LINEAR TRANSFORMATIONS ON ALGEBRAS OF MATRICES

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1. Introduction. Let M_n denote the algebra of n-square matrices over the complex numbers; and let U_n, H_n, and R_k denote respectively the unimodular group, the set of Hermitian matrices, and the set of matrices of rank k, in M_n. Let $\mathrm{ev}(A)$ be the set of n eigenvalues of A counting multiplicities. We consider the problem of determining the structure of any linear transformation (l.t.) T of M_n into M_n having one or more of the following properties:

(a) $T(R_k) \subseteq R_k$ for $k = 1, \ldots, n$.
(b) $T(U_n) \subseteq U_n$
(c) $\det T(A) = \det A$ for all $A \in H_n$.
(d) $\mathrm{ev}(T(A)) = \mathrm{ev}(A)$ for all $A \in H_n$.

We remark that we are not in general assuming that T is a multiplicative homomorphism; more precisely, T is a mapping of M_n into itself, satisfying

$$T(aA + bB) = aT(A) + bT(B)$$

for all A, B in M_n and all complex numbers a, b.

We shall show first that if T satisfies property (a), then there exist non-singular matrices U and V such that either

$$T(A) = UAV$$

or

$$T(A) = UA'V,$$

for all $A \in M_n$, where A' is the transpose of A. We shall then show that any T satisfying (b), (c), or (d) must in turn satisfy (a), and determine the additional restrictions on U and V required in these cases.

2. Rank Preservers. In this section we shall characterize all linear transformations of M_n which preserve rank. To this end it is convenient to consider each matrix of M_n as an n^2-vector, and to represent the l.t. T as an $n^2 \times n^2$ matrix.

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & & & \\ T_{n1} & & \cdots & T_{nn} \end{pmatrix} \tag{1}$$

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where each T_{ij} is an n-square matrix. If $v_j(A)$ denotes the jth column of A, then T maps $A = (v_1(A), v_2(A), \ldots, v_n(A))$ into the matrix

$$\left(\sum_{j=1}^{n} T_{1j}v_j(A), \ldots, \sum_{j=1}^{n} T_{nj}v_j(A)\right).$$

Let $\rho(A)$ denote the rank of A. If T preserves rank, $T(x, 0, \ldots, 0) = (T_{11}x \ldots, T_{n1}x)$ has rank 1 for any non-zero vector x where 0 is the zero vector. We shall call m n-square matrices $A_1, \ldots, A_m$ *collinear* if, for every non-zero n-vector x,

$$\rho(A_1x, \ldots, A_mx) = 1.$$

LEMMA 1. *If $A_1, \ldots, A_m$ are collinear, there exist non-zero vectors $z_1, \ldots, z_n$* such that

(2) $$v_j(A_i) = k_{ij}z_j, \qquad i = 1, \ldots, m;\ j = 1, \ldots, n;$$

where the k_{ij} are scalars. Moreover, for each j, $k_{ij} \neq 0$ for some i.

Proof. Let e_j denote the unit vector with jth entry equal to 1. Then $A_ie_j = v_j(A_i)$. The lemma follows from the fact that $\rho(v_j(A_1), \ldots, v_j(A_m)) = 1$.

LEMMA 2. *If the matrices $A_1, \ldots, A_m$ are collinear, and z_1, z_β are linearly independent for some β* (cf. (2)), *then there exists a non-singular matrix A and scalars l_i, not all zero, such that*

(3) $$A_i = l_iA, \qquad i = 1, \ldots, m$$

Proof. The matrix $(A_1(e_1 + e_\beta), \ldots, A_m(e_1 + e_\beta)) = (k_{11}z_1 + k_{1\beta}z_\beta, \ldots, k_{m1}z_1 + k_{m\beta}z_\beta)$ has rank 1. For some s, $k_{s1} \neq 0$, by Lemma 1. The Grassmann products

$$(k_{s1}z_1 + k_{s\beta}z_\beta)\wedge (k_{i1}z_1 + k_{i\beta}z_\beta) = 0,$$

for $i = 1, \ldots, m$. Since $z_1\wedge z_\beta \neq 0$, it follows that $k_{s1}k_{i\beta} - k_{s\beta}k_{i1} = 0$, or

(4) $$k_{i\beta} = \frac{k_{s\beta}k_{i1}}{k_{s1}}, \qquad i = 1, \ldots, m.$$

Moreover, $k_{s\beta} \neq 0$ (otherwise all $k_{i\beta} = 0$); and (4) holds for all β such that z_1 and z_β are independent.

Suppose now that z_1 and z_γ are dependent; then z_β and z_γ are independent. By the preceding argument,

$$k_{i\gamma} = \frac{k_{s\gamma}k_{i\beta}}{k_{s\beta}} = \frac{k_{s\gamma}}{k_{s\beta}}\left(\frac{k_{s\beta}k_{i1}}{k_{s1}}\right) = \frac{k_{s\gamma}k_{i1}}{k_{s1}}, \qquad i = 1, \ldots, m.$$

Thus equations (4) hold for all $1 \leqslant \beta \leqslant n$. It follows that $A_i = l_iA_s$, $i = 1, \ldots, m$, where $l_i = k_{i1}/k_{s1}$. In particular $l_s = 1$.

The matrix A_s cannot be singular, for then $\rho(A_1x, \ldots, A_mx) = 0$ when x is an eigenvector of A_s corresponding to the eigenvalue 0.

An immediate consequence of Lemmas 1 and 2 is

LEMMA 3. *If the matrices $A_1, \ldots, A_n$ are all singular and collinear, then there exist scalars k_{ij} and a non-zero vector z such that $v_j(A_i) = k_{ij} z$, $i, j = 1, \ldots, n$.*

LEMMA 4. *Let T be a rank preserver on M_n. If some block $T_{\alpha\beta}$ in the representation (1) of T is non-singular, then there exist scalars c_{ij} such that*

$$T_{ij} = c_{ij} T_{\alpha\beta}; \qquad i, j = 1, \ldots, n. \tag{5}$$

Proof. First note that $T_{1\beta}, \ldots, T_{n\beta}$ are collinear. Since $T_{\alpha\beta}$ is non-singular, the vectors $z_1, \ldots, z_n$ of Lemma 1 are linearly independent. Hence $T_{i\beta} = c_{i\beta} T_{\alpha\beta}$, $i = 1, \ldots, n$.

Suppose $T_{\sigma\gamma}$ is also non-singular, $\gamma \neq \beta$. Then $T_{i\gamma} = l_{i\gamma} T_{\sigma\gamma}$, $i = 1, \ldots, n$. If $T_{\sigma\gamma}$ is not a multiple of $T_{\alpha\beta}$, choose a vector x so that $T_{\alpha\beta}x$ and $T_{\sigma\gamma}x$ are linearly independent; and let X be the matrix with $v_j(X) = x$ for $j = \beta, \gamma$, and $v_j(X) = 0$ for $j \neq \beta, \gamma$. Then $\rho(T(X)) = 1$. This implies that

$$(T_{i\beta}x + T_{i\gamma}x) \wedge (T_{t\beta}x + T_{t\gamma}x) = 0, \qquad i, t = 1, \ldots, n.$$

Since $T_{\alpha\beta}x \wedge T_{\sigma\gamma}x \neq 0$,

$$c_{i\beta} l_{t\gamma} - l_{i\gamma} c_{t\beta} = 0 \text{ for all } i, t. \tag{6}$$

Let Y be a matrix for which $v_\beta(Y)$ and $v_\gamma(Y)$ are independent and $v_j(Y) = 0$ for $j \neq \beta, \gamma$. Then $\rho(Y) = 2$, while $\rho(T(Y)) \leqslant 1$ by (6). This contradiction shows that $T_{\sigma\gamma}$ is a multiple of $T_{\alpha\beta}$, and (5) holds for $T_{i\gamma}$, $i = 1, \ldots, n$.

Finally suppose that $T_{i\gamma}$ is singular for some γ and all i. By Lemma 3 there exist scalars k_{ij} and a non-zero vector z such that $v_j(T_{i\gamma}) = k_{ij}z$. Thus $T_{i\gamma}x$ is a multiple of z for any vector x. Choose y so that $T_{\alpha\beta}y = z$, and choose x independent of y. For the matrix Y above with $v_\beta(Y) = y$ and $v_\gamma(Y) = x$, $\rho(Y) = 2$, while $\rho(T(Y)) \leqslant 1$. Hence this case cannot arise. This completes the proof of the lemma.

Not every rank preserver need have a non-singular block in its representation (1). For example, the transformation T_1, which maps each matrix onto its transpose, is linear and preserves rank. In its matrix, $T_{ij} = E_{ji}$, where E_{ij} is the matrix with 1 in the i, j position and 0's elsewhere. We have, however, the following result.

LEMMA 5. *Let T be a rank preserver. If every T_{ij} in the representation (1) is singular, then the $n^2 \times n^2$ matrix TT_1 has a non-singular block.*

Proof. By Lemma 3, there exist vectors $z_1, \ldots, z_n$ such that each column of T_{ij} is a multiple of z_j for $i, j = 1, \ldots, n$. For any matrix A, $v_i(T(A))$ is a linear combination of the columns of the T_{ij}. Hence the columns of $T(A)$ are linear combinations of the vectors z_j. This implies that $z_1, \ldots, z_n$ are linearly independent; for, if not, the columns of $T(A)$ would be linearly dependent, which is not the case when A is non-singular. Denote the blocks of TT_1 by W_{ij}, $i, j = 1, \ldots, n$. Then

$$W_{ij} = \sum_{k=1}^{n} T_{ik}E_{jk},$$

and $v_k(W_{ij}) = v_j(T_{ik})$. Thus the kth column of each W_{ij} is a multiple of z_k. Since TT_1 preserves rank, the blocks $W_{11}, \ldots, W_{n1}$ are collinear. The result then follows from Lemma 2.

THEOREM 1. *Let T be a* l.t. *of M_n into M_n. T is a rank preserver if and only if there exist non-singular matrices U and V such that either*:

(7) $$T(A) = UAV \quad \text{for all } A,$$

or

(8) $$T(A) = UA'V \quad \text{for all } A.$$

Proof. The sufficiency of the condition is obvious. For the necessity, if the representation (1) of T has a non-singular block $T_{\alpha\beta}$, choose $U = T_{\alpha\beta}$ and $V = (c_{ji})$ in Lemma 4. If T has no non-singular block, define the rank preserver T_2 by $T_2(A') = T(A)$. By Lemma 5, T_2 has a non-singular block; hence there exist U and V non-singular such that $T(A) = T_2(A') = UA'V$ for all A.

3. Determinant Preservers. We shall show that, if a linear transformation T of M_n maps unimodular matrices into unimodular matrices, it preserves determinant; that if it preserves determinant, it preserves rank; and determine the appropriate forms of U and V in Theorem 1.

LEMMA 6. *If the* l.t. *T maps U_n into U_n, then* $\det T(A) = \det A$ *for all matrices A.*

Proof. If $\det A \neq 0$, $\det [A/(\det A)^{1/n}] = 1$; hence $\det T(A) = (\det A). \det [T(A/(\det A)^{1/n})] = \det A$. Now $\det T(A)$ is a polynomial in the entries a_{ij} of A which is equal to $\det A$ for all non-singular A; thus this relation is an identity so that $\det T(A) = \det A$ for all A.

LEMMA 7. *If T preserves determinant, then T is non-singular and hence onto.*

Proof. Suppose $T(A) = 0$; then $\rho(A) < n$. There exist non-singular matrices M and N such that $MAN = I_r \dotplus 0_{n-r}$, where $r = \rho(A)$, I_r is the $r \times r$ unit matrix, 0_{n-r} is the $(n-r) \times (n-r)$ zero matrix and $\dotplus$ denotes the direct sum. For any X, $[\det (MAN + X)]/\det MN = \det (A + M^{-1}XN^{-1}) = \det T(A + M^{-1}XN^{-1}) = \det T(M^{-1}XN^{-1}) = \det M^{-1}XN^{-1}$. Hence $\det (MAN + X) = \det X$. Set $X = 0_r \dotplus I_{n-r}$. Then $\det (MAN + X) = 1$, while $\det X = 0$ unless $r = 0$. Hence $A = 0$.

LEMMA 8. *If T preserves determinant, then T preserves rank.*

Proof. Let A be an arbitrary matrix. There exist non-singular matrices M_1, N_1, M_2, N_2, such that $M_1AN_1 = Y_1 = I_r \dotplus 0_{n-r}$ and $M_2T(A)N_2 = Y_2 = I_s \dotplus 0_{n-s}$ where $r = \rho(A)$ and $s = \rho(T(A))$. Define a mapping ϕ of M_n by:

$$\phi(X) = M_2T(M_1^{-1}XN_1^{-1})N_2.$$

Then ϕ is linear with the property: $\det \phi(X) = k \det X$, where

$$k = \det (M_2 M_1^{-1} N_1^{-1} N_2);$$

also $\phi(Y_1) = Y_2$. Set $Y_3 = 0_r \dot{+} I_{n-r}$. For any scalar λ, $\det(\lambda Y_1 + Y_3) = \lambda^r$. On the other hand, $\det \phi(\lambda Y_1 + Y_3) = \det(\lambda Y_2 + \phi(Y_3)) = p(\lambda)$, a polynomial in λ of degrees $\leqslant s$. Since $p(\lambda) \equiv k\lambda^r$, $k \neq 0$, identically in λ, it follows that $r \leqslant s$, and $\rho(A) \leqslant \rho(T(A))$.

By Lemma 7, T^{-1} exists; moreover, since T preserves determinant, $\det B = \det (TT^{-1}(B)) = \det T^{-1}(B)$ for all B in M_n. Thus T^{-1} preserves determinant, and $\rho(T(A)) \leqslant \rho(T^{-1}T(A)) = \rho(A)$. Therefore $\rho(A) = \rho(T(A))$.

THEOREM 2. *Let T be a* l.t. *of M_n. The following conditions are equivalent*:

(i) *T maps U_n into U_n.*

(ii) *T preserves determinant.*

(iii) *There exist unimodular matrices U and V such that either* (7) *or* (8) *holds.*

Proof. Lemma 6 gives (i) $\leftrightarrow$ (ii); (iii) $\rightarrow$ (ii) is obvious. If T preserves determinant, then by Lemma 8 and Theorem 1, there exist non-singular matrices U_1 and V_1 such that $T(A) = U_1 A V_1$ or $T(A) = U_1 A' V_1$. Since $\det T(I) = 1$, $\det U_1 V_1 = 1$. Choose $U = U_1/(\det U_1)^{1/n}$ and $V = V_1/(\det V_1)^{1/n}$. Thus (ii) $\rightarrow$ (iii).

We shall show in the next section that preservation of determinant for Hermitian matrices is also equivalent to conditions (i)—(iii).

4. Eigenvalue Preservers.

LEMMA 9. *Let T be a* l.t. *of M_n. If* $\text{ev}(T(H)) = \text{ev}(H)$ *for all Hermitian matrices H, then* $\text{ev}(T(A)) = \text{ev}(A)$ *for all A in M_n.*

Proof. Note first that if H is Hermitian and satisfies the given condition, then $\text{tr}\{[T(H)]^m\} = tr\{H^m\}$ for $m = 1, 2, \ldots$, where $tr(X)$ denotes the trace of X. For any matrix A there exist Hermitian matrices K, L such that $A = K + iL$. For real α, $K + \alpha L$ is Hermitian and

$$tr\{[T(K + \alpha L)]^m\} = tr\{(K + \alpha L)^m\}. \tag{9}$$

For each m, equation (9) is a polynomial equation in α of degree $\leqslant m$ satisfied by all real α. Hence (9) is satisfied by all complex α, and in particular by $\alpha = i$. If the eigenvalues of A and $T(A)$ are λ_j and μ_j, respectively, $j = 1, \ldots, n$, then

$$\sum_{j=1}^{n} \lambda_j^m = \sum_{j=1}^{n} \mu_j^m \qquad m = 1, 2, \ldots.$$

It follows that the corresponding elementary symmetric functions of the λ_j and the μ_j are equal, and that $\text{ev}(T(A)) = \text{ev}(A)$.

LEMMA 10. *If* $\mathrm{ev}(T(A)) = \mathrm{ev}(A)$ *for all* $A \in M_n$, *then* $T(I) = I$, *where* I *is the unit matrix of order* n.

Proof. T preserves determinant. Hence, for λ complex and $A \in M_n$, $\det(\lambda I - A) = \det(\lambda T(I) - T(A)) = \det(\lambda I - CT(A))$, where $C = (T(I))^{-1}$. Thus $\mathrm{ev}(T(A)) = \mathrm{ev}(A) = \mathrm{ev}(CT(A))$. Since T is non-singular by Lemma 7, $T(A)$ ranges over all of M_n as A does. Hence $\mathrm{ev}(A) = \mathrm{ev}(CA)$ for all A. Choose U unitary so that $CU = H$, where H is positive definite Hermitian. Then $\mathrm{ev}(U) = \mathrm{ev}(CU) = \mathrm{ev}(H)$, so that U has positive eigenvalues. Hence $U = 1$ and $C = H$. Since the eigenvalues of $C = (T(I))^{-1}$ are all 1, $C = I$, $C^{-1} = I$, and $T(I) = I$.

From Lemmas 9 and 10 and Theorem 2 we obtain

THEOREM 3. *Let* T *be a* l.t. *of* M_n. *The following conditions are equivalent*:

(i) T *preserves eigenvalues for all Hermitian matrices in* M_n.

(ii) T *preserves eigenvalues for all matrices in* M_n.

(iii) *There exists a unimodular matrix* U *such that either* $T(A) = UAU^{-1}$ *for all* $A \in M$ *or* $T(A) = UA'U^{-1}$ *for all* $A \in M_n$.

THEOREM 4. *Let* T *be a* l.t. *of* M_n. *If* $\mathrm{ev}(T(H)) = \mathrm{ev}(H)$ *and* $T(H)$ *is Hermitian for all Hermitian* H *in* M_n, *then the matrix* U *in Theorem* 3 (iii) is *unitary.*

Proof. $T(H) = (T(H))^*$ implies $UHU^{-1} = U^{-1*}HU^*$ and $U^*UH = HU^*U$. for all Hermitian H. It follows easily that $U^*U = I$.

THEOREM 5. *Let* T *be a* l.t. *of* M_n. *Then* T *preserves determinant if and only if it preserves determinant for Hermitian matrices.*

Proof. Define $\phi(A) = CT(A)$, where $C = (T(I))^{-1}$. If T preserves determinant for Hermitian H, then $\det(\lambda I - H) = \det(\lambda T(I) - T(H)) = \det(\lambda I - \phi(H))$ for all real λ. Hence $\mathrm{ev}(\phi(H)) = \mathrm{ev}(H)$, and by Lemma 9, $\mathrm{ev}(\phi(A)) = \mathrm{ev}(A)$ for all A. Thus $\det A = \det \phi(A) = \det \mathrm{T}(A)$ for all A.

Professor N. Jacobson communicated to us the following information while this paper was in press: Theorem 1 was obtained by L. K. Hua (Science Reports of the National Tsing Hua University, Ser. A, *5* (1948) pp. 150–81) and in more general form by H. Jacob (Amer. J. Math., *77* (1955) pp. 177–89). In both these papers T is assumed non-singular; actually our proof of Theorem 1 requires only that $T(R_i) \subseteq R_i$, for $i = 1, 2, n$ without the assumption that T be non-singular. Also Dieudonné (Archiv. d. Math., *1* (1948) pp. 282–7) shows that if T preserves the cone $\det A = 0$ and T is non-singular then T has the form indicated in Theorem 2 (iii). Again, our result does not require the assumption that T be non-singular: this follows if T preserves all determinants (Lemma 7).

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