J. Appl. Prob. **42,** 426–445 (2005)
Printed in Israel

MULTIPERIOD CONDITIONAL DISTRIBUTION FUNCTIONS FOR CONDITIONALLY NORMAL GARCH(1, 1) MODELS

RAYMOND BRUMMELHUIS,* *Birkbeck College*
DOMINIQUE GUÉGAN,** *École Normale Supérieure de Cachan*

Abstract

We study the asymptotic tail behavior of the conditional probability distributions of r_{t+k} and $r_{t+1} + \cdots + r_{t+k}$ when $(r_t)_{t\in\mathbb{N}}$ is a GARCH(1, 1) process. As an application, we examine the relation between the extreme lower quantiles of these random variables.

Keywords: Generalized autoregressive heteroskedastic process; conditional probability density function; large deviation probability; asymptotics; Laplace integral; quantile estimation; value at risk

2000 Mathematics Subject Classification: Primary 60G35
Secondary 62E17; 62P05; 91B28; 91B30

1. Introduction and main results

GARCH(1, 1) processes are popular in empirical finance, since they provide parsimonious models capable of successfully modeling phenomena like volatility clustering, excess kurtosis, and fat tails in time series of financial returns. They are often used, in particular, to provide estimates of one-day value at risk. The GARCH(1, 1) model, as introduced by Bollerslev [3], is the 2-component stochastic process (r_t, σ_t) defined recursively by

$$r_{t+1} = \sigma_{t+1}\varepsilon_{t+1},$$
$$\sigma_{t+1}^2 = a_0 + a_1 r_t^2 + b_1\sigma_t^2, \qquad \sigma_{t+1} \geq 0, \tag{1}$$

where ε_t is a white noise (that is, an independent and identically distributed) process and a_0, a_1, and b_1 are parameters. The original idea of conditional heteroskedasticity goes back to Engle [11], whose ARCH(1) model corresponds to setting $b_1 = 0$ in (1). In typical financial applications, r_t will be the daily log-return on some risky asset with price P_t, i.e. $r_t = \log(P_t/P_{t-1})$. We fix an initial time t, to be interpreted as 'today', and let

$$r_t = \rho_0 \quad \text{and} \quad \sigma_t = s_0$$

be today's observed return and volatility, respectively. Throughout the paper, we will suppose that the ε_t are independent and identically normally distributed with mean 0 and variance 1, unless stated otherwise.

Received 12 September 2002; revision received 10 May 2004.
* Postal address: School of Economics, Mathematics and Statistics, Birkbeck College, University of London, Malet Street, London WC1E 7HX, UK. Email address: r.brummelhuis@statistics.bbk.ac.uk
** Postal address: Département d'Économie et de Gestion, ENS Cachan, 61 av. du President Wilson, 94235 Cachan Cedex, France. Email address: guegan@ecogest.ens-cachan.fr

An investor or financial institution should, for risk-assessment purposes, be interested in both the conditional and the unconditional probability distributions of future returns r_{t+k}. The conditional distribution will be important in situations of high market volatility and for high-risk market players like hedge funds or emerging market investors. The investor, or institution, will then be interested in conditional probability densities like that of the one-day return, k periods into the future, i.e.

$$p_{t,k}(x;\,\rho_0, s_0) := \mathrm{P}(r_{t+k} = x \mid r_t = \rho_0, \sigma_t = s_0), \tag{2}$$

and that of the total return

$$r_{t+k,t} := r_{t+1} + \cdots + r_{t+k} = \log\left(\frac{P_{t+k}}{P_t}\right)$$

over the entire k-day period, i.e.

$$q_{t,k}(x;\,\rho_0, s_0) := \mathrm{P}(r_{t+k,t} = x \mid r_t = \rho_0, \sigma_t = s_0). \tag{3}$$

Here P is the 'physical' or 'objective' (as opposed to the risk-neutral) probability, and we will be using the informal notation $\mathrm{P}(X = x \mid A)$ for the conditional probability density of a random variable X given the event A: see also the beginning of Section 2, below.

From these densities, various risk measures like conditional value at risk (VaR) or conditional expected shortfall (see [1]) can be computed. In this paper, we will analyze the asymptotic properties of (2) and (3) for large $|x|$ and arbitrary, but *fixed*, k, under the assumption that the returns are modeled by the GARCH process (1). As an application, we will examine the k-day VaR at asymptotically large confidence levels. Our approach should be contrasted with most of the mathematical literature on GARCH processes (see, for example, [5], [10], [17], and references therein), which emphasizes the study of the stationary distribution of (1). This amounts to studying the limit of $p_{t,k}(x;\,\rho_0, s_0)$ as $k \to \infty$, and is important for the estimation of the unconditional VaR. Stationarity considerations will not play a role in this paper.

We note in passing that, as the process (1) is time homogeneous, we could have taken $t = 0$ without loss of generality. However, in practice there is still a dynamic time dependence in the ever-changing initial conditions (ρ_0, s_0) at t, which are today's realized return and volatility. To retain this dynamical flavor, important for applications, we have opted for the present notation.

The following three theorems are the main results of the paper.

Theorem 1. *Let $(r_t)_t$ be the GARCH$(1,1)$ process (1) with independent, Gaussian distributed ε_t of mean 0 and variance 1. Fix a time t and a time horizon $t + k$, and let $r_t = \rho_0$ and $\sigma_t = s_0$ be given. Let $\sigma_{t+1} = (a_0 + a_1\rho_0^2 + b_1 s_0^2)^{1/2}$ be the volatility over $[t, t+1]$ as predicted by the GARCH$(1,1)$ model. Then, as $x \to \pm\infty$,*

$$p_{t,k}(x;\,\rho_0, s_0) \simeq C_k \frac{\exp(-c_k|x|^{2/k})}{|x|^{1-1/k}}, \tag{4}$$

with constants c_k and C_k given by

$$c_k = \tfrac{1}{2} k a_1^{-(1-1/k)} \sigma_{t+1}^{-2/k} \tag{5}$$

and

$$C_k = \frac{\exp((k-1)b_1/2a_1)}{\sqrt{2\pi}} a_1^{-(1-1/k)/2} \sqrt{\frac{2^{k-1}}{k}} \sigma_{t+1}^{-1/k}, \tag{6}$$

and with an error of $O(|x|^{-1}\exp(-c_k|x|^{2/k}))$.

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

Note the substantial thickening of the tails with increasing k, even when going from $k = 1$ to $k = 2$. From $k = 3$ onwards, the conditional probability density of r_{t+k}, given r_t and σ_t, is already subexponential. Also note that the conditional probability densities of a GARCH(1, 1) process and an ARCH(1) process have the same qualitative asymptotic behavior; the only quantitative difference is the multiplicative prefactor $\exp((k-1)b_1/2a_1)$.

The asymptotic relation (4) can be generalized to a complete asymptotic expansion – see Remark 3, below.

We next turn to the k-day returns $r_{t+k,t} = r_{t+1} + \cdots + r_{t+k}$. Intuitively, we expect the 'biggest' term r_{t+k} to dominate the others, as far as the probability of extremal events is concerned. This intuition is confirmed by our next result.

Theorem 2. *Let $(r_t)_t$ be the GARCH$(1,1)$ process (1), with an independent and identically distributed standard normal $(\varepsilon_t)_t$, where we further suppose that $b_1 > 0$. Fix k and let $\rho_0 \in \mathbb{R}$ and $s_0 > 0$. Then, there exist constants $c'_k, c''_k, C'_k, C''_k > 0$, depending on k, a_0, a_1, b_1, ρ_0, and s_0, such that*

$$C'_k |x|^{-(1-1/k)} \exp(-c'_k |x|^{2/k}) \leq q_{t,k}(x; \rho_0, s_0) \leq C''_k |x|^{-(1-1/k)} \exp(-c''_k |x|^{2/k}) \qquad (7)$$

for $|x| \geq 1$.

For technical reasons, we have limited ourselves to genuine GARCH processes, in the sense that $b_1 \neq 0$; these are, in any case, the most important ones in practice. Theorem 2 is primarily a qualitative result, which would in fact extend to a slightly larger class of processes (as will become clear from its proof in Section 5, below). The fact that we have inequalities in (7) instead of asymptotic equivalences is, by itself, not necessarily a handicap in practical applications. What is more serious is that we have not given explicit values for the constants, let alone optimal ones. 'Chasing the constants' in the proof of Theorem 2 does not give very good values for these and, therefore, Theorem 2 is mostly of theoretical interest. However, we can obtain a slightly more explicit result for the cumulative distribution function

$$F_{r_{t+k,t} \mid \rho_0, s_0}(x) := \mathrm{P}(r_{t+k,t} < x \mid r_t = \rho_0, \sigma_t = s_0).$$

Using Theorem 1, we will prove the following result.

Theorem 3. *With the same notation as in Theorem 1, we have*

$$\lim_{x \to -\infty} \frac{\log F_{r_{t+k,t} \mid \rho_0, s_0}(x)}{|x|^{2/k}} = -c_k,$$

with c_k defined by (5).

Note that Theorem 3 does not allow us to draw straightforward conclusions about the constants c'_k and c''_k in Theorem 2, even asymptotically for negative x, since asymptotic relations cannot, in general, be differentiated. In this respect, Theorem 2 contains more information.

We finally note that the present paper's standing hypothesis of normally distributed innovations is known not always to be realistic for empirical time series of financial returns. In risk-management practice, heavy-tailed distributions, like Student t-distributions with a small number of degrees of freedom, are increasingly used – compare also Frey and McNeil [12], who used extreme value theory to estimate the tails of the ε_t by a generalized Pareto distribution. We remark that Theorems 1 and 3 can be generalized to GARCH(1, 1) processes with Pareto-tailed ε_t; these will be treated elsewhere.

The remainder of the paper is organized as follows. In Section 2, we derive suitable representation formulae for $p_{t,k}$ and $q_{t,k}$, which are a consequence of the Markov property of (1). In Section 3, we prove an asymptotic expansion lemma for Laplace transforms of functions of the form $x^{-\beta}\exp(-cx^{-\alpha})$, with $\alpha > 0$; observe that these are flat at $x = 0$. Sections 4, 5, and 6 are devoted to the proofs of, respectively, Theorems 1, 2, and 3. Finally, in Section 7, we discuss some consequences of Theorem 3 for extreme lower quantiles of $r_{t,t+k}$ or, in financial terminology, for the k-day value at risk at confidence levels tending to 1.

2. Representation formulae for r_{t+k} and $r_{t+k,t}$

The following notational conventions will be used throughout the paper. All random variables will be defined on some sufficiently rich probability space $(\Omega, \mathcal{F}, \mathrm{P})$. If X is a random variable on Ω and $A \in \mathcal{F}$ an event, we will denote the cumulative distribution function of X, relative to the conditional probability measure $\mathrm{P}(\cdot \mid A)$, by $F_{X \mid A}$; that is, $F_{X \mid A}(x) := \mathrm{P}(X < x \mid A)$. In fact, A will mostly be of the form $\{r_t = \rho_0, \sigma_t = s_0\}$, in which case we will simply write $F_{X \mid \rho_0, s_0}$.

We will assume, for convenience, that all random variables we deal with possess a probability density function (PDF). This assumption could be relaxed in many places but, since it is satisfied by the majority of the GARCH models used in practice, we have limited ourselves to this case. We will write $X \sim f$ to indicate that the random variable X has PDF f. Following a custom from the physics literature, we will often use the mathematically incorrect, but nevertheless very convenient, notation $\mathrm{P}(X = x \mid A)$ for the PDF of X relative to $\mathrm{P}(\cdot \mid A)$. More explicitly,

$$\mathrm{P}(X = x \mid A) := \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{P}(X < x \mid A),$$

assuming $F_{X \mid A}$ to be absolutely continuous.

Now consider a general nonparametric GARCH(1, 1) process given by

$$r_{t+1} = \sigma_{t+1}\varepsilon_{t+1} \quad \text{and} \quad \sigma_{t+1} = \varphi(r_t, \sigma_t). \tag{8}$$

Here, the random shocks or innovations $(\varepsilon_t)_t$ satisfy the usual hypothesis of being independent and identically distributed with mean 0 and variance 1. Moreover, we suppose that they possess a PDF, i.e.

$$\varepsilon_t \sim f, \tag{9}$$

where $0 \le f \in L^1(\mathbb{R})$. As for the function $\varphi\colon \mathbb{R}^2 \to (0, \infty)$, we suppose that it is Borel measurable and satisfies the following condition: for each $\sigma > 0$, the random variable $\varphi(\sigma\varepsilon, \sigma)$ possesses a density h_σ, i.e.

$$\varphi(\sigma\varepsilon, \sigma) \sim h_\sigma, \qquad 0 \le h_\sigma \in L^1(\mathbb{R}). \tag{10}$$

This class of nonparametric GARCH models contains most of the models proposed in the literature – see [4] and [13] for overviews and [6], [7], and [14] for additional examples. Exceptions are the threshold GARCH models of Zakoian and of Gouriéroux and Montfort – see [4] and references therein.

We are interested in the properties of r_{t+k}, given that $r_t = \rho_0$ and $\sigma_t = s_0$, for an arbitrary, but fixed, $k \ge 1$. We would like, in particular, to determine and analyze its conditional PDF $p_{t,k}(x; \rho_0, s_0)$, which, with our notational conventions, can be written as

$$p_{t,k}(x; \rho_0, s_0) = \mathrm{P}(r_{t+k} = x \mid r_t = \rho_0, \sigma_t = s_0).$$

We define two integral operators

$$F(u)(x) = \int_0^\infty \frac{1}{s} f\left(\frac{x}{s}\right) u(s)\, \mathrm{d}s,$$
$$H(u)(s) = \int_0^\infty h_{s'}(s) u(s')\, \mathrm{d}s',$$

from $L^1(0, \infty)$ to $L^1(\mathbb{R})$ and from $L^1(0, \infty)$ to itself, respectively. Here, $f(\cdot)$ and $h_{s'}(\cdot)$ are PDFs, so these operators are positivity preserving, and an application of Fubini's theorem shows that they are of norm 1. We will find it convenient to let F and H act formally on delta measures: $F(\delta_{s_0}) = s_0^{-1} f(\cdot/s_0)$, and similarly for H; cf. the statement of Theorem 4 below.

The GARCH(1, 1) process (8) can be interpreted as a two-component Markov process for (r_t, σ_t), with an almost-sure evolution for the σ_t component. It is often convenient to note that σ_t, by itself, follows a scalar Markov process, i.e.

$$\sigma_{t+1} = \varphi(\sigma_t \varepsilon_t, \sigma_t), \tag{11}$$

with transition probability densities

$$\begin{aligned} \mathrm{P}(\sigma_{t+1} = s \mid \sigma_t = \sigma) &= \mathrm{P}(\varphi(\sigma_t \varepsilon_t, \sigma_t) = s \mid \sigma_t = \sigma) \\ &= h_\sigma(s), \end{aligned} \tag{12}$$

since ε_t is independent of σ_t. This observation will be used in the proof of the next theorem.

Theorem 4. *Let $(r_t)_{t\in\mathbb{N}}$ be defined by (8). Under the hypotheses (9) and (10) on ε_t and φ, respectively, we have*

$$p_{t,k}(x; \rho_0, s_0) = F \circ H^{k-1}(\delta_{\varphi(\rho_0, s_0)}). \tag{13}$$

Proof. If $k = 1$ then (13) is trivially true. If $k > 1$ then, writing P_{ρ_0,s_0} for $\mathrm{P}(\cdot \mid r_t = \rho_0, \sigma_t = s_0)$ and using the fact that $r_{t+k} = \sigma_{t+k}\varepsilon_{t+k}$, with ε_{t+k} independent of σ_{t+k}, we have

$$\begin{aligned} \mathrm{P}_{\rho_0,s_0}(r_{t+k} = x) &= \int_0^\infty \mathrm{P}_{\rho_0,s_0}(\sigma_{t+k}\varepsilon_{t+k} = x \mid \sigma_{t+k} = s_k)\, \mathrm{P}_{\rho_0,s_0}(\sigma_{t+k} = s_t)\, \mathrm{d}s_k \\ &= \int_0^\infty \frac{1}{s_k} f\left(\frac{x}{s_k}\right) \mathrm{P}_{\rho_0,s_0}(\sigma_{t+k} = s_k)\, \mathrm{d}s_k. \end{aligned} \tag{14}$$

By the Markov nature of (11), and using (12), we have

$$\mathrm{P}_{\rho_0,s_0}(\sigma_{t+k} = s_k) = H^{k-1}(\delta_{\varphi(\rho_0,s_0)}),$$

where we have used the fact that $\mathrm{P}_{\rho_0,s_0}(\sigma_{t+1} = s_1) = \delta_{\varphi(\rho_0,s_0)}(s_1)$. Substitution of this into (14) completes the proof of (13).

Remarks 1. (i) For a nonparametric ARCH(1) process, by which we mean a process of the form (8) with a φ which does not depend on σ, there is a simpler formula for $p_{t,k}(x; \rho_0)$. Since we now simply have a one-component Markov process, we immediately obtain

$$p_{t,k}(x; \rho_0) = \widetilde{F}^k(\delta_{\rho_0}),$$

where $\widetilde{F}$ is, by definition, the integral operator on $L^1(\mathbb{R})$ whose kernel is given by the transition probability densities $(r, \rho) \mapsto \mathrm{P}(r_{t+1} = r \mid r_t = \rho) = \varphi(\rho)^{-1} f(r/\varphi(\rho))$. This can be used to slightly shorten the proof of Theorem 1 in the case of an ARCH(1) process, but does not lead to any essential simplifications.

(ii) The condition that the ε_t be independent and identically distributed can easily be weakened to independence only, and we can also let φ explicitly depend on t. Independence of the ε_t, however, is essential, and the results of this paper are not expected to hold for, for example, the important class of weak GARCH processes introduced by Drost and Nijman [9]. This is unfortunate, since it would have greatly simplified the discussion for multiperiod returns, the Drost–Nijman class, unlike ordinary GARCH processes, being closed under temporal aggregation.

The functions h_σ defined in (10) can be easily computed for a GARCH(1, 1) process. If ε_t has density f then

$$h_\sigma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{P}((a_0 + a_1\sigma^2\varepsilon^2 + b_1\sigma^2)^{1/2} < s)$$
$$= s(a_1\sigma^2(s^2 - a_0 - b_1\sigma^2))^{-1/2} \sum_{i=+1,-1} f\left(i\left(\frac{s^2 - a_0 - b_1\sigma^2}{a_1\sigma^2}\right)^{1/2}\right)$$

if $s \geq (a_0 + b_1\sigma^2)^{1/2}$, and $h_\sigma(s) = 0$ otherwise. If f is symmetric, this simplifies to

$$2s(a_1\sigma^2(s^2 - a_0 - b_1\sigma^2))^{-1/2} f\left(\left(\frac{s^2 - a_0 - b_1\sigma^2}{a_1\sigma^2}\right)^{1/2}\right)\mathbf{1}_{\{s>\sqrt{a_0+b_1\sigma^2}\}}, \tag{15}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. We will apply this formula in Section 3, with f the standard normal density.

Formula (13) can be used for fast numerical computation of the densities $p_{t,k}(\cdot \mid \rho_0, s_0)$. This is because we are dealing with an operator product, so that we need only evaluate a succession of one-dimensional integrals, instead of a single k-dimensional one (a similar remark applies to all Markov processes, of course).

We next turn to the k-period returns $r_{t+k,t}$. Since the class of GARCH(1, 1) models is not closed under temporal aggregation [9], the preceding results are not directly applicable. We will therefore proceed differently, by using the two-component Markov process $Z_t = (r_t, \sigma_t)$. Write $z_0 = (\rho_0, s_0)$ and let P_{z_0} be the probability conditional on $Z_t = z_0$. Then, the joint PDF of $(Z_{t+1}, \dots, Z_{t+k})$ with respect to P_{z_0} can be evaluated to be

$$\mathrm{P}_{z_0}((Z_{t+1}, \dots, Z_{t+k}) = (z_1, \dots, z_k)) = \prod_{j=1}^{k} \mathrm{P}_{z_0}(Z_{t+j} = z_j \mid Z_{t+j-1} = z_{j-1}),$$

by the Markov property. It follows that this joint PDF equals

$$\mathrm{P}((r_{t+j}, s_{t+j}) = (x_j, s_j), 1 \leq j \leq k \mid (r_t, s_t) = (\rho_0, s_0))$$
$$= \prod_{j=1}^{k} \frac{1}{s_j} f\left(\frac{x_j}{s_j}\right)\delta(s_j - \varphi(x_{j-1}, s_{j-1})), \tag{16}$$

with $\delta(s - v)$ the Dirac delta measure and $x_0 = \rho_0$; the delta measures come from the fact that (8) is deterministic in the second component (this would be different for a stochastic volatility model). The conditional PDF of $r_{t+k,t} = x$ is now found by integrating (16) against $\delta(x - (x_1 + \dots + x_k))$. Evaluating the integrals over $s_1, \dots, s_k$ that involve the delta functions, we obtain the following result.

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

Theorem 5. *Recursively define the functions* $\hat{s}_j = \hat{s}_j(x_1, \dots, x_{j-1})$, $j \geq 1$, *by*

$$\hat{s}_1 = \varphi(\rho_0, s_0), \qquad \hat{s}_j = \varphi(x_{j-1}, \hat{s}_j).$$

Then,

$$\begin{aligned} &\mathrm{P}(r_{t+k,t} = x \mid r_t = \rho_0, \sigma_t = s_0) \\ &\quad = \int \cdots \int \frac{1}{\hat{s}_k} f\left(\frac{x - (x_1 + \cdots + x_{k-1})}{\hat{s}_k}\right) \prod_{j=1}^{k-1} \frac{1}{\hat{s}_j} f\left(\frac{x_j}{\hat{s}_j}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1}. \end{aligned}$$

This formula no longer has the simple recursive structure of an operator product, and this will complicate its analysis in Section 5, below. This also limits its usefulness in numerical evaluation of the PDF.

3. Asymptotics of Laplace transforms

In this section, we prove the following technical lemma on asymptotic expansions of certain degenerate Laplace integrals, which will form the basis of the proofs of Theorems 1 and 2.

Lemma 1. *Let* $\alpha > 0$, $s > 0$, $c > 0$, *and* $\beta \in \mathbb{R}$. *Then, as* $s \to \infty$,

$$\begin{aligned} &\int_0^\infty x^{-\beta} \exp(-cx^{-\alpha}) \exp(-sx) \, \mathrm{d}x \\ &\quad \simeq \left(\frac{s}{\alpha}\right)^{(\beta-\alpha/2-1)/(\alpha+1)} \exp\left(-(\alpha+1)c^{1/(\alpha+1)}\left(\frac{s}{\alpha}\right)^{\alpha/(\alpha+1)}\right) \sum_{j=0}^{\infty} C_j s^{-j\alpha/(\alpha+1)}, \end{aligned} \tag{17}$$

with $C_0 = (2\pi/\alpha(\alpha+1))^{1/2} c^{(1-2\beta)/2(\alpha+1)}$.

We refer to the integral in (17) as degenerate, since all derivatives of $x^{-\beta}\exp(-cx^{-\alpha})$ vanish at 0. In particular, we cannot simply apply Watson's lemma.

Remarks 2. (i) The meaning of the asymptotic expansion (17) is the usual one: if we cut off the sum after $N-1$ terms, then there exists, for each $R > 0$, a constant $C_{N,R} \equiv C_{N,R}(c, \alpha, \beta)$ such that the error we make can be estimated by

$$C_{N,R} s^{(\beta-(N+1/2)\alpha-1)/(\alpha+1)} \exp\left(-(\alpha+1)c^{1/(\alpha+1)}\left(\frac{s}{\alpha}\right)^{\alpha/(\alpha+1)}\right)$$

for $s > R$.

To simplify the formulae, we will henceforth only record the main term of the various asymptotic series we encounter, and write (17) as

$$\begin{aligned} &\int_0^\infty x^{-\beta} \exp(-sx - x^{-\alpha}) \, \mathrm{d}x \\ &\simeq \exp\left(-(\alpha+1)\left(\frac{s}{\alpha}\right)^{\alpha/(\alpha+1)}\right)\left(\sqrt{\frac{2\pi}{\alpha(\alpha+1)}}\left(\frac{s}{\alpha}\right)^{(\beta-\alpha/2-1)/(\alpha+1)} + O(s^{(\beta-3\alpha/2-1)/(\alpha+1)})\right), \end{aligned} \tag{18}$$

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

where we have taken $c = 1$ for simplicity. We will often even leave out the O term altogether. This should cause no confusion.

(ii) A closely related result is the Abelian part of de Bruijn's Tauberian theorem (see Theorem 4.12.9 in Bingham *et al.* [2]), which states the asymptotic equivalence of the logarithm of the left-hand side of (18) with the logarithm of the exponential on the right-hand side (we thank Paul Embrechts for bringing this to our attention). De Bruijn's theorem holds, in fact, for a much larger class of phase functions, namely those behaving asymptotically as $x^{-\alpha}$ as $x \to 0+$, in the sense of regular variation. The class of phase functions in (17) could be similarly generalized, but the present case is all that we will need in this paper.

Proof of Lemma 1. By scaling, it suffices to prove (17) only for $c = 1$. We first split the integral into two parts, according to

$$\begin{aligned}\int_0^\infty & x^{-\beta}\exp(-sx - x^{-\alpha})\,\mathrm{d}x \\ &= \int_0^{s^{-1/(1+\alpha)}} x^{-\beta}\exp(-sx - x^{-\alpha})\,\mathrm{d}x + \int_{s^{-1/(1+\alpha)}}^\infty x^{-\beta}\exp(-sx - x^{-\alpha})\,\mathrm{d}x \\ &=: I + II.\end{aligned}$$

Observe that $sx = x^{-\alpha}$ precisely when $x = s^{-1/(\alpha+1)}$. We next analyze the two parts separately, using Laplace's method. We start with the integral II. Making the change of variable $x = s^{-1/(1+\alpha)}y$, we find that

$$II = s^{(\beta-1)/(\alpha+1)}\int_1^\infty \exp(-s^{\alpha/(\alpha+1)}(y + y^{-\alpha}))y^{-\beta}\,\mathrm{d}y,$$

which, apart from the prefactor, is a classical Laplace integral of the form

$$\int_1^\infty \mathrm{e}^{-\lambda\varphi(y)}a(y)\,\mathrm{d}y.$$

The main contribution to the asymptotics will come from the absolute minimum of the phase function $\varphi(y) = y + y^{-\alpha}$ in $[1, \infty)$ or from the boundary point $y = 1$. It is easily seen that $\varphi(y)$ has an absolute minimum in $[0, \infty)$ at $y = y_\mathrm{c} = \alpha^{1/(\alpha+1)}$. We distinguish the following three cases.

Case (i): $\alpha > 1$. In this case, $y_\mathrm{c} \in (1, \infty)$ and we get a contribution

$$\mathrm{e}^{-\lambda\varphi(y_\mathrm{c})}\bigg(\bigg(\frac{2\pi}{\lambda}\bigg)^{1/2}\frac{a(y_\mathrm{c})}{\varphi''(y_\mathrm{c})^{1/2}} + O(\lambda^{-3/2})\bigg),$$

where the O term actually stands for a complete asymptotic series in the powers $\lambda^{-(1/2)-j}$. Computing $\varphi(y_\mathrm{c}) = \alpha^{1/(\alpha+1)} + \alpha^{-\alpha/(\alpha+1)} = (\alpha+1)\alpha^{-\alpha/(\alpha+1)}$ and $\varphi''(y_\mathrm{c}) = \alpha(\alpha+1)/\alpha^{(\alpha+2)/(\alpha+1)}$, and remembering the prefactor and the fact that $\lambda = s^{\alpha/(\alpha+1)}$, we find the following contribution to II:

$$\exp\bigg(-(\alpha+1)\bigg(\frac{s}{\alpha}\bigg)^{\alpha/(\alpha+1)}\bigg)\bigg(\sqrt{\frac{2\pi}{\alpha(\alpha+1)}}\bigg(\frac{s}{\alpha}\bigg)^{(\beta-\alpha/2-1)/(\alpha+1)} + \cdots\bigg), \qquad (19)$$

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

the dots indicating lower-order terms. We must compare this with the contribution from the boundary point $y_{\rm c} = 1$, which is

$$\exp(-2s^{\alpha/(\alpha+1)})((1-\alpha)^{-1}s^{(\beta-\alpha-1)/(\alpha+1)} + \cdots). \tag{20}$$

However, these will all be dominated by (19), as follows from the elementary observation that, for all $\alpha > 0$,

$$(\alpha+1)\alpha^{-\alpha/(\alpha+1)} \le 2 \tag{21}$$

with equality if and only if $\alpha = 1$.

To prove (21), we must show that

$$\log(\alpha+1) - \left(\frac{\alpha}{\alpha+1}\right)\log\alpha \le \log 2$$

for $\alpha > 0$. This follows from the fact that the left-hand side has an absolute maximum, equal to $\log 2$, at $\alpha = 1$.

Continuing with the analysis of II, we consider the two remaining cases.

Case (ii): $\alpha = 1$. The minimum $y_{\rm c}$ coincides with the boundary point, and we obtain a contribution equal to half that of (19).

Case (iii): $\alpha < 1$. In this case, $y_{\rm c} < 1$ and the asymptotics of II will be given by (20), since only $y = 1$ will contribute.

We next repeat the analysis for the integral I. Making the substitution $x = s^{-1/(\alpha+1)}u^{-1}$, we find that

$$I = s^{(\beta-1)/(\alpha+1)}\int_1^\infty \exp(-s^{\alpha/(\alpha+1)}(u^\alpha + u^{-1}))u^{\beta-2}\,{\rm d}u.$$

In this case, the phase function equals $\varphi(u) = u^\alpha + u^{-1}$, which has an absolute minimum at $u = u_{\rm c} = \alpha^{-1/(\alpha+1)}$. We compute that $\varphi(u_{\rm c}) = (\alpha+1)\alpha^{-\alpha/(\alpha+1)}$ (as for II) and that

$$\begin{aligned}\varphi''(u_{\rm c}) &= \alpha(\alpha-1)\alpha^{-(\alpha-2)/(\alpha+1)} + 2\alpha^{3/(\alpha+1)}\\ &= \alpha^{2/(\alpha+1)}(\alpha(\alpha+1))\alpha^{-\alpha/(\alpha+1)}.\end{aligned}$$

We then have to consider the same three cases as for II.

Case (i′): $\alpha > 1$. Since $u_{\rm c} < 1$, the only contribution to the asymptotics will come from the boundary point $u = 1$, which will give (20).

Case (ii′): $\alpha = 1$. In this case, $u_{\rm c} = 1$ and we get a contribution of half that of (19), as before.

Case (iii′): $\alpha < 1$. Now the critical point $u_{\rm c} > 1$ is in the integration range of I, and will give a contribution to the asymptotics that turns out to be the same as (19). By observation (21), this contribution will again dominate that coming from the boundary point.

It now suffices to add up the asymptotics of I and II and observe once more that, in cases (i) and (i′) and cases (iii) and (iii′), the contribution of the interior minimum dominates, by (21).

4. Proof of Theorem 1

We will now prove Theorem 1. We will start from Theorem 4 and use induction on k. For this, we need to know how the operators F and H will affect the asymptotic behavior of the functions on which they act. This will be analyzed in the following two lemmas. Let us start with H. Its kernel is given by (15) with $f(x) = (2\pi)^{-1/2}\exp(-\frac{1}{2}x^2)$, the standard normal density. Therefore,

$$H(v)(s) = \frac{2s\mathrm{e}^{b_1/2a_1}}{\sqrt{2\pi a_1}} \int_0^{\sqrt{(s^2-a_0)/b_1}} \frac{1}{\sqrt{t^2(s^2-a_0-b_1t^2)}} \exp\biggl(\frac{-(s^2-a_0)}{2a_1t^2}\biggr) v(t)\,\mathrm{d}t \tag{22}$$

if $s^2 > a_0$, and $H(v)(s) = 0$ otherwise. If $b_1 = 0$, the integral is to be taken over the whole positive real axis. We then have the following lemma.

Lemma 2. *Suppose that* $v(s) \simeq Cs^{\beta}\exp(-cs^{\alpha})$ *for* $0 < s \to \infty$*, where* $\beta \in \mathbb{R}$*,* $c > 0$*, and* $\alpha > 0$*. Then,*

$$H(v)(s) \simeq C' s^{(2\beta-\alpha)/(\alpha+2)} \exp(-c' s^{2\alpha/(\alpha+2)}), \qquad s \to \infty,$$

where

$$c' = \tfrac{1}{2}(\alpha+2)(\alpha a_1)^{-\alpha/(\alpha+2)} c^{2/(\alpha+2)} \tag{23}$$

and

$$C' = \frac{2C\exp(b_1/2a_1)}{\sqrt{\alpha+2}} (c\alpha a_1)^{-(\beta+1)/(\alpha+2)}. \tag{24}$$

Proof. Making the change of variable $z = 1/t^2$ in (22) and letting $\gamma(s) = 2s\mathrm{e}^{b_1/2a_1}/(2\pi a_1)^{1/2}$ and $\tilde{s} = (s^2 - a_0)/2a_1$, we obtain

$$H(v)(s) = \tfrac{1}{2}\gamma(s) \int_{b_1/2a_1\tilde{s}^{-1}}^{\infty} \sqrt{\frac{z}{2a_1\tilde{s}z - b_1}}\, \mathrm{e}^{-\tilde{s}z} \frac{1}{z} v\biggl(\frac{1}{\sqrt{z}}\biggr)\,\mathrm{d}z.$$

We can easily see that, since $z^{-1}v(z^{-1/2}) \simeq Cz^{-(\beta/2)-1}\exp(-cz^{-\alpha/2})$ as $z \to 0$, this integral is asymptotically equivalent to

$$\begin{aligned} H(v)(s) &\simeq \tfrac{1}{2}\gamma(s)c \int_{b_1/2a_1\tilde{s}}^{\infty} \sqrt{\frac{z}{2a_1\tilde{s}z - b_1}}\, z^{-(\beta/2)-1} \exp(-\tilde{s}z - cz^{-\alpha/2})\,\mathrm{d}z \\ &= \tfrac{1}{2}\gamma(s)\tilde{s}^{(\beta-1)/2} c \int_{b_1/2a_1}^{\infty} \sqrt{\frac{w}{2a_1w - b_1}}\, w^{-(\beta/2)-1} \exp(-w - c\tilde{s}^{\alpha/2} w^{-\alpha/2})\,\mathrm{d}w \\ &= \frac{\gamma(s)\tilde{s}^{(\beta-1)/2}}{\alpha\sqrt{2a_1}} c \int_0^A \frac{1}{\sqrt{1-(A^{-1}y)^{2/\alpha}}}\, y^{(\beta/\alpha)-1} \exp(-y^{-2/\alpha} - c\tilde{s}^{\alpha/2}y)\,\mathrm{d}y, \end{aligned}$$

where we have successively made the changes of variable $w = \tilde{s}z$ and $y = w^{-\alpha/2}$, and introduced $A := (b_1/2a_1)^{-\alpha/2}$. The integrand clearly has an integrable singularity at A, and we split the integral into $\int_0^{A/2} + \int_{A/2}^{A}$ (schematically). The second integral can be trivially estimated as $K\exp(-ka^{\alpha})$ for suitable constants k and K. For the first integral, we Taylor expand $(1-(A^{-1}y)^{2/\alpha})^{-1/2}$, and then extend the integration from $(0, \frac{1}{2}A]$ to $(0, \infty]$, thereby introducing a further error $Ks^p\exp(-ks^{\alpha})$, for some suitable power p and possibly larger constants k and K. The conclusion is that $H(v)(s)$ will be asymptotically equivalent to

$$\begin{aligned} H(v)(s) \simeq{}& \frac{\gamma(s)\tilde{s}^{(\beta-1)/2}}{\alpha\sqrt{2a_1}} c \int_0^{\infty} y^{(\beta/\alpha)-1} \exp(-y^{2/\alpha} - c\tilde{s}^{\alpha/2}y)\,\mathrm{d}y \\ &+ O\biggl(\int_0^{\infty} y^{\beta/\alpha} \exp(-y^{2/\alpha} - c\tilde{s}^{\alpha/2}y)\,\mathrm{d}y\biggr) + O(s^p\exp(-ks^{\alpha})), \end{aligned}$$

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

for some positive p and k. The two integrals are Laplace integrals of the type studied in Lemma 1, with $c = 1$; s replaced by $c\tilde{s}^{\alpha/2}$ (with c as in the hypothesis on v); α replaced by $2/\alpha$; and β by, respectively, $-(\beta/\alpha) + 1$ and $-\beta/\alpha$. A straightforward application of Lemma 1 then gives, after some computations,

$$H(v)(s) \simeq C' s^{(2\beta-\alpha)/(\alpha+2)} \exp(-c' s^{2\alpha/(\alpha+2)})(1 + O(s^{-\alpha/(\alpha+2)}) + O(s^p \exp(-ks^\alpha)))$$

with c' and C' given by (23) and (24), and where we have used the fact that

$$\exp(-c(s^2 - a_0)^{\alpha/(\alpha+2)}) \simeq \exp(-cs^{2\alpha/\alpha+2})\biggl(1 + \sum_\nu c_\nu s^{-2\nu}\biggr) \quad \text{as } s \to \infty,$$

which holds because $\alpha/(\alpha+2) < 1$. The main term clearly dominates the first error term and, since $2\alpha/(\alpha+2) < \alpha$ for $\alpha > 0$, it also dominates the second.

We next perform a similar analysis for F.

Lemma 3. *Suppose that* $v(s) \simeq Cs^\beta \exp(-cs^\alpha)$ *for* $0 < s \to \infty$, *where* $\beta \in \mathbb{R}$, $c > 0$, *and* $\alpha > 0$. *Then,*

$$F(v)(x) \simeq C'|x|^{(2\beta-\alpha)/(\alpha+2)} \exp(-c' x^{2\alpha/(\alpha+2)}), \qquad x \to \infty,$$

where

$$c' = \tfrac{1}{2}(\alpha+2)c^{2/(\alpha+2)}\alpha^{-\alpha/(\alpha+2)} \tag{25}$$

and

$$C' = \frac{2C}{\sqrt{\alpha+2}}(c\alpha)^{-(\beta+1)/(\alpha+2)}. \tag{26}$$

Proof. By the definition of F, we have

$$\begin{aligned} F(v)(x) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \frac{\exp(-(x^2/2s^2))}{s} v(s)\,\mathrm{d}s \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{u}\exp(-\tfrac{1}{2}ux^2) v\biggl(\frac{1}{\sqrt{u}}\biggr)\frac{1}{u^{3/2}}\,\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-\tfrac{1}{2}ux^2)\frac{1}{u} v\biggl(\frac{1}{\sqrt{u}}\biggr)\,\mathrm{d}u, \end{aligned}$$

making the change of variable $u = 1/s^2$. The integral on the right-hand side is the Laplace transform of $u^{-1}v(u^{-1/2})$ evaluated at $\frac{1}{2}x^2$, whose large-x behavior is completely determined by the small-u behavior

$$\frac{1}{u} v\biggl(\frac{1}{\sqrt{u}}\biggr) \simeq Cu^{-(\beta/2)-1}\exp(-cu^{-\alpha/2}), \qquad u \to 0,$$

by the hypothesis on v. Using Lemma 1 again, the asymptotics of $F(v)(x)$ follow from straightforward calculations.

We can now derive the asymptotic behavior of $H^k(\delta_\sigma)$.

Lemma 4. *Let $\sigma > 0$ and $k \geq 1$. Then, as $s \to \infty$,*

$$H^k(\delta_\sigma)(s) \simeq \widetilde{C}_k s^{-(1-1/k)} \exp\biggl(-\frac{k}{2a_1\sigma^{2/k}} s^{2/k}\biggr), \tag{27}$$

where

$$\widetilde{C}_k = \frac{\exp(kb_1/2a_1)}{\sqrt{2\pi a_1}} \sqrt{\frac{2^{k-1}}{k}} \sigma^{-1/k}.$$

Proof. Define $q_k(s) := H^k(\delta_\sigma)(s)$, $\beta_k := -(1 - 1/k)$, $\tilde{c}_k := k/2a_1\sigma^{2/k}$, and $\alpha_k := 2/k$. We will show, by induction on k, that $q_k(s) \simeq \widetilde{C}_k s^{\beta_k} \exp(-\tilde{c}_k s^{\alpha_k})$ as $s \to \infty$. First, if $k = 1$ then

$$\begin{aligned}
q_1(s) &= H(\delta_\sigma)(s) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2s\exp(b_1/2a_1)}{\sqrt{a_1\sigma^2(s^2 - a_0 - b_1\sigma^2)}} \exp(-(s^2 - a_0)/2a_1\sigma^2)\, \mathbf{1}_{\{s > \sqrt{a_0 + b_1\sigma^2}\}} \\
&\simeq \frac{1}{\sigma\sqrt{2a_1\pi}} \mathrm{e}^{b_1/2a_1} \mathrm{e}^{-s^2/2a_1\sigma^2} \\
&= \widetilde{C}_1 s^{\beta_1} \mathrm{e}^{-c_1 s^{\alpha_1}}, \qquad s \to \infty,
\end{aligned}$$

as required.

Next, assume that the lemma is true for q_{k-1}. Since $q_k(s) = H(q_{k-1})(s)$, we have

$$q_k(s) \simeq C' s^{(2\beta_{k-1} - \alpha_{k-1})/(\alpha_{k-1}+2)} \exp(-c' s^{(2\alpha_{k-1}/(\alpha_{k-1}+2))}),$$

by Lemma 2, with c' and C' given by (23) and (24), respectively (with $\alpha = \alpha_{k-1}$, $\beta = \beta_{k-1}$, $c = \tilde{c}_{k-1}$, and $C = \widetilde{C}_{k-1}$). Now, $2\alpha_{k-1}/(\alpha_{k-1} + 2) = 2/k = \alpha$ and, similarly,

$$\frac{2\beta_{k-1} - \alpha_{k-1}}{\alpha_{k-1} + 2} = -\biggl(1 - \frac{1}{k}\biggr) = \beta_k.$$

Furthermore, by direct computation, we find that $c' = k/2a_1\sigma^{2/k} = \tilde{c}_k$ and

$$\begin{aligned}
C' &= \frac{2\widetilde{C}_{k-1}\mathrm{e}^{b_1/2a_1}}{\sqrt{\alpha_{k-1} + 2}} (c_{k-1}\alpha_{k-1}a_1)^{-(\beta_{k-1}+1)/(\alpha_{k-1}+2)} \\
&= \mathrm{e}^{b_1/2a_1} \sqrt{\frac{2(k-1)}{k}} \sigma^{1/k(k-1)} \widetilde{C}_{k-1} = \widetilde{C}_k,
\end{aligned}$$

which proves (27).

Proof of Theorem 1. By Theorem 4, $p_{t,k} \equiv p_{t,k}(x; \rho_0, s_0) = F \circ H^{k-1}(\delta_{\sigma_{t+1}})$ and, for $k = 1$, we simply obtain $\sigma_{t+1}^{-1}(2\pi)^{-1/2} \exp(-x^2/2\sigma_{t+1}^2)$, which has the correct asymptotic behavior. If $k > 1$ then $p_{t,k} = F(q_{k-1})$ with

$$q_{k-1}(s) := H^{k-1}(\delta_{\sigma_{t+1}})(s) \;\simeq \widetilde{C}_{k-1} s^{\beta_{k-1}} \exp(-\tilde{c}_{k-1} s^{\alpha_{k-1}}),$$

by Lemma 4 with $\sigma = \sigma_{t+1}$, using the notation introduced in the proof of that lemma. Hence, by Lemma 3,

$$p_{t,k}(x; \rho_0, \sigma_0) \simeq C' |x|^{(2\beta_{k-1} - \alpha_{k-1})/(\alpha_{k-1}+2)} \exp(-c' |x| 2\alpha_{k-1}/(\alpha_{k-1} - 2)),$$

with c' and C' now being given by (25) and (26), respectively (with $\alpha = \alpha_{k-1}$, $\beta = \beta_{k-1}$, $c = \tilde{c}_{k-1}$, and $C = \widetilde{C}_{k-1}$). Straightforward computations then yield (4) together with (5) and (6).

Remark 3. A closer look at the proof of Theorem 1 shows that we in fact get a complete asymptotic expansion, i.e.

$$p_{t,k}(x; \rho_0, s_0) \simeq \frac{\mathrm{e}^{-c_k|x|^{2/k}}}{|x|^{1-1/k}} \sum_{\nu \geq 0} C_{\nu,k}|x|^{-\nu/k}, \qquad |x| \to \infty,$$

with $C_{0,k} = C_k$.

5. Proof of Theorem 2

By Theorem 5, with f the standard normal density,

$$\begin{aligned} &\mathrm{P}(r_{t+k,t} = x \mid r_t = \rho_0, \sigma_t = s_0) \\ &\quad = \left(\frac{1}{2\pi}\right)^{(k-1)/2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{k-1} \frac{1}{\hat{s}_j} \mathrm{e}^{-x_j^2/2\hat{s}_j^2} \frac{1}{\hat{s}_k} \\ &\qquad\qquad \times \exp\left(-\frac{(x - (x_1 + \cdots + x_{k-1}))^2}{2\hat{s}_k^2}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1}, \end{aligned} \tag{28}$$

where the standard deviations $\hat{s}_j \equiv \hat{s}_j(x_1, \ldots, x_{j-1})$ are defined recursively by

$$\hat{s}_1^2 \equiv a_0 + a_1\rho_0^2 + b_1 s_0^2, \qquad \hat{s}_j^2 = a_0 + a_1 x_{j-1}^2 + b_1 \hat{s}_{j-1}^2.$$

It easily follows that

$$\hat{s}_j^2 = \sum_{\nu=1}^{j-1} a_1 b_1^{\nu-1} x_{j-\nu}^2 + e_\nu,$$

where $e_1 = \hat{s}_1^2$ and $e_k = a_0 + b_1 e_{k-1}$. We will in fact establish a slightly more general result, and replace the $\hat{s}_j^2$ in (28) by arbitrary functions $L_{j-1} \equiv L_{j-1}(x_1, \ldots, x_{j-1})$, which are affine in $x_1^2, \ldots, x_{j-1}^2$ (note the shift by 1 of the index, relative to $\hat{s}_j$). That is, we let

$$L_j(x_1, \ldots, x_j) = \gamma_0^{(j)} + \sum_{\nu=1}^{j} \gamma_\nu^{(j)} x_\nu^2, \tag{29}$$

with $\gamma_\nu^{(j)}$ a positive constant, $\nu = 0, \ldots, j$. We will also put an adjustable multiplicative constant $\eta > 0$ in the exponent of the final factor of (28), and estimate the functions $q_k(x)$ defined by

$$\begin{aligned} &q_k(x) \equiv q_k(x; \eta, L_0, \ldots, L_{k-1}) \\ &\quad = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{j=1}^{k-1} \frac{\mathrm{e}^{-x_j^2/2L_{j-1}}}{\sqrt{2\pi L_{j-1}}} \frac{1}{\sqrt{2\pi L_{k-1}}} \\ &\qquad\qquad \times \exp\left(\frac{-\eta(x - (x_1 + \cdots + x_{k-1}))^2}{2L_{k-1}}\right) \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1}, \end{aligned} \tag{30}$$

under the following conditions on the coefficients of L_j:

$$\gamma_\nu^{(j)} > 0, \qquad 0 \le \nu \le j. \tag{31}$$

This will be satisfied if $L_{j-1} = \hat{s}_j^2$ for a GARCH(1, 1) process with $b_1 > 0$.

We will then prove the following set of inequalities, of which Theorem 2 will be an immediate consequence.

Assertion 1. *For given affine forms $L_0, \dots, L_{k-1}$ (as in (29)) satisfying (31), and given $\eta > 0$, there exist strictly positive constants c, c', C, and C' such that*

$$C|x|^{-(1-1/k)}\mathrm{e}^{-c|x|^{2/k}} \le q_k(x) \le C'|x|^{-(1-1/k)}\mathrm{e}^{-c'|x|^{2/k}}. \tag{32}$$

The constants c, c', C, and C' can be chosen locally uniformly in η and $\gamma_\nu^{(j)}$, $0 \le \nu \le j$, $j \le k$.

In the remainder of this section, we will prove this claim by induction on k. The idea is to estimate $q_k(x)$ from above and from below by a Laplace transform of a q_{k-1} with slightly modified η and L_j (modulo a negligible error) and then use Lemma 1 again. To accomplish this, we will first eliminate x_1 from all factors under the integral sign of (30) except the first one, using the following elementary inequality.

Lemma 5. *For all ε, $0 < \varepsilon \le 1$, and all $a, b \in \mathbb{R}$, we have*

$$C_{b,\varepsilon}^{-}\mathrm{e}^{-(1+\varepsilon)a^2} \le \mathrm{e}^{-(a+b)^2} \le C_{b,\varepsilon}^{+}\mathrm{e}^{-(1-\varepsilon)a^2}, \tag{33}$$

where $C_{b,\varepsilon}^{-} = \exp(-(\varepsilon^{-1}+1)b^2)$ and $C_{b,\varepsilon}^{+} = \exp((\varepsilon^{-1}-1)b^2)$.

Proof. To prove, for example, the upper bound, write

$$\exp((1-\varepsilon)a^2)\exp(-(a+b)^2) = \exp(-(\varepsilon a^2 + 2ab + b^2))$$

and maximize over a. The lower bound is proven in the same way.

It is clear that (32) holds for $k = 1$. We now suppose that it holds for $k-1$, and aim to prove it for k. We first establish the upper bound. Apply the second inequality in (33) with $a = \eta^{1/2}(x - (x_2 + \cdots + x_k))/(2L_{k-1})^{1/2}$ and $b = -\eta^{1/2}x_1/(2L_{k-1})^{1/2}$. The constant $C_{b,\varepsilon}^{+}$ then becomes

$$C_{b,\varepsilon}^{+} = \exp\biggl(\frac{(\varepsilon^{-1}-1)\eta x_1^2}{2L_{k-1}}\biggr) \le \exp\biggl(\frac{(\varepsilon^{-1}-1)\eta x_1^2}{2\gamma_0^{(k-1)}}\biggr)$$

and we see that it can be absorbed in the numerator of the first factor in the integrand of (30), namely $\exp(-x_1^2/2L_0)$, provided that ε is sufficiently close to 1. In fact, $C_{b,\varepsilon} < \exp(x_1^2/4L_0)$ if

$$(1 + \gamma_0^{(k-1)}/2\eta L_0)^{-1} < \varepsilon < 1.$$

With such a choice of ε, we then have

$$q_k(x) \le \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} \prod_{j=2}^{k-1} \frac{\mathrm{e}^{-x_j^2/2L_{j-1}}}{\sqrt{2\pi L_{j-1}}} \frac{1}{\sqrt{2\pi L_{k-1}}} \times \exp\biggl(-\frac{(1-\varepsilon)\eta(x-(x_2+\cdots+x_{k-1}))^2}{2L_{k-1}}\biggr)\,\mathrm{d}x_1\cdots\mathrm{d}x_{k-1}.$$

We now split this integral as follows, and estimate the two pieces separately:

$$\int_{|x_1|\le 1} \mathrm{d}x_1 \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} \int_{\mathbb{R}^{k-2}} (\cdots) + \int_{|x_1|>1} \mathrm{d}x_1 \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} \int_{\mathbb{R}^{k-2}} (\cdots) =: I + II.$$

We first show that I is of the same order as some $q_{k-1}(x; \eta^*, L_1^*, \dots, L_{k-1}^*)$ for some suitable choice of η^* and L_ν^*. In fact, if $|x_1| \le 1$ then

$$\begin{aligned} L_j(x_1, \dots x_j) &\le \gamma_0^{(j)} + \gamma_1^{(j)} + \gamma_2^{(j)} x_2^2 + \cdots + \gamma_j^{(j)} x_j^2 \\ &=: L_j^*(x_2, \dots, x_j), \end{aligned}$$

where L_1^* is just a constant, independent of $x_2, \dots, x_{k-1}$. We also have

$$\frac{L_j^*}{L_j} \le \max\biggl\{1, \frac{\gamma_0^{(j)} + \gamma_1^{(j)}}{\gamma_0^{(j)}}\biggr\},$$

this without any restriction on $(x_1, \dots, x_j)$. It follows that, for a suitable constant $C > 0$,

$$\begin{aligned} |I| \le C \int_{|x_1|\le 1} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} \prod_{j=2}^{k-1} \frac{\mathrm{e}^{-x_j^2/2L_{j-1}^*}}{\sqrt{2\pi L_{j-1}^*}} \\ \times \frac{1}{\sqrt{2\pi L_{k-1}^*}} \exp\biggl(\frac{-(1-\varepsilon)\eta(x - (x_2 + \cdots + x_{k-1}))^2}{2L_{k-1}^*}\biggr) \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1}. \end{aligned}$$

We recognize the integral over $\mathrm{d}x_2 \cdots \mathrm{d}x_{k-1}$ as being a constant times $q_{k-1}(x; (1-\varepsilon)\eta, L_1^*, \dots, L_{k-1}^*)$ and, therefore, by the induction hypothesis,

$$|I| \le C|x|^{-(1-1/(k-1))} \mathrm{e}^{-c|x|^{2/(k-1)}}$$

for suitable constants c and C. For $|x| \to \infty$, this is of strictly lower order than the inequality we are trying to establish for $q_k(x)$.

We next turn to integral II. If $|x_1| > 1$ then

$$\begin{aligned} L_j(x_1, \dots, x_j) &\le (\gamma_0^{(j)} + \gamma_1^{(j)}) x_1^2 + \gamma_2^{(j)} x_2^2 + \cdots + \gamma_j^{(j)} x_j^2 \\ &= x_1^2 \biggl(\gamma_0^{(j)} + \gamma_1^{(j)} + \gamma_2^{(j)} \frac{x_2^2}{x_1^2} + \cdots + \gamma_j^{(j)} \frac{x_j^2}{x_1^2}\biggr) \\ &=: x_1^2 \tilde{L}_j\biggl(\frac{x_2}{x_1}, \dots, \frac{x_j}{x_1}\biggr), \end{aligned}$$

the last equation defining $\tilde{L}_j$. Similarly, for $|x_1| > 1$ we can estimate

$$\begin{aligned} L_j(x_1, \dots, x_j) &\ge \gamma_1^{(j)} x_1^2 + \gamma_2^{(j)} x_2^2 + \cdots + \gamma_j^{(j)} x_j^2 \\ &= x_1^2 \biggl(\gamma_1^{(j)} + \gamma_2^{(j)} \frac{x_2^2}{x_1^2} + \cdots + \gamma_j^{(j)} \frac{x_j^2}{x_1^2}\biggr) \\ &\ge c\, x_1^2 \tilde{L}_j\biggl(\frac{x_2}{x_1}, \dots, \frac{x_j}{x_1}\biggr), \end{aligned} \tag{34}$$

provided that

$$c \le \frac{\gamma_1^{(j)}}{\gamma_0^{(j)} + \gamma_1^{(j)}}.$$

Note that, to have (34) with a $c > 0$, we must have $\gamma_1^{(j)} > 0$, which is ensured by condition (31). By substituting these inequalities into (30), we find that, for a suitable $C > 0$,

$$II \le C \int_{|x_1|>1} \int_{\mathbb{R}^{k-2}} \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} \prod_{j=2}^{k-1} \frac{\exp(-x_j^2/2x_1^2 \tilde{L}_{j-1})}{|x_1|\sqrt{2\pi \tilde{L}_{j-1}}} \times \frac{1}{|x_1|\sqrt{2\pi \tilde{L}_{k-1}}} \exp\biggl(\frac{-(1-\varepsilon)\eta(x-(x_2+\cdots+x_{k-1}))^2}{2x_1^2 \tilde{L}_{k-1}}\biggr)\, \mathrm{d}x_1 \cdots \mathrm{d}x_{k-1}$$

(we can in fact take $C = [\min_j (\gamma_1^{(j)}/(\gamma_0^{(j)} + \gamma_1^{(j)}))]^{-(k-1)/2}$). If we now change variable to $y_j := x_j/|x_1|$, $2 \le j \le k-1$, we see that the previous inequality can be written as

$$II \le C \int_{|x_1|>1} \frac{1}{|x_1|} \frac{\mathrm{e}^{-x_1^2/4L_0}}{\sqrt{2\pi L_0}} q_{k-1}\biggl(\frac{x}{|x_1|}; (1-\varepsilon)\eta, \tilde{L}_1, \ldots, \tilde{L}_{k-1}\biggr)\, \mathrm{d}x_1.$$

By the induction hypothesis, the $q_{k-1}(x/x_1)$ contained in the integrand is less than or equal to

$$C\biggl(\frac{|x|}{|x_1|}\biggr)^{\beta} \mathrm{e}^{-c(|x|/|x_1|)^{\alpha}},$$

with

$$\alpha = \frac{2}{k-1}, \qquad \beta = -1 + \frac{1}{k-1}, \tag{35}$$

and, thus, after a rescaling, and with different constants c and C,

$$II \le |x|^{\beta} C \int_{|x_1|>1} |x_1|^{-\beta-1} \mathrm{e}^{-c(|x|/|x_1|)^{\alpha}} \mathrm{e}^{-x_1^2}\, \mathrm{d}x_1. \tag{36}$$

We now assume that $x > 0$ and write the integral as twice that over the range $[1, \infty)$. We again want to use Lemma 1 and for this we rewrite our integral as a Laplace transform with large parameter, by introducing the new variable $z = x_1^{-\alpha}$. Then, the right-hand side of (36) is less than or equal to a constant times

$$x^{\beta} \int_0^1 z^{(\beta/\alpha)-1} \mathrm{e}^{-z^{-2/\alpha}} \mathrm{e}^{-cx^{\alpha} z}\, \mathrm{d}z$$

and, by Lemma 1 with $s = cx^{\alpha}$ and α and β replaced respectively by $2/\alpha$ and $1 - (\beta/\alpha)$, we find that

$$II \le Cx^{(2\beta-\alpha)/(\alpha+2)} \exp(-cx^{2\alpha/(\alpha+2)}),$$

again with different c and C. Since, using (35), the exponents of x corresponding to $(2\beta-\alpha)/(\alpha+2)$ and $2\alpha/(\alpha+2)$ in this formula turn out to be, respectively, $-(1-1/k)$ and $2/k$, this proves the desired upper bound for II and, thus, for $q_k(x)$.

We next turn to the lower bound for q_k. By the first inequality of (33) we see, in the same way as before, that

$$\exp\biggl(-\frac{\eta(x-(x_1+\cdots+x_{k-1}))^2}{2L_{k-1}}\biggr) \geq C^-_{b,\varepsilon}\exp\biggl(-\frac{(1+\varepsilon)\eta(x-(x_2+\cdots+x_{k-1}))^2}{2L_{k-1}}\biggr),$$

where

$$C^-_{b,\varepsilon} = \exp\biggl(-\frac{\eta(1+\varepsilon^{-1})x_1^2}{2L_{k-1}}\biggr) \geq \exp\biggl(-\frac{\eta(1+\varepsilon^{-1})x_1^2}{2\gamma_0^{(k-1)}}\biggr).$$

We can combine $C^-_{b,\varepsilon}$ with the numerator of the first factor of the integrand of (30) to produce a factor $\mathrm{e}^{-\kappa x_1^2}$. By doing so, and limiting the x_1 integration in (30) to $|x_1| > 1$, we find that

$$\begin{aligned} q_k(x) \geq \int_{|x_1|>1}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} \frac{\mathrm{e}^{-\kappa x_1^2}}{\sqrt{2\pi L_0}} \prod_{j=2}^{k-1}\frac{\mathrm{e}^{-x_j^2/2L_{j-1}}}{\sqrt{2\pi L_{j-1}}} \\ \times \frac{1}{\sqrt{2\pi L_{k-1}}}\exp\biggl(-\frac{\eta(1+\varepsilon)(x-(x_2+\cdots+x_{k-1}))^2}{2L_{k-1}}\biggr)\,\mathrm{d}x_1\cdots\mathrm{d}x_{k-1}. \end{aligned} \tag{37}$$

As before, we next eliminate x_1 from the L_j. First, if $j \geq 1$ then

$$\begin{aligned} L_j(x_1,\dots,x_j) &\geq x_1^2\biggl(\gamma_1^{(j)} + \gamma_2^{(j)}\frac{x_2^2}{x_1^2} + \cdots + \gamma_j^{(j)}\frac{x_j^2}{x_1^2}\biggr) \\ &=: x_1^2\hat{L}_j\biggl(\frac{x_2}{x_1},\dots,\frac{x_j}{x_1}\biggr). \end{aligned}$$

Next, if $|x_1| > 1$ then

$$\begin{aligned} L_j(x_1,\dots,x_j) &\leq (\gamma_0^{(j)} + \gamma_1^{(j)})x_1^2 + \cdots + \gamma_j^{(j)}x_j^2 \\ &\leq c x_1^2\hat{L}_j\biggl(\frac{x_2}{x_1},\dots,\frac{x_j}{x_1}\biggr), \end{aligned} \tag{38}$$

provided that $c \geq (\gamma_0^{(j)} + \gamma_1^{(j)})/\gamma_1^{(j)}$; there exists such a (finite) c since $\gamma_1^{(j)} > 0$, by (31). Substituting these inequalities into (37) and making the same change of variable $y_j = x_j/x_1$ as before (with $j \geq 2$), we find that, for a suitable constant $C > 0$,

$$q_k(x) \geq C\int_{|x_1|>1}\frac{\mathrm{e}^{-\kappa x_1^2}}{|x_1|}q_{k-1}\biggl(\frac{x}{x_1};(1+\varepsilon)\eta,\hat{L}_2,\dots,\hat{L}_{k-1},(1+\varepsilon)\eta\biggr).$$

Using the induction hypothesis and Lemma 1, we find the required lower bound for $q_k(x)$.

6. Proof of Theorem 3

Theorem 3 is much easier to prove than Theorem 2, and is in fact a fairly straightforward consequence of the following corollary to Theorem 1. Let

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y$$

denote the complementary error function. By integrating (4), we then find the following result.

https://doi.org/10.1239/jap/1118777180 Published online by Cambridge University Press

Corollary 1. *Let*

$$\hat{C}_k = \exp\biggl(\frac{k-1}{2}\biggl(\log 2 + \frac{b_1}{a_1}\biggr)\biggr).$$

Then,

$$F_{r_{t+k} \mid \rho_0, s_0}(x) \simeq \hat{C}_k \operatorname{erfc}(\sqrt{2c_k}|x|^{1/k}), \qquad x \to -\infty,$$

where c_k is given by (5).

By symmetry, we have the same asymptotics for $\overline{F}_{r_{t+k} \mid \rho_0, s_0}(x) := 1 - F_{r_{t+k} \mid \rho_0, s_0}(x)$ as $x \to \infty$.

We now prove Theorem 3. First, observe that if $\lambda_1 + \cdots + \lambda_k = 1$ with $\lambda_j \geq 0$ $(1 \leq j \leq k)$, then, for any $x \in \mathbb{R}$,

$$\{r_{t+k,t} < x\} \subset \bigcup_{j=1}^{k} \{r_{t+j} < \lambda_j x\}$$

and, therefore, we have

$$\mathrm{P}_{\rho_0, s_0}(r_{t+k,t} < x) \leq \sum_{j=1}^{k} \mathrm{P}_{\rho_0, s_0}(r_{t+j} < \lambda_j x),$$

where, recall, P_{ρ_0, s_0} stands for $\mathrm{P}(\cdot \mid r_t = \rho_0, \sigma_t = s_0)$. We choose $\lambda_k = 1 - \varepsilon$ and $\lambda_j = \varepsilon/(k-1)$, $1 \leq j \leq k-1$, for some $0 < \varepsilon < 1$, which will tend to 0 at the end of the proof. By Corollary 1,

$$\mathrm{P}_{\rho_0, s_0}(r_{t+j} < x) \simeq \hat{C}_j \operatorname{erfc}(\sqrt{2c_j}|x|^{1/j}), \qquad x \to -\infty,$$

and a moment's thought then shows that, for all $\eta > 0$, there exists an $R \equiv R(\eta, \varepsilon, k) > 0$ such that, for all $x < -R$,

$$F_{r_{t+k,t} \mid r_t = \rho_0, \sigma_t = s_0}(x) \leq (1 + \eta)\hat{C}_k \operatorname{erfc}((1 - \varepsilon)\sqrt{2c_k}|x|^{1/k}).$$

Taking logarithms, it follows that

$$\limsup_{x \to -\infty} \frac{\log F_{r_{t+k,t} \mid r_t = \rho_0, \sigma_t = s_0}(x)}{|x|^{2/k}} \leq -(1 - \varepsilon)^2 c_k$$

(since $\log(\operatorname{erfc}(x)) \simeq -\frac{1}{2}x^2$ as $|x| \to \infty$). Hence, letting $\varepsilon \to 0$, we have

$$\limsup_{x \to -\infty} \frac{\log F_{r_{t+k,t} \mid r_t = \rho_0, \sigma_t = s_0}(x)}{|x|^{2/k}} \leq -c_k. \tag{39}$$

To obtain a lower bound, we simply note that if $r_{t+k} < x$ and $r_{t+k-1,t} < 0$, then $r_{t+k,t} < x$. Hence,

$$\mathrm{P}_{\rho_0, s_0}(r_{t+k,t} < x) \geq \mathrm{P}_{\rho_0, s_0}(r_{t+k} < x, r_{t+k-1,t} < 0).$$

We can easily check, using the symmetry properties of a GARCH(1, 1) process, that

$$\mathrm{P}_{\rho_0, s_0}(r_{t+k} < x, r_{t+k-1,t} < 0) = \mathrm{P}_{\rho_0, s_0}(r_{t+k} < x, r_{t+k-1,t} > 0).$$

It follows that

$$\mathrm{P}_{\rho_0, s_0}(r_{t+k} < x, r_{t+k-1} < 0) = \tfrac{1}{2}\,\mathrm{P}_{\rho_0, s_0}(r_{t+k} < x) = \tfrac{1}{2} F_{r_{t+k} \mid \rho_0, s_0}(x)$$

and, therefore, that

$$F_{r_{t+k,t}\,|\,\rho_0,s_0}(x) \geq \tfrac{1}{2} F_{r_{t+k}\,|\,\rho_0,s_0}.$$

Using Corollary 1 and taking logarithms, we find that

$$\liminf_{x\to-\infty} \frac{\log F_{r_{t+k,t}\,|\,r_t=\rho_0,\sigma_t=s_0}(x)}{|x|^{2/k}} \geq -c_k,$$

which, together with (39), proves Theorem 3.

7. Application to extreme lower quantiles

The above theorems have theoretical implications for financial risk management, in particular VaR estimations in GARCH models. Again, consider a risky asset (or portfolio of assets) with price P_t, $t = 0, 1, 2, \dots$, where, to be specific, we assume that t is measured in days. Recall that $\mathrm{VaR}_\alpha(t, k)$, the *conditional* (or *dynamic*) *value at risk at confidence* $1 - \alpha$ *and over the time window* $[t, t + k]$, is defined as the αth lower quantile of the profit-and-loss distribution $\Delta P_{t,k} = P_{t+k} - P_t$, given the information available at time t. Here, α and k are given parameters; in practice, α equals 0.05 or 0.01 for k equal to 1 or 10, respectively. We refer to [8], [15], and [16] (which was at the origin of the concept) for further information on VaR and its uses in financial risk management, and to [1] for a critique of its suitability as a risk measure. As quantile function, we will take the left inverse of the distribution function as defined, for example, in [10, Definitions 3.3.4 and 3.3.5]; cf. (40) below.

The conditional VaR should be carefully distinguished from the unconditional VaR, which is computed from the stationary return distribution – see [12] for more discussion of this point.

Let $r_t = \log(P_t/P_{t-1})$ be the log-return over $[t - 1, t]$, and assume that r_t follows a GARCH(1, 1) process. It is convenient to introduce the conditional return at risk, $\mathrm{RaR}_\alpha(t, k)$, which will simply be the αth lower quantile of $r_{t+k,t}$, given that $r_t = \rho_0$ and $\sigma_t = s_0$. Explicitly,

$$\mathrm{RaR}_\alpha(t, k) = \inf\{x : F_{r_{t+k,t}\,|\,\rho_0,s_0}(x) = \alpha\}, \tag{40}$$

where we have suppressed the dependence of the left-hand side on ρ_0 and s_0, to simplify the notation. This quantity is related to the conditional VaR by

$$\mathrm{VaR}_\alpha(t, k) = (\exp(\mathrm{RaR}_\alpha(t, k)) - 1)P_t.$$

In practice, one often approximates the right-hand side of this by $\mathrm{RaR}_\alpha(t, k)P_t$, although this might give rise to sizeable errors. (Of course, such an approximation would have been exact were we to have used percentage returns instead of logarithmic ones, but that would have caused problems with temporal aggregation of one-day returns to k-day returns.) Observe that we are recording losses using negative numbers.

The prediction of the one-day conditional return at risk is trivial in a GARCH(1, 1) model. However, one often also needs to know the VaR over multiple-day time windows. A well-known example is given by the Bank of International Settlements capital adequacy requirements, which ask banks to estimate their 10-day VaR at the 99% confidence level. As a rule of thumb, practitioners simply rescale one-day VaR (or, more precisely, one-day RaR) by $k^{1/2}$. The origin of this heuristic rule lies in the simple random walk model for log-prices and is, therefore, strictly speaking, not applicable to GARCH models. Our asymptotic results allow us to investigate this point more closely for asymptotically vanishing α. Theorem 3 has the following corollary.

Corollary 2. *For k fixed and $\alpha \to 0$, we have*

$$\mathrm{RaR}_\alpha(t,k) \simeq -k^{-k/2} a_1^{(k-1)/2} \sigma_{t+1} (\log \alpha^{-2})^{k/2}.$$

Hence, $|\mathrm{RaR}_\alpha(t,k)|$ tends to infinity as $(\log \alpha^{-2})^{k/2}$ when α tends to 0. In particular, for any $k \geq 2$,

$$\lim_{\alpha \to 0} \frac{\mathrm{RaR}_\alpha(t,k)}{\sqrt{k}\, \mathrm{RaR}_\alpha(t,1)} = \infty,$$

which shows that the $k^{1/2}$ rule fails spectacularly for very small α, even if $k = 2$. It remains to be seen to what extent this asymptotic result is relevant for the αs used in practical risk assessment. This question is probably most easily investigated using numerical simulations, since good explicit error bounds have turned out to be hard to obtain from our proofs.

Acknowledgement

The authors gratefully acknowledge the support of the Fondation de la Banque de France for this research, as part of the project 'Mesures de risques extrêmes en finance et choix du temps', proposal no. 4.

References

[1] Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Math. Finance* **9,** 203–228.
[2] Bingham, N. H., Goldie, G. M. and Teugels, J. L. (1987). *Regular Variation*. Cambridge University Press.
[3] Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *J. Econometrics* **31,** 307–325.
[4] Bollerslev, T., Engle, R. F. and Nelson, D. B. (1994). ARCH models. In *Handbook of Econometrics* (Handbook Econom. **2**), Vol. 4, eds R. F. Engle and D. McFadden, North-Holland, Amsterdam, pp. 2959–3038.
[5] De Haan, L., Resnick, S. I., Rootzén, H. and de Vries, C. G. (1989). Extremal behavior of solutions to a stochastic difference equation with applications to ARCH processes. *Stoch. Process. Appl.* **32,** 213–224.
[6] Diebolt, J. and Guégan, D. (1991). Le modèle β-ARCH. *C. R. Acad. Sci. Paris* **312,** 625–630.
[7] Diebolt, J. and Guégan, D. (1993). Tail behavior of the stationary density of general nonlinear autoregressive process of order 1. *J. Appl. Prob.* **30,** 315–329.
[8] Dowd, K. (1998). *Beyond Value at Risk*. John Wiley, Chichester.
[9] Drost, F. C. and Nijman, T. E. (1993). Temporal aggregation of GARCH processes. *Econometrica* **61,** 909–927.
[10] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events for Insurance and Finance*. Springer, Berlin.
[11] Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50,** 907–1007.
[12] Frey, R. and McNeil, A. (2000). Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *J. Empirical Finance* **7,** 271–300.
[13] Guégan, D. (1994). *Séries Chronologiques Non Linéaires à Temps Discrèt*. Economica, Coll. Statistique Mathématique et Probabilité, Paris.
[14] Guégan, D. (2000). A new model: the k-factor GIGARCH process. *J. Signal Process.* **4,** 265–271.
[15] Jorion, P. (1997). *Value at Risk: The New Benchmark for Controlling Market Risk*. McGraw-Hill, New York.
[16] Kim, J. and Mina, J. (1996). *RiskMetrics Technical Document*. Available at: http://www.riskmetrics.com/techdoc.html.
[17] Mikosch, T. and Stărică, C. (2000). Limit theory for the sample autocorrelations and extremes of a GARCH(1, 1) process. *Ann. Statist.* **28,** 1427–1451.