COUNTABLE PARACOMPACTNESS AND SOUSLIN'S PROBLEM

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1. Introduction. A linearly ordered space S in which neighborhoods are segments is called a Souslin space if

(i) S is not separable, but

(ii) every collection of disjoint segments of S is countable.

Whether a Souslin space exists is not known; this is the problem referred to in the title and was proposed by Souslin in **(2)**.

A *covering* of a topological space T is a collection of open subsets of T whose sum is T. A covering is *locally finite* if every point of T is in some open set which intersects only a finite number of sets of the covering. A topological space T is said to be *countably paracompact* if every countable covering of T has a locally finite refinement **(1)**.

A topological space T is *normal* if for every two disjoint closed sets K_1 and K_2 of T there are disjoint open sets H_1 and H_2 of T containing K_1 and K_2, respectively; if, in addition, every point of T is a closed set, then T is a *normal Hausdorff space*. It is not known whether every normal Hausdorff space is countably paracompact.

In this paper the following will be proved:

THEOREM. *If there exists a Souslin space, then there exists a normal Hausdorff space which is not countably paracompact.*

Let us say that a topological space has property D if the following is true: *whenever $C_1, C_2, C_3, \ldots$ is a decreasing sequence of closed sets having no common part, then there exists a sequence $D_1, D_2, D_3, \ldots$ of open sets having no common part such that, for each n, D_n covers C_n.*

In the proof of our theorem we shall use the following result due to Dowker **(1**, p. 220**)**: *A normal space is countably paracompact if and only if it has property* D.

2. Preliminary constructions and notation. We now assume that there is a Souslin space S. If S contains any separable segments, the collection of all maximal separable segments is at most countable; and the sum of this collection of segments is not dense in S, since S would otherwise be separable. Hence *there is a segment s in S none of whose subsegments is separable.*

Let R_1 be a collection of disjoint subsegments of s. Suppose R_α has been defined for all ordinals $\alpha < \beta$, where β is a countable ordinal. If β is not a limit

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ordinal, let S_β be the sum of the segments of $R_{\beta-1}$ and E_β be a set consisting of one point from each term of $R_{\beta-1}$; then let R_β be the set of all maximal segments of $S_\beta - E_\beta$. If β is a limit ordinal, let S_β be the set of all points which belong to some segment of R_α for every $\alpha < \beta$; and let R_β be the collection of all maximal segments in S_β. Since each E_α is countable for all non-limit ordinals $\alpha < \beta$, the closure of the sum of all such $E_\alpha s$ is separable; hence R_β exists for all β.

Let R be the sum of all R_β where β is a countable ordinal.

For each term x of R let $\phi(x)$ denote the ordinal such that x belongs to $R_{\phi(x)}$.

If β is a countable limit ordinal, then for each x in R_β we consider a sequence $f_1(x), f_2(x), f_3(x), \ldots$ of segments in R_β such that

(a) $f_i(x_1) = f_j(x_2)$ if and only if $i = j$ and $x_1 = x_2$.

(b) if x is a subsegment of a segment y of R where $\phi(y) < \beta$, then, for some n, $f_i(x)$ is a subsegment of y whenever $i > n$.

From now on we will assume that whenever one of the letters x, y, z, or w is used it stands for a segment of R; whenever one of the letters i, j, m, or n is used it stands for a positive integer; and whenever a greek letter is used it stands for a countable ordinal.

3. Construction of T. The points of T are the ordered pairs (x, n) where x is in R and n is a positive integer. We now define a neighborhood system in T.

3.1. If $\phi(x)$ is not a limit ordinal, a neighborhood of (x, n) consists of the point (x, n) alone.

3.2. If $\phi(x)$ is a limit ordinal and $\beta < \phi(x)$, a neighborhood of $(x, 1)$ consists of all points $(y, 1)$ such that x is a subsegment of y and $\beta < \phi(y) \leqslant \phi(x)$.

3.3. To define neighborhoods of $p = (x, n)$ if $\phi(x) = \beta$ is a limit ordinal and $n > 1$, we proceed inductively, assuming that neighborhoods have been defined for all (y, m) with $m < n$ and for all (y, n) with $\phi(y) < \beta$.

Let $F_i(p)$, for each i, be the set of all points $(f_j(x), n-1)$ where $j > i$.

Let $G_\alpha(p)$, for each α, be the set of all points (y, n) such that x is a subsegment of y and $\alpha < \phi(y) < \beta$.

A set N is a neighborhood of $p = (x, n)$ if N is the sum of

(a) a neighborhood of each point of some $F_i(p)$,

(b) a neighborhood of each point of some $G_\alpha(p)$, and

(c) the point p itself.

3.4. For better orientation and later use we mention the following facts about T.

(i) The set C_n consisting of all (x, m) with $m \geqslant n$ is closed.

(ii) For any β, the set of all (x, n) with $\phi(x) \leqslant \beta$ is countable, open, and closed.

(iii) The neighborhoods of points of the form $(x, 1)$ as described in 3.2 are open and closed.

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(iv) If $\phi(x)$ is a limit ordinal, $n > 1$, and x is a subsegment of z, then every neighborhood of (x, n) contains points of the form (y, m), for any $m < n$, such that $\phi(y) = \phi(x)$ and y is a subsegment of z.

4. To prove that T has the desired properties we introduce more machinery.

Let Q be a subcollection of R. If x has the property that every subsegment y of x (in R) contains a segment z such that z is in Q, then we say that *x is Q-full.* If x is Q-full, then, for any β, we let $L(x, \beta, Q)$ be the collection of all y such that

(a) y is a subsegment of x belonging to Q and $\phi(y) \geqslant \beta$;

(b) y is not a proper subsegment of any segment of R for which (a) holds.

Since the segments of $L(x, \beta, Q)$ are disjoint, they are countable, and there exists a smallest ordinal $\delta(x, \beta, Q)$ such that $\phi(y) < \delta(x, \beta, Q)$ for every y in $L(x, \beta, Q)$.

We note that $\beta < \delta(x, \beta, Q)$.

5. Proof that T does not have property D. We let C_n be the closed set consisting of all points (x, m) where $m \geqslant n$. Suppose that, for each n, D_n is an open set containing C_n.

LEMMA 5.1. *Let n be fixed. For every x there exists a subsegment y such that, if z is any subsegment of y, $(z, 1)$ is a point of D_n.*

Let Q_n be the set of all x for which $(x, 1)$ is not in D_n. The lemma is equivalent to the assertion: no x is Q_n-full.

To prove the Lemma, suppose x is Q_n-full. Let $\beta_1 = \phi(x)$, $\beta_i = \delta(x, \beta_{i-1}, Q_n)$ for $i > 1$, and β be the limit ordinal of $\beta_1, \beta_2, \beta_3, \ldots$. By 3.3, there is a segment y of R_β such that y is a subsegment of x and $(y, 1)$ is in D_n.

For each i, let z_i be the segment of $L(x, \beta_i, Q_n)$ of which y is a subsegment. Since z_i is in Q_n, $(z_i, 1)$ is not in D_n. By 3.2, $(y, 1)$ is a limit point of the sequence $(z_1, 1), (z_2, 1), (z_3, 1), \ldots$. Consequently $(y, 1)$ is not in D_n, and this contradiction proves Lemma 5.1.

5.2. Let P_n be the set of all x such that $(y, 1)$ is in D_n if y is a subsegment of x. Lemma 5.1 shows that every x is P_n-full for every n.

Let x be a segment of R_1; pick γ such that $\delta(x, 1, P_n) < \gamma$ for every n. Let y be a subsegment of x belonging to R_γ. We see that $(y, 1)$ is in D_n for every n.

Consequently $D_1 \cdot D_2 \cdot D_3 \cdot \ldots$ exists and T does not have property D.

6. Proof that T is a normal Hausdorff space. It is clear that every point of T is a closed set.

Let H and K be disjoint closed subsets of T. For each n let H_n and K_n be the sets of all x for which (x, n) is in H or K, respectively.

LEMMA 6.1. *Suppose i and j are integers and x and y are segments of R where y is a subsegment of x. Then, if x is H_i-full, y is not K_j-full.*

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Suppose, on the contrary, that x is H_i-full and y is K_j-full. Choose $\gamma_0 > \phi(y)$. For odd n, let $\gamma_n = \delta(y, \gamma_{n-1}, K_j)$; if n is even, let $\gamma_n = \delta(x, \gamma_{n-1}, H_i)$. Let γ be the limit ordinal of $\gamma_1, \gamma_2, \gamma_3, \ldots$.

There exists a subsegment z of y in R_γ. If $i = j$ the point (z, j) is clearly a limit point of both H and K which is impossible since H and K are closed and disjoint. Assume $i < j$. Then clearly (z, j) is a limit point of K. But every neighborhood of (z, j) contains points of the form (w, i) where w is a subsegment of y in R_γ; and (w, i) is a limit point of H. So (z, j) is also a limit point of H which is a contradiction.

The case $j < i$ is treated similarly.

LEMMA 6.2. *There is an ordinal μ with the following property: no x in R_μ contains two segments y and z such that y is in H_n and z is in K_m, for any choice of m and n.*

Let P_1 be the set of all x none of whose subsegments in R belongs to H_n for any n. Let P_2 be the set of all x none of whose subsegments in R belongs to K_m for any m. Let P_i', for i equal to 1 or 2, be the collection of all maximal segments in P_i. Then P_i' is countable, and we can choose μ so that $\phi(y) < \mu$ for all y in P_i'. Choose x in R_μ.

Case 1: *No subsegment of x is K_m-full for any m.*

Then every segment is I_m-full where I_m is the set of all segments none of whose subsegments belongs to K_m. Put $\alpha_0 = \mu$, $\alpha_m = \delta(x, \alpha_{m-1}, I_m)$, and let α be the limit ordinal of $\alpha_1, \alpha_2, \alpha_3, \ldots$.

If z is a subsegment of x in R_α, then z is in P_2; so z is a subsegment of w for some w in P_2'. By our choice of μ, we see that x is a subsegment of w, and it follows that no subsegment of x in R belongs to K_m for any m.

Case 2: *Some subsegment y of x is K_m-full for some m.*

By 6.1 no subsegment of y is then H_n-full for any n. Proceeding as in Case 1, we see that y is a subsegment of a segment w of P_1', so that x is also a subsegment of w, and no subsegment of x belongs to H_n for any n.

This completes the proof of Lemma 6.2.

6.3. Choosing μ in accordance with Lemma 6.2, we let X be the set of all (x, n) for which $\phi(x) \leqslant \mu$. Then X is a countable, closed and open subset of T; and we can order the points of X in a simple countable sequence $p_1, p_2, p_3, \ldots$.

Let $A_0 = H \cdot X$ and $B_0 = K \cdot X$. We shall construct nonintersecting sequences $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ of closed sets. Having constructed A_{m-1} and B_{m-1}, consider the point $p_m = (x, n)$.

I. *Suppose that p_m is not in B_{m-1}.*

Case 1. If $\phi(x)$ is not a limit ordinal, let $A_m = A_{m-1} + p_m$, and let $B_m = B_{m-1}$.

Case 2. If $\phi(x)$ is a limit ordinal and $n = 1$, let J be a neighborhood of p_m of the type described in 3.2 such that J does not intersect B_m. Put $A_m = A_{m-1} + J$ and $B_m = B_{m-1}$.

Case 3. If $\phi(x)$ is a limit ordinal and $n > 1$, choose i and β so that no point of $F_i(p_m)$ or of $G_\beta(p_m)$ (using the notation introduced in 3.3) is a point of A_{m-1}. We put $A_m = A_{m-1} + p_m + F_i(p_m) + G_\beta(p_m)$ and $B_m = B_{m-1}$.

II. *If p_m is in B_{m-1}, then perform the operations of cases* 1, 2, *and* 3 *above interchanging A and B.*

Then A_m and B_m are closed and disjoint and the induction is complete. We now prove

6.4. *The sets $A = A_1 + A_2 + A_3 + \ldots$ and $B = B_1 + B_2 + B_3 + \ldots$ are open.*

Proof. If $p = (x, n)$ is in A, then p is certainly an interior point of A whenever $\phi(x)$ is not a limit ordinal and whenever $n = 1$. Suppose A is not open. Then there is a point $(x, n) = p$ of A which is not an interior point of A, but such that every point (y, i) of A where $i < n$ and every point (z, n) where $\phi(z) < \phi(x)$ is an interior point of A. If m is the integer such that $p_m = p$, the above construction of A_m and the definition of neighborhood in 3.3 show that a neighborhood of p_m is included in A.

Hence A is open and the same is, of course, true of B.

6.5. Let V' and W' be the collections of all x in R_μ which have a subsegment in some H_i or in some K_j, respectively. Let V and W be the sets of all points (y, n) with y a proper subsegment of a segment of V' or W', respectively.

Then V and W are open disjoint subsets of T by Lemma 6.2.

Finally, the sets $A + V$ and $B + W$ are disjoint and open and cover H and K, respectively. Hence T is normal and the proof is complete.

References

1. C. H. Dowker, *On countably paracompact spaces*, Can. J. Math., *3* (1951), 219–224.
2. M. Souslin, *Problème* 3, Fund. Math., *1* (1920), 223.