \odot \mathrm{B}(\alpha' \downarrow \mathcal{C})^{\mathrm{op}} \to \coprod_{\alpha\in\mathrm{Ob}(\mathcal{C})} G(\alpha) \odot \mathrm{B}(\alpha \downarrow \mathcal{C})^{\mathrm{op}}.$$

The result follows therefore from Remark 5.8. □

The Quillen equivalences (15) and (18) provide an equivalence of homotopy categories

$$\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) = \mathsf{Ho}(\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))) \simeq \mathsf{Ho}(\mathfrak{M}\mathscr{M}(X)), \tag{19}$$

and by Lemma 5.2 the left-hand side contains $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ as a full triangulated subcategory.

6. Perverse realization of motives

In this last section we give the construction of the realization functors. Let us briefly sketch it as a guide.

Given an affine scheme $Y \in \mathsf{SmAff}/X$, we have, associated with every stratification of Y, a complex of objects in $\mathscr{M}(X)$ that computes, for cellular stratifications, its $\mathscr{M}$-homology. The first step is to get rid of choices by taking a homotopy colimit over all stratifications. For functoriality it is necessary to consider all stratifications, but only the cellular ones yield the right answer (fortunately Lemma 4.7 shows that there are enough of them).

The realization is so far only defined over SmAff/X. The next step is to extend it to all smooth quasi-projective X-schemes by a homotopy left Kan extension inspired by the affine replacement functor introduced by Morel in [Mor12, § A.2].

One then uses Proposition 5.7 to extend it further to a left Quillen functor on the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ with its projective model structure. We check that it is compatible with the $(\mathbf{A}^1, \text{ét})$-Bousfield localization (see Proposition 6.6). The final step is to stabilize the construction (see Proposition 6.21).

6.1 Recall that we have an exact fully faithful functor

$$\mathsf{i} : \mathsf{Ch}(\mathscr{M}(X)) \to \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$$

and the constant simplicial functor

$$\mathsf{cc} : \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

For $Y \in \mathsf{SmAff}/X$ and a stratification $Y_\bullet$ of Y, we denote by $\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)$ the image of $\mathsf{r}_X^{\mathscr{M}}(Y, Y_\bullet)$ in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Definition 6.1. Let $Y \in \mathsf{SmAff}/X$. We set

$$\mathsf{ra}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)).$$

This provides a functor

$$\mathsf{ra}_X^{\mathscr{M}} : \mathsf{SmAff}/X \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

Indeed, let $f : Y \to Y'$ be a morphism in SmAff/X. There is a functor $f_\sharp : \mathsf{Strat}_Y \to \mathsf{Strat}_{Y'}$. Hence, by [Hir03, Proposition 19.1.8], we have a canonical morphism

$$\operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y', f_\sharp(Y_\bullet))) \to \operatorname*{hocolim}_{Y'_\bullet \in \mathsf{Strat}_{Y'}} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y', Y'_\bullet)) =: \mathsf{ra}_X^{\mathscr{M}}(Y').$$

On the other hand, we have a morphism $\mathsf{r}_X^{\mathscr{M}}(Y, -) \to \mathsf{r}_X^{\mathscr{M}}(Y', f_\sharp(-))$ of functors on Strat_Y, which induces a map

$$\mathsf{ra}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) \to \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y', f_\sharp(Y_\bullet))).$$

The composition provides a map $\mathsf{ra}_X^{\mathscr{M}}(Y) \to \mathsf{ra}_X^{\mathscr{M}}(Y')$, and functoriality is easy to check.

Remark 6.2. For every stratification $Y_\bullet$ of Y, the object $\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)$ is cofibrant in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ by Remark B.4. Hence, since cc is a left Quillen functor, it follows from [Hir03, Theorem 18.5.2] that $\mathsf{ra}_X^{\mathscr{M}}(Y)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Let us mention the following important consequence of Lemma 4.7.

Lemma 6.3. *Let $Y \in \mathsf{SmAff}/X$. Then the canonical morphism*

$$\operatorname*{hocolim}_{Y_\bullet \in \mathsf{Cell}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) \to \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet))$$

is a quasi-isomorphism.

Proof. By [Hir03, Theorem 19.6.7], it is enough to check that the inclusion functor $\mathsf{I} : \mathsf{Cell}_Y \to \mathsf{Strat}_Y$ is homotopy right cofinal. We have to check that, for every $Y_\bullet$ in Strat_Y, the nerve $\mathrm{B}(Y_\bullet \downarrow \mathsf{I})$ is contractible. This follows from Lemma 4.7, which implies that the category $Y_\bullet \downarrow \mathsf{I}$ is filtered. □

The next step is to extend the functor $\mathsf{ra}_X^{\mathscr{M}}$ to smooth quasi-projective X-schemes that may not be affine. For this we use a homotopy left Kan extension inspired by the affine replacement functor introduced by Morel in [Mor12, § A.2].

Definition 6.4. Let $Y \in \mathsf{Sm}/X$. We set

$$\mathsf{r}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{(Z \to Y) \in (\mathsf{SmAff}/X) \downarrow Y} \mathsf{ra}_X^{\mathscr{M}}(Z).$$

Let $I_Y : (\mathsf{SmAff}/X) \downarrow Y \to \mathsf{SmAff}/X$ be the forgetful functor defined by $I_Y(Z \to Y) = Z$. The above homotopy colimit may then be rewritten as

$$\mathsf{r}_X^{\mathscr{M}}(Y) := \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y.$$

Since $\mathsf{ra}_X^{\mathscr{M}}(Z)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ for every $Z \in \mathsf{SmAff}/X$, it follows from [Hir03, Theorem 18.5.2] that $\mathsf{r}_X^{\mathscr{M}}(Y)$ is cofibrant in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ as well.

Let $f : Y' \to Y$ be a morphism of smooth quasi-projective X-schemes. There is a functor

$$f_* : (\mathsf{SmAff}/X) \downarrow Y' \to (\mathsf{SmAff}/X) \downarrow Y$$

which maps a morphism $(Z \to Y')$ to the morphism $(Z \to Y)$ obtained by composition with f. Note that by definition, $I_Y \circ f_* = I_{Y'}$. Hence, by [Hir03, Proposition 19.1.8], we have a canonical morphism

$$\mathsf{r}_X^{\mathscr{M}}(Y') := \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y'} \mathsf{ra}_X^{\mathscr{M}} \circ I_{Y'} \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y =: \mathsf{r}_X^{\mathscr{M}}(Y).$$

This provides a functor

$$\mathsf{r}_X^{\mathscr{M}} : \mathsf{Sm}/X \to \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

Remark 6.5. Denote again by $\mathsf{r}_X^{\mathscr{M}}$ the restriction of $\mathsf{r}_X^{\mathscr{M}}$ to the subcategory SmAff/X. There is a canonical morphism of functors

$$\mathsf{r}_X^{\mathscr{M}} \to \mathsf{ra}_X^{\mathscr{M}}.$$

For every $Y \in \mathsf{SmAff}/X$, the induced morphism $\mathsf{r}_X^{\mathscr{M}}(Y) \to \mathsf{ra}_X^{\mathscr{M}}(Y)$ is a weak equivalence. Indeed, this follows from [Hir03, Corollary 19.6.8] since $(\mathrm{Id} : Y \to Y)$ is a final object in the over category $(\mathsf{SmAff}/X) \downarrow Y$.

6.2 We may apply the construction explained in § 5.5 to the functor $\mathsf{r}_X^{\mathscr{M}}$. Since $\mathsf{r}_X^{\mathscr{M}}(Y)$ is cofibrant for every $Y \in \mathsf{Sm}/X$, Proposition 5.7 yields a Quillen adjunction

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}} := (\mathsf{r}_X^{\mathscr{M}})^* : \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) : (\mathsf{r}_X^{\mathscr{M}})_* =: \mathsf{RRQ}_X^{\mathscr{M},\mathrm{eff}}$$

such that the functors $\mathsf{r}_X^{\mathscr{M}}$ and $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(- \otimes \mathbb{Q})$, are canonically isomorphic. Note that in the previous adjunction, the category of presheaves $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ is endowed with the projective model structure. To go further, we need to see that the adjunction is also compatible with the $(\mathbf{A}^1, \text{ét})$-model structure obtained by Bousfield localization.

Theorem 6.6. *The adjunction* $(\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}, \mathsf{RRQ}_X^{\mathscr{M},\mathrm{eff}})$ *induces a Quillen adjunction*

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}} : \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) : \mathsf{RRQ}_X^{\mathscr{M},\mathrm{eff}}$$

where $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ *is endowed with the* $(\mathbf{A}^1, \text{ét})$*-local projective model structure.*

The proof of Theorem 6.6 relies on the universal property of Bousfield localization and Proposition 2.1. Theorem 6.6 provides realization functors for effective étale motives. Let $\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}$ be the left derived functor of $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}$ and $\mathsf{RR}_X^{\mathscr{M},\mathrm{eff}}$ be the right derived functor of $\mathsf{RRQ}_X^{\mathscr{M},\mathrm{eff}}$. By Theorem 6.6 we have an adjunction

$$\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}} : \mathbf{DA}^{\mathrm{eff},\text{ét}}(X, \mathbb{Q}) \leftrightarrows \mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))) : \mathsf{RR}_X^{\mathscr{M},\mathrm{eff}}.$$

Recall that we have an equivalence of triangulated categories (provided by the Quillen equivalence (15))

$$\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) = \mathsf{Ho}(\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))) \rightleftarrows \mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))).$$

Remark 6.7. For every $Y \in \mathsf{Sm}/X$, the presheaf $Y \otimes \mathbb{Q}$ is cofibrant for the projective model structure on $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$. In particular,

$$\mathsf{RL}_X^{\mathscr{M},\mathrm{eff}}(Y \otimes \mathbb{Q}) = \mathsf{RLQ}_X^{\mathscr{M}}(Y \otimes \mathbb{Q}) \simeq \mathsf{r}_X^{\mathscr{M}}(Y).$$

6.3 In the remainder of this section we prove the properties of the functor $\mathsf{RL}_X^{\mathscr{M}}$ needed to prove Theorem 6.6. For this it will be handy to consider the following objects.

Definition 6.8. Let $Y \in \mathsf{SmAff}/X$. We set

$$\mathsf{TH}_X^{\mathscr{M}}(Y) := \operatorname*{colim}_{Y_\bullet \in \mathsf{Cell}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y, Y_\bullet).$$

This defines a complex of objects of the category $\mathbf{Sh}(\mathscr{M}(X), \mathbb{Q})$, that is, an object in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Remark 6.9. By Lemma 4.7, the complex $\mathsf{TH}_X^{\mathscr{M}}(Y)$ is also given by the colimit over all stratifications (in that case, however, the transition morphisms are not always quasi-isomorphisms):

$$\mathsf{TH}_X^{\mathscr{M}}(Y) := \operatorname*{colim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y, Y_\bullet).$$

In particular, the term of degree $-i$ of the complex $\mathsf{iTH}_X^{\mathscr{M}}(Y)$ is

$$\operatorname*{colim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) = \operatorname*{colim}_{Y_\bullet \in \mathsf{Cell}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i).$$

Remark 6.10. Note that the canonical maps

$$\begin{tikzcd}
\mathsf{ra}_X^{\mathscr{M}}(Y) \arrow[r] & \operatorname{colim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{ccTH}_X^{\mathscr{M}}(Y)[-2d] \\
\operatorname*{hocolim}_{Y_\bullet \in \mathsf{Cell}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) \arrow[u] \arrow[r] & \operatorname*{colim}_{Y_\bullet \in \mathsf{Cell}_Y} \mathsf{cc}(\mathsf{ir}_X^{\mathscr{M}}(Y, Y_\bullet)) = \mathsf{ccTH}_X^{\mathscr{M}}(Y)[-2d] \arrow[u]
\end{tikzcd}$$

are weak equivalences (the vertical one on the right even being an isomorphism).

From this we obtain a functor

$$\mathsf{TH}_X^{\mathscr{M}} : \mathsf{SmAff}/X \to \mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})).$$

We now prove some elementary properties of the functor $\mathsf{TH}_X^{\mathscr{M}}$. Recall that an affine vector bundle torsor over a quasi-projective k-scheme Y is an affine scheme T and an affine morphism $T \to Y$ which is a E-torsor for some vector bundle E over Y. Recall that every quasi-projective k-scheme Y admits an affine vector bundle torsor $T \to Y$ (see [Jou73, Lemme 1.5] or [Wei89, Proposition 4.3]). If Y is an affine scheme, then by [EGAIII, Théorème (1.3.1)] an affine vector bundle torsor $T \to Y$ is simply a vector bundle.

Lemma 6.11. *Let $Y \in \mathsf{SmAff}/X$ and $T \to Y$ be a vector bundle. Then the canonical morphism $\mathsf{TH}_X^{\mathscr{M}}(T) \to \mathsf{TH}_X^{\mathscr{M}}(Y)$ is a quasi-isomorphism.*

Proof. We may assume that Y is non-empty. Let $Y_\bullet$ be a stratification of Y and n the smallest integer such that $Y_n = Y$. Consider the stratification $T^\bullet$ of T given by

$$T_i := \begin{cases} Y_i & \text{if } i \leqslant n, \\ T & \text{if } i > n, \end{cases}$$

where Y_i is embedded in T using the zero section $Y \hookrightarrow T$. We have, by Lemma 3.9,

$$\mathsf{TH}_X^{\mathscr{M}}(T, Y, i) = 0$$

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for all $i \in \mathbb{Z}$. Hence, if $Y_\bullet$ is cellular, then $T_\bullet$ is cellular and

$$\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet) = \mathsf{TH}_X^{\mathscr{M}}(Z, Z^\bullet).$$

This implies that projection induces a quasi-isomorphism $\mathsf{TH}_X^{\mathscr{M}}(T) \to \mathsf{TH}_X^{\mathscr{M}}(Y)$ and the result follows. $\square$

Lemma 6.12. *Let $Y \in \mathsf{SmAff}/X$ and let $p : Y' \to Y$ be a Galois covering with Galois group G. Then the canonical morphism*

$$\mathsf{TH}_X^{\mathscr{M}}(Y')^G \to \mathsf{TH}_X^{\mathscr{M}}(Y)$$

is a quasi-isomorphism.

Proof. We may assume that Y is non-empty. Let $Y_\bullet$ be a stratification of Y. Let Y_i' be the inverse image of Y_i under p. Since p is finite étale, we have $\dim(Y_i') = \dim(Y_i) \leqslant i$ and $Y_\bullet'$ is a stratification of Y' invariant under the action of G. Lemma 3.8 implies that p induces an isomorphism of complexes

$$\mathsf{TH}_X^{\mathscr{M}}(Y', Y_\bullet')^G \to \mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet).$$

The result follows from this. $\square$

Remark 6.13. If $Y' \to Y$ is an étale morphism and $Y_\bullet$ is a stratification of Y, then $\dim(Y_i') = \dim(Y_i)$ for every $i \in \mathbb{Z}$ where $Y_i' := Y_i \times_Y Y'$. In particular, $Y_\bullet'$ is stratification of Y. We call it the induced stratification.

Let $a : Y \to X$ be a smooth affine morphism of quasi-projective k-varieties. Consider an elementary affine Nisnevich square

$$\begin{tikzcd} V \arrow[r, "u'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

Let $V_\bullet$, $U_\bullet$, $E_\bullet$ and $Y_\bullet$ be stratifications of the schemes V, U, E and Y, respectively. If $U_\bullet$, $E_\bullet$ and $V_\bullet$ are induced by $Y_\bullet$, then for $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$ the long exact sequence (8) yields the exact sequence

$$\begin{tikzcd} \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i+1) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \arrow[d] \\ & & \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \arrow[d] \\ & & \mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i-1). \end{tikzcd}$$

For perverse Nori motives we just have an exact sequence (see Corollary 3.7)

$$\mathsf{TH}_X^{\mathscr{N}}(V_i, V_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{N}}(U_i, U_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{N}}(E_i, E_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{N}}(Y_i, Y_{i-1}, i).$$

If the stratifications are just compatible, by which we mean that $Y_\bullet$ is finer than $u_\sharp(U_\bullet)$ and $e_\sharp(E_\bullet)$, $U_\bullet$ is finer than $e'_\sharp(V_\bullet)$, and $E_\bullet$ is finer than $u'_\sharp(V_\bullet)$, then we just have morphisms

$$\mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i).$$

This is a complex which may not be exact.

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PROPOSITION 6.14. *Let $Y \in \mathsf{SmAff}/X$ and*

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

be an elementary affine Nisnevich square. The short sequence in $\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$*,*

$$0 \to \mathsf{TH}_X^{\mathscr{M}}(V) \to \mathsf{TH}_X^{\mathscr{M}}(U) \oplus \mathsf{TH}_X^{\mathscr{M}}(E) \to \mathsf{TH}_X^{\mathscr{M}}(Y) \to 0,$$

is exact.

Proof. A sequence of complexes being exact if and only if it is degreewise exact, this amounts to showing that, for every integer $i \in \mathbb{Z}$, the sequence

$$\begin{tikzcd} 0 \arrow[d] & \\ \operatorname*{colim}_{V_\bullet \in \mathsf{Strat}_V} \mathsf{iTH}_X^{\mathscr{M}}(V_i, V_{i-1}, i) \arrow[r] & \operatorname*{colim}_{U_\bullet \in \mathsf{Strat}_U} \mathsf{iTH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \operatorname*{colim}_{E_\bullet \in \mathsf{Strat}_E} \mathsf{iTH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \arrow[d] \\ & \operatorname*{colim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{iTH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \arrow[d] \\ & 0 \end{tikzcd} \tag{20}$$

is exact in $\mathbf{Sh}(\mathscr{M}(X), \mathbb{Q})$ (see Remark 6.9). For $W \in \{V, U, E, Y\}$, let

$$\mathscr{F}_W := \operatorname*{colim}_{W_\bullet \in \mathsf{Strat}_W} \mathsf{iTH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)$$

where the colimit is taken in the category $\mathbf{PSha}(\mathscr{M}(X), \mathbb{Q})$ and not in the category of sheaves $\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})$. For every $A \in \mathscr{M}(X)$, one has

$$\begin{aligned} \mathscr{F}_W(A) &= \operatorname*{colim}_{W_\bullet \in \mathsf{Strat}_W} \Gamma(A, \mathsf{iTH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)) \\ &= \operatorname*{colim}_{W_\bullet \in \mathsf{Strat}_W} \operatorname{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(W_i, W_{i-1}, i)). \end{aligned}$$

Note that the sequence (20) is the induced sequence

$$0 \to a_{\mathrm{epi}}\mathscr{F}_V \to a_{\mathrm{epi}}\mathscr{F}_U \oplus a_{\mathrm{epi}}\mathscr{F}_E \to a_{\mathrm{epi}}\mathscr{F}_Y \to 0,$$

so we may use Remark 5.1 to show its exactness. Let us prove the exactness on the right and on the left (the exactness at the center is proved similarly using Lemma 3.6 or Corollary 3.7).

Let $A \in \mathscr{M}(X)$ and $\alpha \in \mathscr{F}_Y(A)$. There exist a stratification $Y_\bullet$ of Y and an element $\alpha_{Y_\bullet} \in \operatorname{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i))$ that lifts α. Let $U_\bullet$, $E_\bullet$ and $V_\bullet$ be the induced stratifications. Let $V'_\bullet$ be a cellular stratification of V finer than $V_\bullet$, and let $Y''_\bullet$ be a stratification of Y such that $h(V'_i) \subseteq Y''_i$ for every $i \in \mathbb{Z}$. Let $E''_\bullet$, $U''_\bullet$ and $V''_\bullet$ be the stratifications induced by $Y''_\bullet$. Let us show that the morphism

$$\mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \to \mathsf{TH}_X^{\mathscr{M}}(Y''_i, Y''_{i-1}, i)$$

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factors through the image of the morphism

$$\mathsf{TH}_X^{\mathscr{M}}(U_i'', U_{i-1}'', i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i'', E_{i-1}'', i) \to \mathsf{TH}_X^{\mathscr{M}}(Y_i'', Y_{i-1}'', i).$$

Using the exact and faithful functor $\mathscr{N}(X) \to \mathscr{P}(X)$, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. In that case, there is by Lemma 3.6 a commutative diagram in $\mathscr{M}(X)$ with exact rows:

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(U_i'', U_{i-1}'', i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i'', E_{i-1}'', i) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(Y_i'', Y_{i-1}'', i) \arrow[r] & \mathsf{TH}^{\mathscr{M}_X}(V_i'', V_{i-1}'', i-1) \\
 & & \mathsf{TH}_X^{\mathscr{M}}(V_i', V_{i-1}', i-1) \arrow[u] \\
\mathsf{TH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \arrow[r] \arrow[uu] & \mathsf{TH}_X^{\mathscr{M}}(Y_i, Y_{i-1}, i) \arrow[r] \arrow[uu] & \mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i-1) \arrow[u]
\end{tikzcd}$$

The result then follows from the fact that $\mathsf{TH}_X^{\mathscr{M}}(V_i', V_{i-1}', i-1) = 0$ since $V_\bullet'$ is a cellular stratification. This implies the existence of an epimorphism $B \twoheadrightarrow A$ in $\mathscr{M}(X)$ and elements

$$(\beta_{U_\bullet''}, \gamma_{E_\bullet''}) \in \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}_X^{\mathscr{M}}(U_i'', U_{i-1}'', i)) \oplus \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}_X^{\mathscr{M}}(E_i'', E_{i-1}'', i))$$

such that the image of $(\beta_{U_\bullet''}, \gamma_{E_\bullet''})$ in

$$\Gamma(B, \mathsf{i}\mathsf{TH}_X^{\mathscr{M}}(Y_i'', Y_{i-1}'', i)) = \mathrm{Hom}_{\mathscr{M}(X)}(B, \mathsf{TH}_X^{\mathscr{M}}(Y_i'', Y_{i-1}'', i))$$

is equal to the image of $\alpha_{Y_\bullet}$. Let (β, γ) the image of $(\beta_{U_\bullet''}, \gamma_{E_\bullet''})$ in $\mathscr{F}_U(B) \oplus \mathscr{F}_E(B)$. Then the image of (β, γ) in $\mathscr{F}_Y(B)$ is equal to the image of α. This shows the exactness on the right.

Let $A \in \mathscr{M}(X)$ and $\alpha \in \mathscr{F}_V(A)$ such that $\alpha = 0$ in $\mathscr{F}_U(A) \oplus \mathscr{F}_E(A)$. Let $V_\bullet$ be a stratification of V and $\alpha_{V_\bullet}$ an element in $\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i))$ that lifts α. There exist a stratification $U_\bullet$ of U and a stratification $E_\bullet$ of E, both compatible with $V_\bullet$, such that $\alpha_{V_\bullet} = 0$ in

$$\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(U_i, U_{i-1}, i)) \oplus \mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(E_i, E_{i-1}, i)).$$

Let $Y_\bullet$ be a stratification of Y compatible with $U_\bullet$ and $V_\bullet$. Let $Y_\bullet'$ be a cellular stratification finer than $Y_\bullet$, and let $V_\bullet'$, $U_\bullet'$ and $E_\bullet'$ be the induced stratifications. The morphism

$$\mathsf{TH}_X^{\mathscr{M}}(V_i', V_{i-1}', i) \to \mathsf{TH}_X^{\mathscr{M}}(U_i', U_{i-1}', i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i', E_{i-1}', i)$$

is a monomorphism. Indeed, using the faithful exact functor $\mathscr{N}(X) \to \mathscr{P}(X)$, we may assume that $\mathscr{M} \in \{\mathscr{H}, \mathscr{P}\}$. In that case, by Lemma 3.6, one has the commutative diagram in which the top row is exact:

$$\begin{tikzcd}
\mathsf{TH}_X^{\mathscr{M}}(Y_i', Y_{i-1}', i+1) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(V_i', V_{i-1}', i) \arrow[r] & \mathsf{TH}_X^{\mathscr{M}}(U_i', U_{i-1}', i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i', E_{i-1}', i) \\
 & \mathsf{TH}_X^{\mathscr{M}}(V_i, V_{i-1}, i) \arrow[r] \arrow[u] & \mathsf{TH}_X^{\mathscr{M}}(U_i, U_{i-1}, i) \oplus \mathsf{TH}_X^{\mathscr{M}}(E_i, E_{i-1}, i) \arrow[u]
\end{tikzcd}$$

Since $Y_\bullet'$ is cellular, $\mathsf{TH}_X^{\mathscr{M}}(Y_i', Y_{i-1}', i+1) = 0$ and the claim follows. Hence the image of $\alpha_{V_\bullet}$ in $\mathrm{Hom}_{\mathscr{M}(X)}(A, \mathsf{TH}_X^{\mathscr{M}}(V_i', V_{i-1}', i)$ vanishes and therefore $\alpha = 0$ in $\mathscr{F}_V(A)$. This shows the exactness on the left. $\square$

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6.4 Let us state two consequences of the previous results.

Corollary 6.15. *Let $Y \in \mathsf{SmAff}/X$ and $T \to Y$ be a vector bundle. Then the canonical morphism*

$$\mathsf{ra}_X^{\mathscr{M}}(T) \to \mathsf{ra}_X^{\mathscr{M}}(Y)$$

is a weak equivalence in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Proof. This follows from Remark 6.10 and Lemma 6.11. □

Corollary 6.16. *Let*

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\ U \arrow[r, "u"] & Y \end{tikzcd}$$

be an elementary affine Nisnevich square. Then the following square is homotopy cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$:

$$\begin{tikzcd} \mathsf{ra}_X^{\mathscr{M}}(V) \arrow[r] \arrow[d] & \mathsf{ra}_X^{\mathscr{M}}(E) \arrow[d] \\ \mathsf{ra}_X^{\mathscr{M}}(U) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(Y) \end{tikzcd}$$

Proof. This follows immediately from Proposition 6.14 using a classical result of homological algebra (see, for example, [KS94, Proposition 1.7.5]) and Remark 6.10. □

Proposition 6.17. *Let $Y \in \mathsf{Sm}/X$ and $T \to Y$ be an affine vector bundle torsor. Then the morphism*

$$\mathsf{r}_X^{\mathscr{M}}(T) \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

is a weak equivalence in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

Proof. The proof of the proposition follows the proof of [Wen10, Proposition 3.11]. Let $p : T \to Y$ be an affine vector bundle torsor. We have to show that the morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y$$

induced by the functor p_* is a weak equivalence ($I_Y \circ p_* = I_T$). Consider the functor obtained by base change along p,

$$\begin{aligned} p^* : (\mathsf{SmAff}/X) \downarrow Y &\to (\mathsf{SmAff}/X) \downarrow T \\ (Z \to Y) &\mapsto (T \times_Y Z \to T). \end{aligned}$$

Note that this functor is well defined. Indeed, $T \times_Y Z \to Z$ is an affine vector bundle torsor over an affine scheme Z and therefore $T \times_Y Z$ is also an affine scheme. As shown in [Wen10, Proof of Proposition 3.11], the functor p^* is homotopy right cofinal, and the canonical morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \tag{21}$$

is therefore a weak equivalence by [Hir03, Theorem 19.6.7]. On the other hand, the morphisms of affine schemes $(I_T \circ p^*)(Z \to Y) = T \times_Y Z \to Z = I_Y(Z \to Y)$ define a morphism of functors $I_T \circ p^* \to I_Y$ and thus yield morphisms of functors

$$\mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \mathsf{ra}_X^{\mathscr{M}} \circ I_Y, \quad \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_*. \tag{22}$$

Since $p_Z : T \times_Y Z \to Z$ is an affine vector bundle torsor, by Lemma 6.15, the morphisms (22) are weak equivalences of diagrams and therefore, by [Hir03, Theorem 19.4.2], the maps

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y, \tag{23}$$

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \tag{24}$$

are weak equivalences. We have a commutative square

$$\begin{tikzcd}
\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \arrow[r] \arrow[d, "(24)"] & \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \arrow[d, "(23)"] \\
\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \circ p_* \arrow[r] & \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y
\end{tikzcd}$$

Since (23) and (24) are weak equivalences, it is enough to show that the top horizontal map is a weak equivalence. The composition

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \circ p_* \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \circ p^* \xrightarrow{(21)} \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow T} \mathsf{ra}_X^{\mathscr{M}} \circ I_T \tag{25}$$

of this map with (21) is the canonical map induced by the functor

$$p^* \circ p_* : (\mathsf{SmAff}/X) \downarrow T \to (\mathsf{SmAff}/X) \downarrow T.$$

Since this functor is homotopy right cofinal (see [Wen10, proof of Proposition 3.11]), the composition (25) is a weak equivalence by [Hir03, Theorem 19.6.7]. This concludes the proof since (21) is a weak equivalence. □

Proposition 6.18. *Let $Y \in \mathsf{Sm}/X$.*

(i) *Let U, E be an open cover of Y and $V = U \times_Y E$. Then the square*

$$\begin{tikzcd}
\mathsf{r}_X^{\mathscr{M}}(V) \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(E) \arrow[d] \\
\mathsf{r}_X^{\mathscr{M}}(U) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y)
\end{tikzcd}$$

is homotopy cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$.

(ii) *The morphism*

$$\mathsf{r}_X^{\mathscr{M}}(Y \times_k \mathbf{A}_k^1) \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

is a weak equivalence.

Proof. Proposition 6.17 allows the use of Jouanolou's trick. The proof of the first statement is then completely similar to the proof of the Mayer–Vietoris property for homotopy invariant K-theory given in [Wei89, Theorem 5.1]. The details are left to the reader. Let us prove the second statement. Let $T \to Y$ be an affine vector bundle torsor (since Y is quasi-projective over k, such a torsor exists by [Jou73, Lemme 1.5]). We have commutative squares

$$\begin{tikzcd}
\mathsf{ra}_X^{\mathscr{M}}(T \times_k \mathbf{A}_k^1) \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(T \times_k \mathbf{A}_k^1) \arrow[l] \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(Y \times_k \mathbf{A}_k^1) \arrow[d] \\
\mathsf{ra}_X^{\mathscr{M}}(T) & \mathsf{r}_X^{\mathscr{M}}(T) \arrow[l] \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y)
\end{tikzcd}$$

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By Proposition 6.17 and Remark 6.5, the horizontal morphisms are weak equivalences. The result then follows from Lemma 6.15 which ensures that the vertical arrow on the left is a weak equivalence. $\square$

Remark 6.19. Let $Y \in \mathsf{Sm}/X$ and $\mathscr{U} := \{U_i \hookrightarrow Y\}_{i\in I}$ be a finite open cover of Y. Let U be the disjoint union of the U_i. We have the usual Čech simplicial object $\check{\mathrm{C}}(\mathscr{U}) : \Delta^{\mathrm{op}} \to \mathsf{SmAff}/X$ such that for every $n \in \Delta$, $\check{\mathrm{C}}(\mathscr{U})_n$ is the fiber product over X of n copies of U:

$$\check{\mathrm{C}}(\mathscr{U})_n = U \times_X \cdots \times_X U.$$

One can show by induction on the number of open subsets in $\mathscr{U}$ (see [Wei89, Theorem 6.3]) that the canonical morphism

$$\mathsf{r}_X^{\mathscr{M}}(Y, \mathscr{U}) := \operatorname*{hocolim}_{\Delta} \mathsf{r}_X^{\mathscr{M}}(\check{\mathrm{C}}(\mathscr{U})) \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

is a weak equivalence.

Lemma 6.20. *Let $Y \in \mathsf{SmAff}/X$ and let $p : Y' \to Y$ be a Galois covering with Galois group G. Then the canonical morphism*

$$\mathsf{r}_X^{\mathscr{M}}(Y')^G \to \mathsf{r}_X^{\mathscr{M}}(Y)$$

is a weak equivalence.

Proof. Let $T \to Y$ be an affine vector bundle torsor and $p : T' \to T$ the Galois cover obtained by base change. We have commutative squares.

$$\begin{tikzcd}
\mathsf{ra}_X^{\mathscr{M}}(T')^G \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(T')^G \arrow[l] \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(Y')^G \arrow[d] \\
\mathsf{ra}_X^{\mathscr{M}}(T) & \mathsf{r}_X^{\mathscr{M}}(T) \arrow[l] \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y)
\end{tikzcd}$$

By Proposition 6.17 and Remark 6.5 the horizontal morphisms are weak equivalences. The result then follows from Remark 6.10 and Lemma 6.12 which ensure that the vertical arrow on the left is a weak equivalence. $\square$

Proof of Theorem 6.6. Let us first remark that $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\emptyset \otimes \mathbb{Q}) = 0$. Let $Y \in \mathsf{Sm}/X$ and

$$\begin{tikzcd}
V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd}$$

be either a Zariski square or an affine Nisnevich square. By Proposition 6.18 and Corollary 6.16, the square

$$\begin{tikzcd}
\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(V \otimes \mathbb{Q}) \arrow[r] \arrow[d] & \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(E \otimes \mathbb{Q}) \arrow[d] \\
\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(U \otimes \mathbb{Q}) \arrow[r] & \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(Y \otimes \mathbb{Q})
\end{tikzcd}$$

is cocartesian in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ (here we have also used Remarks 6.5 and 6.7). One the other hand, by Proposition 6.18 and Remark 6.7, the morphism

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathbf{A}_Y^1 \otimes \mathbb{Q}) \to \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(Y \otimes \mathbb{Q})$$

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is an isomorphism in $\mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})))$. If $p: Y' \to Y$ is a Galois covering with Galois group G, then by Remark 6.7 and Lemma 6.20 the morphism

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}((Y' \otimes \mathbb{Q})^G) \to \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(Y \otimes \mathbb{Q})$$

is an isomorphism in $\mathsf{Ho}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})))$. It follows that $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}$ sends the morphisms in (2) to isomorphisms in the homotopy category and Theorem 6.6 follows from the universal property of Bousfield localizations. □

6.5 It remains to stabilize the above construction in order to obtain a realization functor also for motives that may not be effective. The key result that we need is the following proposition.

PROPOSITION 6.21. *There exists a natural transformation*

$$\begin{tikzcd}
\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \arrow[r, "\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}"] \arrow[d, "T_X\otimes -"'] \arrow[dr, phantom, "\Downarrow\rho"] & \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q})) \arrow[d, "\mathsf{T}_X^{\mathscr{M}}"] \\
\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q})) \arrow[r, "\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}"] & \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))
\end{tikzcd}$$

such that

$$\rho_{\mathscr{X}} : (\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}})(\mathscr{X}) \to \mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(T_X \otimes \mathscr{X})$$

is a weak equivalence for every presheaf $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$.

The pair $(\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}, \rho)$ is therefore a Quillen map of pairs in the sense of [Hov01, Definition 5.4], and [Hov01, Proposition 5.5] provides a Quillen adjunction

$$\mathsf{RLQ}_X^{\mathscr{M}} : \mathsf{Sp}_{T_X}(\mathbf{PSh}(\mathsf{Sm}/X), \mathsf{Ch}(\mathbb{Q})) \leftrightarrows \mathfrak{M}\mathscr{M}(X) : \mathsf{RRQ}_X^{\mathscr{M}}$$

with respect to the $(\mathbf{A}^1, \text{ét})$-local stable projective model structure on the left-hand side and the stable model structure on the right-hand side. Note that by construction (see [Hov01, Proposition 5.5]), for every $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$, the $\mathsf{T}_X^{\mathscr{M}}$-spectra

$$\mathsf{Sus}^0_{\mathsf{T}_X^{\mathscr{M}}}(\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X})), \quad \mathsf{RLQ}_X^{\mathscr{M}}(\mathsf{Sus}^0_{T_X}(\mathscr{X}))$$

are canonically equivalent. Using the above Quillen adjunction and the equivalences (19), one gets an adjunction on the homotopy categories

$$\mathsf{RL}_X^{\mathscr{M}} : \mathbf{DA}^{\text{ét}}(X, \mathbb{Q}) \rightleftarrows \mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q})) : \mathsf{RR}_X^{\mathscr{M}}.$$

Recall that the full triangulated category $\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q})$ of constructible motives is defined as the smallest triangulated subcategory of $\mathbf{DA}^{\text{ét}}(X, \mathbb{Q})$ stable by direct factors and containing the homological motives of smooth quasi-projective X-schemes (or equivalently, smooth affine X-schemes by Mayer and Vietoris). Since by construction, for every affine smooth X-scheme Y, the image lands in the full triangulated category $\mathrm{D}^{\mathrm{b}}(\mathscr{M}(X))$ of $\mathrm{D}(\mathbf{Sha}(\mathscr{M}(X), \mathbb{Q}))$, the above functor induces a triangulated functor

$$\mathbf{DA}^{\text{ét}}_{\mathsf{ct}}(X, \mathbb{Q}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{M}(X)).$$

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6.6 It remains to prove Proposition 6.21. The proof is slightly technical, as we have to unwind the construction of the functor $\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}$ to construct the natural transformation ρ step by step. It essentially boils down to properties of cellular complexes associated with specific stratifications. That is to say, we have the following lemma.

LEMMA 6.22. *Let $Y \in \mathsf{SmAff}/X$. There exists a morphism*

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$ such that the induced morphism

$$\mathsf{ra}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

is a weak equivalence (here the morphism $\mathsf{ra}_X^{\mathscr{M}}(Y) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$ is the morphism induced by the unit section of $\mathbf{G}_{m,Y}$).

Proof. Let $Y_\bullet$ be a stratification of Y. Consider the stratification $\mathsf{G}(Y_\bullet)$ of the quasi-projective k-scheme $\mathbf{G}_{m,Y}$ defined by the closed subsets $\mathsf{G}(Y_\bullet)_i := Y_{i-1} \times_k \mathbf{G}_{m,k}$. By Lemma 3.10, the complex $\mathsf{TH}_X^{\mathscr{M}}(Y, Y_\bullet)(1)[1]$ is a direct summand of the complex $\mathsf{TH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}, \mathsf{G}(Y_\bullet))$. The inclusion as a direct factor induces a morphism of functors on Strat_Y,

$$\mathsf{TH}_X^{\mathscr{M}}(Y, -)(1)[1] \to \mathsf{TH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}, \mathsf{G}(-)),$$

and thus a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y, -))) \to \mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}, \mathsf{G}(-))).$$

Taking homotopy colimits, we obtain a morphism in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$,

$$\operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y, -))) \to \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}, \mathsf{G}(-))) \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}),$$

where the second morphism is the canonical morphism associated with the functor $\mathsf{G} : \mathsf{Strat}_Y \to \mathsf{Strat}_{\mathbf{G}_{m,Y}}$ (see [Hir03, Proposition 19.1.8]).

By Lemma 5.9, there is a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) := \mathsf{T}_X^{\mathscr{M}}(\operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y, -))) \simeq \operatorname*{hocolim}_{Y_\bullet \in \mathsf{Strat}_Y} \mathsf{T}_X^{\mathscr{M}}(\mathsf{cc}(\mathsf{iTH}_X^{\mathscr{M}}(Y, -))).$$

This provides the desired morphism. □

Remark 6.23. The morphisms constructed in the proof of Lemma 6.22 are functorial in Y and define a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \to \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,-})$$

on SmAff/X.

To prove Proposition 6.21 we will also need the following lemma.

LEMMA 6.24. *Let $\mathsf{c}_X^{\mathscr{M}}$ be the cokernel of the natural transformation $\mathsf{r}_X^{\mathscr{M}} \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-})$ given by the unit section. Then there is an isomorphism of functors*

$$(\mathsf{c}_X^{\mathscr{M}})^* \simeq (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes -).$$

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Proof. By definition, $\mathsf{c}_X^{\mathscr{M}}$ is a functor $\mathsf{Sm}/X \to \Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{M}(X), \mathsf{Ch}(\mathbb{Q}))$, and for every smooth quasi-projective X-scheme Y one has an exact sequence

$$0 \to \mathsf{r}_X^{\mathscr{M}}(Y) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \to \mathsf{c}_X^{\mathscr{M}}(Y) \to 0. \tag{26}$$

The endofunctor $(\mathbf{G}_{m,X} \otimes \mathbb{Q}) \otimes -$ of the category $\mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ admits $\mathscr{H}om(\mathbf{G}_{m,X} \otimes \mathbb{Q}, -)$ as right adjoint (here $\mathscr{H}om$ denotes the internal Hom in the category of presheaves on Sm/X). For $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$, the presheaf $\mathscr{H}om(\mathbf{G}_{m,X} \otimes \mathbb{Q}, \mathscr{X})$ is nothing more than the presheaf $Y \mapsto \mathscr{X}(\mathbf{G}_{m,Y})$. It follows that the functor $(\mathsf{r}_X^{\mathscr{M}})^*((\mathbf{G}_{m,X} \otimes \mathbb{Q}) \otimes -)$ is left adjoint to the functor $\mathscr{F} \mapsto \underline{\mathrm{Hom}}(\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}), \mathscr{F})$ and is therefore isomorphic to the functor $(\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}))^*$. For every $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$, this isomorphism fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathsf{r}_X^{\mathscr{M}})^*((X \otimes \mathbb{Q}) \otimes \mathscr{X}) \arrow[r] \arrow[d, no head, "\mathrm{iso.}"'] & (\mathsf{r}_X^{\mathscr{M}})^*((\mathbf{G}_{m,X} \otimes \mathbb{Q}) \otimes \mathscr{X}) \arrow[r] \arrow[d, no head, "\mathrm{iso.}"'] & (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes \mathscr{X}) \arrow[r] & 0 \\
& (\mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X}) \arrow[r] & (\mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,-}))^*(\mathscr{X}) \arrow[r] & (\mathsf{c}_X^{\mathscr{M}})^*(\mathscr{X}) \arrow[r] & 0
\end{tikzcd}$$

The rows in this diagram are exact. For the upper row this follows from the fact that $(r_X^{\mathscr{M}})^*$ is right exact (it is a left adjoint). For the lower row it follows from the exact sequences (26) and the definition of (17) as a colimit. This provides an isomorphism of functors

$$(\mathsf{c}_X^{\mathscr{M}})^* \simeq (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes -)$$

as desired. □

Proof of Proposition 6.21. By construction,

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(\mathscr{X})) = \mathsf{T}_X^{\mathscr{M}}((\mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X})) = (\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}})^*(\mathscr{X})$$

and

$$\mathsf{RLQ}_X^{\mathscr{M},\mathrm{eff}}(T_X \otimes \mathscr{X}) = (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes \mathscr{X}),$$

hence it is enough to construct a natural transformation

$$\vartheta : (\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}})^* \to (\mathsf{r}_X^{\mathscr{M}})^*(T_X \otimes -)$$

such that $\vartheta_{\mathscr{X}}$ is a weak equivalence for every $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$. By Lemma 6.24, it is therefore enough to construct a natural transformation

$$\varrho : \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{r}_X^{\mathscr{M}} \to \mathsf{c}_X^{\mathscr{M}}$$

such that ϱ_Y is a weak equivalence for every $Y \in \mathsf{Sm}/X$.

Let us first extend Lemma 6.22 to smooth quasi-projective X-schemes which may not be affine. For $Y \in \mathsf{Sm}/X$, we construct a morphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

as follows. Consider the functor

$$\begin{aligned} \mathbf{G}_m : (\mathsf{SmAff}/X) \downarrow Y &\to (\mathsf{SmAff}/X) \downarrow \mathbf{G}_{m,Y} \\ (Z \to Y) &\mapsto (\mathbf{G}_{m,Z} \to \mathbf{G}_{m,Y}) \end{aligned}$$

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and the induced morphism (see [Hir03, Proposition 19.1.8])

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow \mathbf{G}_{m,Y}} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} =: \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}).$$

Note that $I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m = \mathbf{G}_{m,-} \circ I_Y$. By Remark 6.23, the morphisms of Lemma 6.22 thus induce a morphism of functors

$$\mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \to \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} \circ \mathbf{G}_m.$$

This provides a morphism

$$\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y \to \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow \mathbf{G}_{m,Y}} \mathsf{ra}_X^{\mathscr{M}} \circ I_{\mathbf{G}_{m,Y}} =: \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}).$$

By Lemma 5.9, there is a canonical isomorphism

$$\mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) := \mathsf{T}_X^{\mathscr{M}}(\operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{ra}_X^{\mathscr{M}} \circ I_Y) = \operatorname*{hocolim}_{(\mathsf{SmAff}/X)\downarrow Y} \mathsf{T}_X^{\mathscr{M}} \circ \mathsf{ra}_X^{\mathscr{M}} \circ I_Y.$$

Note that for every affine scheme $Y \in \mathsf{SmAff}/X$ the square

$$\begin{tikzcd}
\mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \arrow[d] \\
\mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(Y)) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})
\end{tikzcd}$$

is commutative, where the vertical morphisms are the weak equivalences of Remark 6.5 and the lower horizontal morphism is the morphism constructed in Lemma 6.22.

It follows from Lemma 6.22 and Jouanolou's trick that the induced morphism

$$\mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

(given by the unit section on the first summand) is a weak equivalence. Indeed, let $T \to Y$ be an affine vector bundle torsor. We then have a commutative diagram

$$\begin{tikzcd}
\mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \\
\mathsf{r}_X^{\mathscr{M}}(T) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(T)) \arrow[r] \arrow[u] \arrow[d] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,T}) \arrow[u] \arrow[d] \\
\mathsf{ra}_X^{\mathscr{M}}(T) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{ra}_X^{\mathscr{M}}(T)) \arrow[r] & \mathsf{ra}_X^{\mathscr{M}}(\mathbf{G}_{m,T})
\end{tikzcd}$$

The vertical morphisms are weak equivalences by Proposition 6.17 and Remark 6.5, and so the result follows from Lemma 6.22 which ensures that the lower horizontal morphism is a weak equivalence.

Let $1_Y : Y \to \mathbf{G}_{m,Y}$ be the unit section and $p : \mathbf{G}_{m,Y} \to Y$ be the projection. Since $p \circ 1_Y = \mathrm{Id}_Y$, the morphisms induced by the unit section

$$\mathsf{RQ}_X^{\mathscr{M}}(Y \otimes \mathbb{Q}) \to \mathsf{RQ}_X^{\mathscr{M}}(\mathbf{G}_{m,Y} \otimes \mathbb{Q}), \quad \mathsf{r}_X^{\mathscr{M}}(Y) \to \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y})$$

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are monomorphisms. We then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y) \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(\mathbf{G}_{m,Y}) \arrow[r] & \mathsf{c}_X^{\mathscr{M}}(Y) \arrow[r] & 0 \\
0 \arrow[r] & \mathsf{r}_X^{\mathscr{M}}(Y) \arrow[r] \arrow[u] & \mathsf{r}_X^{\mathscr{M}}(Y) \oplus \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[u] & \mathsf{T}_X^{\mathscr{M}}(\mathsf{r}_X^{\mathscr{M}}(Y)) \arrow[r] \arrow[u, dotted] & 0
\end{tikzcd}$$

in which all the rows are exact sequences. This provides the desired weak equivalence. □

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In an earlier version of this work, I used Proposition 6.21 to stabilize the construction at the level of symmetric spectra. However, this proposition is too weak to apply known results in the literature such as [Ayo07b, Lemme 4.3.34]. I am grateful to the referees and to Choudhury and Gallauer Alves De Souza for pointing out this mistake. I also thank Steven Zucker for useful conversations at an early stage of this project.

Appendix A. Brown–Gersten property in the Nisnevich topology

A.1 Recall that an elementary Nisnevich square is a cartesian square in Sm/X

$$\begin{tikzcd}
V \arrow[r, "v"] \arrow[d, "e'"'] \arrow[dr, phantom, "\square"] & E \arrow[d, "e"] \\
U \arrow[r, "u"] & Y
\end{tikzcd} \tag{A1}$$

such that u is an open immersion and e is an étale morphism that induces an isomorphism $p^{-1}(Z) \to Z$ for the reduced scheme structures where $Z = Y\backslash U$. If e is also an open immersion then the square is called an elementary Zariski square (an elementary Zariski square is simply the data of a covering of X by two open subschemes U and E). If all the schemes in (A1) are affine then the square is called an elementary affine Nisnevich square.

If $Y \in \mathsf{Sm}/X$ is connected, a morphism of quasi-projective X-schemes $r : Y' \to Y$ is said to be a Galois cover if r is finite étale and $G := \mathrm{Aut}_Y(Y')$ operates transitively and faithfully on the geometric fibers of f. If Y is not connected then $r : Y' \to Y$ is said to be a Galois cover if its restrictions to the connected components consist of Galois covers.

A.2 Recall some definitions from [MV01, Mor12].

Definition A.1. Let $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ be a presheaf.

(i) One says that $\mathscr{X}$ satisfies the B.G. property in the Zariski topology if for every $X \in \mathsf{Sm}/k$ and every covering of X by two open subschemes U, E the following diagram is homotopy cartesian in $\mathsf{Ch}(\mathbb{Q})$:

$$\begin{tikzcd}
\mathscr{X}(Y) \arrow[r] \arrow[d] & \mathscr{X}(E) \arrow[d] \\
\mathscr{X}(U) \arrow[r] & \mathscr{X}(V)
\end{tikzcd}$$

One says that $\mathscr{X}$ satisfies the $\mathbf{A}^1$-B.G. property in the Zarisky topology if $\mathscr{X}$ satisfies the B.G. property in the Zariski topology and for every $X \in \mathsf{Sm}/k$ the map

$$\mathscr{X}(X) \to \mathscr{X}(X \times_k \mathbf{A}^1_k),$$

induced by the projection, is a quasi-isomorphism.

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(ii) One says that $\mathscr{X}$ satisfies the B.G. property (respectively, affine B.G. property) in the Nisnevich topology if, for every $X \in \mathsf{Sm}/k$ and every elementary Nisnevich square (respectively, elementary affine Nisnevich square) (A1), the following diagram is homotopy cartesian in $\mathsf{Ch}(\mathbb{Q})$:

$$\begin{tikzcd} \mathscr{X}(Y) \arrow[r] \arrow[d] & \mathscr{X}(E) \arrow[d] \\ \mathscr{X}(U) \arrow[r] & \mathscr{X}(V) \end{tikzcd}$$

By [Mor12, Theorem A.14], if an object $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ satisfies the $\mathbf{A}^1$-B.G. property in the Zariski topology and the affine B.G. property in the Nisnevich topology, then it satisfies the B.G. property in the Nisnevich topology.

A.3 Let $\mathscr{A}$ be a pseudo-Abelian $\mathbb{Q}$-linear additive category. Given a finite group G and an object A of $\mathscr{A}$, an action of G on A is a morphism of groups

$$\Phi_A : G \to \mathrm{Aut}_{\mathscr{A}}(A)$$

where $\mathrm{Aut}_{\mathscr{A}}(A)$ is the group of automorphisms of A. Since $\mathscr{A}$ is $\mathbb{Q}$-linear, we may consider the projector

$$\Pi_G := \frac{1}{|G|} \sum_{g \in G} \Phi_A(g)$$

for any object of $\mathscr{A}$ with an action of G by automorphisms. The category $\mathscr{A}$ being pseudo-Abelian, Π_G splits, providing a decomposition of A. The subobject A^G of G-invariants under G is the direct summand of A equal to the image of Π_G.

Definition A.2. A presheaf $\mathscr{X} \in \mathbf{PSh}(\mathsf{Sm}/X, \mathsf{Ch}(\mathbb{Q}))$ has elementary Galois descent if, for every Galois cover $Y' \to Y$, the morphism

$$\mathscr{X}(Y) \to \mathscr{X}(Y')^G \tag{A2}$$

is a quasi-isomorphism of $\mathbb{Q}$-vector spaces.

Appendix B. Tools from homotopical algebra

B.1 Recall that the category $\mathsf{Ch}(\mathbb{Q})$ of cochain complexes of $\mathbb{Q}$-vector spaces has a model structure (called the projective model structure) such that the weak equivalences are the quasi-isomorphisms and the fibrations are the epimorphisms (see [Hov99, Theorem 2.3.11]).

Notation B.1. Let $\mathscr{B}$ be an Abelian category. Given $B \in \mathscr{B}$ and an integer $n \in \mathbb{Z}$, we denote by $S^n(B)$ the complex concentrated in degree n with $S^n(B)^n = B$ and by $D^n(B)$ the complex concentrated in degrees n, $n+1$ with $D^n(B)^n = D^n(B)^{n+1} = B$ and the identity as its only non-zero differential. Note that the identity induces a map $S^{n+1}(B) \to D^n(B)$.

Given an object A in $\mathscr{A}$, we denote by $\mathbb{Q}[A]$ the free presheaf of $\mathbb{Q}$-vector spaces associated with A: its sections on $B \in \mathscr{A}$ are given by the free $\mathbb{Q}$-vector space $\mathbb{Q}[\mathrm{Hom}_{\mathscr{A}}(B, A)]$ on the set $\mathrm{Hom}_{\mathscr{A}}(B, A)$.

Let I be the set of maps $S^{n+1}(\mathbb{Q}) \to D^n(\mathbb{Q})$ and J be the set of maps $0 \to D^n(\mathbb{Q})$. The projective model structure on $\mathsf{Ch}(\mathbb{Q})$ is cofibrantly generated (and proper). The set I (respectively, J) is a set of generating cofibrations (respectively, trivial cofibrations). In other words, $\mathsf{Fib} = \mathrm{RLP}(J)$ and $\mathsf{Fib} \cap \mathsf{W} = \mathrm{RLP}(I)$. More generally, the projective model structure on $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is cofibrantly generated (see [Hir03, Theorem 11.6.1]): the maps $S^{n+1}(\mathbb{Q}[A]) \to D^n(\mathbb{Q}[A])$, with $A \in \mathscr{A}$, form a class of generating cofibrations $I_{\mathscr{A}}$, and the maps $0 \to D^n(\mathbb{Q}[A])$, with $A \in \mathscr{A}$, form a class of generating trivial cofibrations $J_{\mathscr{A}}$.

Lemma B.2. *Let* W, Fib *be the classes of maps in* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ *defined as follows: a map belongs to* W *(respectively,* Fib*) if and only if it is a levelwise weak equivalence (respectively, a projective fibration) in* $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. *Let* Cof *be the class of maps in* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ *that have the left lifting property with respect to maps in* $\mathsf{W} \cap \mathsf{Fib}$. *Then the triple* $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$ *defines a model structure on* $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$.

Proof. Note that the class of maps in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which are monomorphisms and quasi-isomorphisms is stable by pushouts, transfinite compositions and retracts. The class $a_{\mathrm{ad}}(J_{\mathscr{A}})$ consists of the morphisms $0 \to D^n(\mathsf{i}(A))$ with $A \in \mathscr{A}$ and $n \in \mathbb{Z}$ which are all monomorphisms and quasi-isomorphisms. Hence every relative $a_{\mathrm{ad}}(J_{\mathscr{A}})$-cell complex is a quasi-isomorphism in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. The lemma then follows from [Hir03, Theorem 11.3.2]. (See also [Cra95, Theorem 3.3]. Note that the smallness assumption in [Hir03, Theorem 11.3.2] follows from the fact that representable presheaves are compact.) □

Remark B.3. The maps in the class $a_{\mathrm{ad}}(I_{\mathscr{A}})$ (respectively, $a_{\mathrm{ad}}(J_{\mathscr{A}})$) are generating cofibrations (respectively, trivial cofibrations) for the projective model structure of Lemma B.2. In particular, since all the maps in $a_{\mathrm{ad}}(I_{\mathscr{A}})$ are monomorphisms and monomorphisms are stable by pushouts, retracts and transfinite compositions, it follows that all cofibrations are monomorphisms.

Remark B.4. The image of a bounded complex of objects in $\mathscr{A}$ under the Yoneda embedding i is cofibrant for the projective model structure on $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$.

B.2 Note that the Quillen adjunction for the projective model structures

$$(-)_{\mathrm{cst}} : \mathsf{Ch}(\mathbb{Q}) \rightleftarrows \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) : \Gamma(0_{\mathscr{A}}, -)$$

implies that the bifunctor

$$- \otimes (-)_{\mathrm{cst}} : \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \tag{B1}$$

is a Quillen bifunctor for the projective model structures. Here we use the fact that the category of presheaves $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ with its projective model structure and the usual tensor product is the symmetric monoidal model category (see, for example, [Bar10, Proposition 4.52] or [Ayo07b, Proposition 4.4.63]).

Remark B.5. For every $\mathscr{F} \in \mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ we have

$$a_{\mathrm{ad}}(\mathscr{F} \otimes K_{\mathrm{cst}}) = a_{\mathrm{ad}}(\mathscr{F}) \otimes K_{\mathrm{cst}}.$$

Lemma B.6. *The bifunctor*

$$- \otimes (-)_{\mathrm{cst}} : \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor for the projective model structures.

Proof. Let $f : \mathscr{F} \to \mathscr{G}$ be a morphism in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $u : K \to L$ be a morphism in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd} \mathscr{F} \otimes K_{\mathrm{cst}} \arrow[r, "\mathscr{F} \otimes u_{\mathrm{cst}}"] \arrow[d, "f \otimes K_{\mathrm{cst}}"'] & \mathscr{F} \otimes L_{\mathrm{cst}} \\ \mathscr{G} \otimes K_{\mathrm{cst}} & \end{tikzcd} \tag{B2}$$

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We have to prove that if f and u are cofibrations in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$, then the map

$$\mathscr{H} \to \mathscr{G} \otimes L_{\mathrm{cst}} \tag{B3}$$

is a cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which is trivial if either f or u is a trivial cofibration. Assume that f is the image under a_{ad} of a cofibration $f' : \mathscr{F}' \to \mathscr{G}'$ in the category $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. Let $\mathscr{H}'$ be the pushout in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram similar to (B2) obtained by replacing f by f'. Since (B1) is a Quillen bifunctor for the projective model structures, the map $\mathscr{H}' \to \mathscr{G}' \otimes L_{\mathrm{cst}}$ is a cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})$ which is trivial if u is a trivial cofibration in $\mathsf{Ch}(\mathbb{Q})$ or f' is a trivial cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$. This implies that its image (B3) under a_{ad} is a cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})$ which is trivial if u is a trivial cofibration in $\mathsf{Ch}(\mathbb{Q})$ or f' is a trivial cofibration in $\mathbf{PSh}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$.

Since $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is cofibrantly generated, with $a_{\mathrm{ad}}(I_{\mathscr{A}})$ and $a_{\mathrm{ad}}(J_{\mathscr{A}})$ as generating cofibrations and trivial cofibrations, the lemma follows from the above case and [Hov99, Corollary 4.2.5]. □

Lemma B.7. *The bifunctor*

$$-\otimes(-)_{\mathrm{cst}} : \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q})) \times \mathsf{Ch}(\mathbb{Q}) \to \mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$$

is a Quillen bifunctor for the projective model structures.

Proof. Since every cofibration in $\mathbf{Sha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ is the image under a_{epi} of a τ-local projective cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ by [Ayo07b, Lemme 4.4.41], it is enough to prove that $-\otimes(-)_{\mathrm{cst}}$ is a Quillen bifunctor for the τ-local projective model structure on $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and the projective model structure on $\mathsf{Ch}(\mathbb{Q})$. Let $f : \mathscr{F} \to \mathscr{G}$ be a τ-local cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ and $u : K \to L$ be a cofibration in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd}
\mathscr{F} \otimes K_{\mathrm{cst}} \arrow[r, "\mathscr{F}\otimes u_{\mathrm{cst}}"] \arrow[d, "f\otimes K_{\mathrm{cst}}"'] & \mathscr{F} \otimes L_{\mathrm{cst}} \\
\mathscr{G} \otimes K_{\mathrm{cst}} &
\end{tikzcd} \tag{B4}$$

Since the τ-local model structure is obtained by a left Bousfield localization, the τ-local cofibrations are the projective cofibrations and it follows from Lemma B.6 that

$$\mathscr{H} \to \mathscr{G} \otimes L_{\mathrm{cst}} \tag{B5}$$

is a τ-local cofibration in $\mathbf{PSha}(\mathscr{A}, \mathsf{Ch}(\mathbb{Q}))$ which is trivial if u is a trivial cofibration. Assume that f is also a τ-local weak equivalence. Since f is a cofibration it is also a monomorphism (see Remark B.3), and therefore $a_{\mathrm{epi}}(f)$ is a monomorphism and a quasi-isomorphism. The square

$$\begin{tikzcd}
a_{\mathrm{epi}}(\mathscr{F}) \otimes K_{\mathrm{cst}} \arrow[r, "a_{\mathrm{epi}}(\mathscr{F})\otimes u_{\mathrm{cst}}"] \arrow[d, "a_{\mathrm{epi}}(f)\otimes K_{\mathrm{cst}}"'] & a_{\mathrm{epi}}(\mathscr{F}) \otimes L_{\mathrm{cst}} \arrow[d] \\
a_{\mathrm{epi}}(\mathscr{G}) \otimes K_{\mathrm{cst}} \arrow[r] & a_{\mathrm{epi}}(\mathscr{H})
\end{tikzcd}$$

being a pushout square, it follows that the map $a_{\mathrm{epi}}(\mathscr{F})\otimes L_{\mathrm{cst}} \to a_{\mathrm{epi}}(\mathscr{H})$ is a quasi-isomorphism. The composition

$$a_{\mathrm{epi}}(\mathscr{F}) \otimes L_{\mathrm{cst}} \to a_{\mathrm{epi}}(\mathscr{H}) \xrightarrow{a_{\mathrm{epi}}(\mathrm{B5})} a_{\mathrm{epi}}(\mathscr{G}) \otimes L_{\mathrm{cst}}$$

being equal to $a_{\mathrm{epi}}(f) \otimes L_{\mathrm{cst}}$ which is a quasi-isomorphism, it follows that a_{epi}(B5) is a quasi-isomorphism and therefore (B5) is a trivial τ-local cofibration. □

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Proof of Proposition 5.5. Let $f:\mathscr{F}\to\mathscr{G}$ be a morphism in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ and $u:K\to L$ be a morphism in $\mathsf{Ch}(\mathbb{Q})$. Let $\mathscr{H}$ be the pushout in the category $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ of the diagram

$$\begin{tikzcd} \mathscr{F}\otimes K_{\mathrm{cst}} \arrow[r, "\mathscr{F}\otimes u_{\mathrm{cst}}"] \arrow[d, "f\otimes K_{\mathrm{cst}}"'] & \mathscr{F}\otimes L_{\mathrm{cst}} \\ \mathscr{G}\otimes K_{\mathrm{cst}} & \end{tikzcd} \tag{B6}$$

We have to prove that if u is a projective cofibration in $\mathsf{Ch}(\mathbb{Q})$ and f is a cofibration in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ for the canonical model structure, then the map

$$\mathscr{H}\to\mathscr{G}\otimes L_{\mathrm{cst}} \tag{B7}$$

is a canonical cofibration in $\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q}))$ which is trivial if either f or u is a trivial cofibration. Since the latching space functor is a left adjoint, the square

$$\begin{tikzcd} \mathrm{L}_n(\mathscr{F})\otimes K_{\mathrm{cst}} \arrow[r, "\mathrm{L}_n(\mathscr{F})\otimes u_{\mathrm{cst}}"] \arrow[d, "f\otimes K_{\mathrm{cst}}"'] & \mathrm{L}_n(\mathscr{F})\otimes L_{\mathrm{cst}} \arrow[d] \\ \mathrm{L}_n(\mathscr{G})\otimes K_{\mathrm{cst}} \arrow[r] & \mathrm{L}_n(\mathscr{H}) \end{tikzcd}$$

is a pushout square.

We have to check that

$$\mathscr{H}_n\sqcup_{\mathrm{L}_n(\mathscr{H})}(\mathrm{L}_n(\mathscr{G})\otimes L_{\mathrm{cst}})\to\mathscr{G}_n\otimes L_{\mathrm{cst}} \tag{B8}$$

is a cofibration. For this remark that $\mathscr{H}_n\sqcup_{\mathrm{L}_n(\mathscr{H})}(\mathrm{L}_n(\mathscr{G})\otimes L_{\mathrm{cst}})$ is the pushout of the diagram

$$\begin{tikzcd} (\mathscr{F}_n\sqcup_{\mathrm{L}_n(\mathscr{F})}\mathrm{L}_n(\mathscr{G}))\otimes K_{\mathrm{cst}} \arrow[r, "\mathrm{Id}\otimes u_{\mathrm{cst}}"] \arrow[d, "f\otimes K_{\mathrm{cst}}"'] & (\mathscr{F}_n\sqcup_{\mathrm{L}_n(\mathscr{F})}\mathrm{L}_n(\mathscr{G}))\otimes L_{\mathrm{cst}} \\ \mathscr{G}_n\otimes K_{\mathrm{cst}} & \end{tikzcd}$$

Since f is a Reedy cofibration, the map $\mathscr{F}_n\sqcup_{\mathrm{L}_n(\mathscr{F})}\mathrm{L}_n(\mathscr{G})\to\mathscr{G}_n$ is a cofibration and therefore, by Lemma B.7, the map (B8) is a cofibration.

Note that since f is a Reedy cofibration, for every $n\in\mathbb{N}$, the induced map $\mathscr{F}_n\to\mathscr{G}_n$ is a cofibration (i.e., Reedy cofibrations are also levelwise cofibrations [Hir03, Proposition 16.3.11]). Hence, for every $n\in\mathbb{N}$, $\mathscr{F}_n\to\mathscr{G}_n$ is a monomorphism and therefore $\mathscr{F}\to\mathscr{G}$ is a monomorphism. This implies that we have an exact sequence

$$0\to\mathscr{F}\otimes K_{\mathrm{cst}}\to(\mathscr{F}\otimes L_{\mathrm{cst}})\oplus(\mathscr{G}\otimes K_{\mathrm{cst}})\to\mathscr{H}\to 0$$

and thus a distinguished triangle in $\mathsf{Ho}_{\mathrm{Reedy}}(\Delta^{\mathrm{op}}\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q})))$,

$$\mathscr{F}\otimes K_{\mathrm{cst}}\to(\mathscr{F}\otimes L_{\mathrm{cst}})\oplus(\mathscr{G}\otimes K_{\mathrm{cst}})\to\mathscr{H}\xrightarrow{+1}.$$

Now since the left derived functor of the colimit functor (see, for example, [Ayo07b, Lemme 4.1.51]) is triangulated, it yields a distinguished triangle in $\mathsf{Ho}(\mathbf{Sha}(\mathscr{A},\mathsf{Ch}(\mathbb{Q})))$,

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}\mathscr{F}\otimes K_{\mathrm{cst}}\to\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F}\otimes L_{\mathrm{cst}})\oplus\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G}\otimes K_{\mathrm{cst}})\to\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}\mathscr{H}\xrightarrow{+1}.$$

Assume that f is a canonical weak equivalence. Then the map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{F} \otimes K_{\mathrm{cst}} \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes K_{\mathrm{cst}})$ is an isomorphism. This implies that the map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F} \otimes L_{\mathrm{cst}}) \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{H}$ is also an isomorphism. Since f is a canonical weak equivalence, the composition

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{F} \otimes L_{\mathrm{cst}}) \to \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathscr{H} \xrightarrow{\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}((\mathrm{B7}))} \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes L_{\mathrm{cst}})$$

is an isomorphism and therefore so is the second map. This shows that (B7) is a canonical weak equivalence.

Assume that u is a trivial cofibration, then the map $\mathscr{F} \otimes K_{\mathrm{cst}} \to \mathscr{G} \otimes B_{\mathrm{cst}}$ is a level weak equivalence and therefore a realization weak equivalence. The map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{F} \otimes K_{\mathrm{cst}} \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes L_{\mathrm{cst}})$ is thus an isomorphism. This implies that the map $\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes K_{\mathrm{cst}}) \to \mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \mathscr{H}$ is an isomorphism. Since $\mathscr{G} \otimes u$ is a canonical weak equivalence, the composition

$$\mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes K_{\mathrm{cst}}) \to \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathscr{H} \xrightarrow{\mathbb{L}\operatorname{colim}_{\Delta^{\mathrm{op}}}((\mathrm{B7}))} \mathbb{L}\operatorname*{colim}_{\Delta^{\mathrm{op}}}(\mathscr{G} \otimes L_{\mathrm{cst}})$$

is an isomorphism, and therefore so is the second map. This shows that (B7) is a canonical weak equivalence. □

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