BULLETIN **13** (3), 1970

ON CARLEMAN INTEGRAL OPERATORS

BY
CHARLES G. COSTLEY

Integral operators on the Hilbert function space $L_2(a, b)$

(1) $$Kf = \int_a^b K(x, y)f(y)\,dy \quad \text{for all } f \in L_2(a, b)$$

with the property

(2) $$\int_a^b |K(x,y)|^2\,dy < \infty \quad \text{for a.a.x}$$

were originally defined by T. Carleman [4]. Here he imposed on the kernel the conditions of measurability and hermiticity,

(3) $$\lim_{x' \to x} \int_a^b |K(x', y) - K(x, y)|^2\,dy = 0$$

for all x with the exception of a countable set with a finite number of limit points and

(4) $$\int_{J_\delta} \int_a^b |K(x, y)|^2\,dy\,dx < \infty \quad \text{for every } \delta > 0$$

where J_δ denotes the interval $[a, b]$ with the exception of subintervals $|x-\xi_v| < \delta$; here ξ_v represents a finite set of points for which (3) fails to hold.

In [5] it is seen that the essential properties of the operator (1) remains valid if we delete (3) and (4) above.

In recent years many extensions and representation problems associated with these Carleman operators have been made ([1], [8]).

However, there exists a class of kernels wider than the classes considered in these works, also introduced by Carleman [4, pp. 137–138] and to which many results can be extended.

This note is concerned with such extensions. We call a kernel $K(x, y)$, of Carleman type if it is measurable, symmetric, and has associated with it a linear operator L_x satisfying the following conditions [1, pp. 137–138]:

(i) $L_x(\xi, K(x, y))$ is in L_2 with respect to y (ξ a parameter).
For approximating kernels $K_\delta(x, y)$,

(5)
(ii) $\lim_{\delta\to 0} L_x(\xi, K_\delta(x, y)) = L_x(\xi, K(x, y))$ and $L_x(\zeta, K_\delta(x, y)) < \gamma(\zeta, y)$
(iii) $\lim_{v\to\infty} L_x(\xi, f_v(x)) = L_x(\xi, f(x))$ if $f_v \in L_2$ and if f_v converges weakly to f.
(iv) $L_x(\xi, K_\delta(x, y)Q(y))\,dy = L_x(\xi, \int_a^b K_\delta(x, y)Q(y))\,dy$ for all $Q(x)$ in L_2.

Received by the editors December 5, 1969.

The linear operator $L_x(\xi, f(x))$ is called closed if $L_x(\xi, f(x)) = 0$ implies $f(x) = 0$.

The equation $\int_a^b L(\xi, K(x,y))Q(y)\,dy = 0$ is called closed if it has no nonzero solutions in L_2.

In this section we review briefly the Carleman development [4, Chap. 1, 2] for kernels satisfying (1)–(4).

Consider

$$Q(x) - \lambda \int_a^t K(x,y)Q(y)\,dy = f(x) \tag{6}$$

with $K(x,y)$ satisfying (2), (3), and (4).

Define approximating kernels.

$$\begin{aligned} K_\delta(x,y) &= 0, \qquad |x-\xi_v| < \delta, \quad v = 1, 2, \ldots, n \\ & \qquad\qquad |y-\xi_v| < \delta \\ &= K(x,y) \quad \text{otherwise.} \end{aligned}$$

For the kernels $K(x,y)$ and nonreal values of λ, the inhomogeneous equation

$$Q(x) - \lambda \int_a^b K_\delta(x,y)Q(x)\,dy = f(x) \tag{7}$$

has a solution $Q_\delta(x)$ satisfying

$$\int_a^b |Q_\delta(x)|^2\,dx \le \frac{|\lambda|^2}{(B)^2} \int_a^b |f(x)|^2\,dx, \qquad B = \operatorname{Im} \lambda. \qquad [4, \text{p. } 53] \tag{8}$$

Consequently,

$$|Q_\delta(x)| < |f(x)| + \frac{|\lambda|^2}{(B)} \left\{ \int_a^b K(x,y)^2\,dx \right\}^{1/2} \left\{ \int_a^b |f(x)|^2\,dx \right\}^{1/2}. \tag{9}$$

The second member of (8) being independent of δ, there exists a sequence of numbers δ_v such that

$$\lim_{v\to\infty} Q_{\delta_v} = Q(x) \in L_2 \quad \text{for } x \neq \xi_v, \quad v = 1, \ldots, n.$$

The existence of a non-null solution of (7) is established with the aid of the following lemmas of M. F. Riesz [6]:

LEMMA 1. *From each sequence* $\{Q_{\delta_v}\}$ *satisfying* (8), *one can always extract a weakly convergent subsequence.*

LEMMA 2 (see also [4, p. 132]). *If* $Q_v(x)$ *converges weakly towards* $Q(x)$, *then*

$$\overline{\lim_{v\to\infty}} \int_a^b Q_v(x)^2\,dx \ge \int_a^b |Q(x)|^2\,dx$$

$$\lim_{v\to\infty} \int_a^b Q_v(x)g(x)\,dx = \int_a^b Q(x)g(x)\,dx, \quad g(x) \in L_2.$$

https://doi.org/10.4153/CMB-1970-066-3 Published online by Cambridge University Press

LEMMA 3. *If $Q_v(x)$ converges weakly to $f(x)$ and $\psi(x)$ converges strongly to $\psi(x)$ then*

$$\lim_{v\to\infty} \int_a^b Q_v(x)\psi_v(x)\,dx = \int_a^b Q(x)\psi(x)\,dx. \tag{10}$$

LEMMA 4. *If $Q_v(x)$ converges weakly to $Q(x)$ and converges in the ordinary sense to $W(x)$ then $Q(x) = W(x)$* a.e.

If for a nonreal value of λ, the homogeneous equation

$$Q(x) - \lambda \int_a^b K(x, y)Q(x)\,dy = 0 \tag{11}$$

admits no nonzero solution in L_2, let T be the necessarily unique solution of (6). With the aid of Lemma 1 we have [4, p. 57]

$$\int_a^b T(f_1)f_2\,dt = \int_a^b T(f_2)f_1\,dt \tag{12}$$

for arbitrary functions f_1 and f_2 in L_2.

For these kernels it is also shown that either all the characteristic values are real or every nonreal λ is a characteristic value.

We associate with equations (6) and (11) the operator equations,

$$L_x(\xi, Q(x)) - \lambda \int_a^b L_x(\xi, K(x, y))Q(y)\,dy = L_x(\xi, f(x))$$

and

$$L_x(\xi, Q(x)) - \lambda \int_a^b L_x(\xi, K(x, y))Q(y)\,dy = 0 \tag{14}$$

so that

$$|L_x(\xi, Q_\delta(x))| \le |\lambda| \left\{ \int_a^b |L_x(\xi, K_\delta(x, y))|^2\,dy \right\}^{1/2} \left\{ \int_a^b |Q_\delta(x)|^2\,dx \right\}^{1/2} + |L_x(\xi, f(x))| \tag{15}$$

and

$$|L_x(\xi, Q_\delta(x))| \le \frac{|\lambda|^2}{|B|} \left\{ \int_a^b \gamma(\xi, y)\,dy \right\}^{1/2} \left\{ \int_a^b |f(x)|^2\,dx \right\}^{1/2} + |L_x(\xi, f(x))| \tag{16}$$

An argument similar to that in [4] shows that there exists a subsequence $\{Q_{\delta_v}\}$ converging to $Q(x)$ and $Q(x)$ is a.e. a solution of (13).

THEOREM 1 (see [4, p. 55]). *Suppose the operator L_x satisfies* (5) *with ξ in some perfect set P and*

$$\int_a^b |L_x(\xi_1, K_\delta(x, y)) - L_x(\xi_2, K_\delta(x, y))|^2\,dy \le \sigma(\xi_1, \xi_2) \tag{17}$$

where $\sigma(\xi_1, \xi_2) \to 0$ as $\xi_1 - \xi_2 \to 0$, then the solution $Q(x)$ of (14) *asserted subsequent to* (16) *is such that $L_x(\xi, Q(x)-f(x))$ is a continuous function in ξ and is an analytic function of λ for all nonreal λ.*

Proof. From (5(i))

$$|L_x(\xi_1, K_\delta(x, y)) - L_x(\xi_2, K_\delta(x, y))|^2 < |\gamma(\xi_1, y) + \gamma(\xi_2, y)|^2 \tag{18}$$

where the latter expression is in L.

In view of (17) we have

$$\int_a^b |L_x(\xi_1, K(x, y)) - L_x(\xi_2, K(x, y))|^2\, dy \le \sigma(\xi_1, \xi_2).$$

With the aid of Schwartz's inequality, from (14) we get

$$\begin{aligned} &|L_x(\xi_1, Q(x)-f(x)) - L_x(\xi_2, Q(x)-f(x))|^2 \\ &\qquad = |\lambda|^2 \left| \int_a^b [L_x(\xi_1, K(x, y)) - L_x(\xi_2, K(x, y))]\, Q(y)\, dy \right|^2 \\ &\qquad \le |\lambda|^2 \int_a^b |Q(y)|^2\, dy \int_a^b |L_x(\xi_1, K(x, y)) - L_x(\xi_2, K(x, y))|^2\, dy. \end{aligned}$$

With the aid of (8) we have

$$\begin{aligned} &|L_x(\xi_1, Q(x)-f(x)) - L_x(\xi_2, Q(x)-f(x))|^2 \\ &\qquad \le \frac{|\lambda|^4}{B^2} \int_a^b |f(x)|^2\, dx \cdot \sigma(\xi_1, \xi_2),\ \xi_1, \xi_2 \text{ in } P. \end{aligned}$$

Therefore $L_x(\xi_1, Q(x)-f(x))$ is a continuous function of ξ. For Q_δ satisfying (7) with the aid of the operator L_x we have

$$\begin{aligned} |L_x(\xi, Q_\delta(x))| \le |\lambda| \left[\int_a^b |L_x(\xi, K_{\delta,r}(x, y))|^2\, dy\right]^{1/2} \left[\int_a^b |Q_\delta(x)|^2\, dx\right]^{1/2} \\ + |L_x(\xi, f(x))| \end{aligned}$$

From (5(i)) and (8) for Q_δ we have

$$|L_x(\xi, Q_\delta(x))| \le \frac{|\lambda|^2}{B^2} \int_b^a |f(x)|^2\, dx \int_b^a \gamma^2(\xi, y)\, dy + |L_x(\xi, f(x))|$$

and

$$|L_x(\xi, Q_\delta(x) - f(x))|^2 \le \frac{|\lambda|^4}{B^2} \int_a^b |f(x)|^2\, dx \int_a^b \gamma^2(\xi, y)\, dy.$$

We also have

$$|L_x(\xi_1, Q_\delta(x)-f(x)) - L_x(\xi_2, Q_\delta(x)-f(x))|^2 \le \frac{|\lambda|^4}{B^2} \int_a^b |f(x)|^2\, dx \cdot \sigma(\xi_1, \xi_2).$$

In view of (15) and the above inequalities, applying Vitali's theorem as in [4, p. 55], it follows that $L_x(\xi, Q(x)-f(x))$ is analytic in λ.

https://doi.org/10.4153/CMB-1970-066-3 Published online by Cambridge University Press

THEOREM 2. *Suppose*

(19) $$\overline{L_x(\xi, Q(x))} = L(\xi, \overline{Q(x)})$$

and

(20) $L_x(\xi, Q(x))$ *is real for* $Q(x)$ *real.*

Then either all values λ *for which the homogeneous equation* (14) *has nonzero solutions are real or for every nonreal value* λ *there exists no nonzero* L_2 *solutions.*

Proof. Let λ_0 be a complex value for which (14) has only zero solutions and λ^* another value λ for which there exists a nonzero solution $Q(x)$ of (14).

Then for such λ, and $Q(x)$, with the aid of (19) we have

$$L_x(\xi, Q(x)) - \lambda_0 \int_a^b L_x(\xi, K(x, y))Q(y)\, dy = L_x(\xi,(1-\lambda_0/\lambda)Q).$$

From (19) it follows that

$$L_x(\xi, \overline{Q(x)}) - \lambda_0 \int_a^b L_x(\xi, K(x, y))\, \overline{Q(y)}\, dy = L(\xi, 1-\lambda_0/\bar{\lambda}^*)\bar{Q}.$$

Applying the equation analogous (12), i.e.

$$\int_a^b T(f_2)f_1\, dx = \int_a^b T(f_1)f_2\, dx, \quad \text{we have}$$

$$(1-\lambda_0/\lambda^*) \int_a^b Q\bar{Q}\, dx = (1-\lambda_0/\bar{\lambda}^*) \int_a^b Q\bar{Q}\, dx.$$

Thus $\lambda^* = \bar{\lambda}^*$, contrary to hypothesis.

THEOREM 3. *If* L_x *is closed then the solutions* $Q_v(x)$, $v=1, 2, \ldots, n$, *corresponding to distinct* λ_v, $v=1, 2, \ldots, n$, *are linearly independent.*

Proof. If untrue, we have

(21) $$C_1Q_1+C_2Q_2+\cdots+C_nQ_n = 0 \quad \text{and} \quad \sum_{v=1}^{n} C_n^2 \neq 0$$

where $Q_1, \ldots, Q_n$ are the L_2 solutions corresponding to distinct $\lambda_1, \ldots, \lambda_n$. Multiply (21) by $L_x(\xi, K(x, y))$. Integrating, with the aid of the equation

(22) $$L_x(\xi, Q_v(x)) = \lambda_v \int_a^b L_x(\xi, K(x, y))\, Q_v(y)\, dy$$

we get

$$\frac{C_1}{\lambda_1} L_x(\xi, Q_1) + \frac{C_2}{\lambda_2} L_x(\xi, Q_2) + \cdots + \frac{C_n}{\lambda_n} L_x(\xi, Q_n)$$
$$= L_x\left(\xi, \left(\frac{C_1}{\lambda_1} Q_1 + \frac{C_2}{\lambda_2} Q_2 + \cdots + \frac{C_n}{\lambda_n} Q_n\right)\right) = 0.$$

https://doi.org/10.4153/CMB-1970-066-3 Published online by Cambridge University Press

Therefore

$$\frac{C_1}{\lambda_1}Q_1+\cdots+\frac{C_n}{\lambda_n}Q_n = 0.$$

Successively repeating the process using (22) we arrive at a system of equations

$$\begin{aligned}
&C_1Q_1+C_2Q_2+\cdots+C_nQ_n = 0\\
&\frac{C_1Q_1}{\lambda_1}+\frac{C_2Q_2}{\lambda_2}+\cdots+\frac{C_nQ_n}{\lambda_n} = 0\\
&\vdots\\
&\frac{C_1Q_1}{\lambda_1^{n-1}}+\frac{C_2Q_2}{\lambda_2^{n-1}}+\cdots+\frac{C_nQ_n}{\lambda_n^{n-1}} = 0
\end{aligned}$$

Since the determinant

$$\begin{vmatrix} 1 & & & 1 \\ \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & & \\ \cdot & \cdot & & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \frac{1}{\lambda_1^{n-1}} & & & \frac{1}{\lambda_n^{n-1}} \end{vmatrix} \neq 0$$

it follows that $C_1=C_2=\cdots=C_n=0$, contrary to hypothesis.

The method of [4, p. 58] shows also that if the operator L is closed the number of linearly independent solutions of (14) is the same for all nonreal λ.

REMARK 1. Results relating to range of the solution $Q(x)$, and existence of an operator T satisfying (12) can be established for the equations with kernels considered here with method used in [4].

REMARK 2. If L_x is closed and (5(iv)) holds for $K(x, y)$, then every solution $Q(x)$ of

$$\int_a^b L_x(\xi, K(x, y))\, Q(y)\, dy = L_x(\xi, f(x)), \quad f \text{ in } L_2$$

is a solution of the first kind equation

$$\int_a^b K(x, y)Q(y)\, dy = f(x).$$

REFERENCES

1. N. I. Achiezer, Upschi Math. Nauk. **5** (1947), (21) (93).

2. M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, American Math. Soc. Colloq. Publ., Vol. **XV**, New York, 1932.

3. V. B. Korotkov, *Integral operators with Carleman kernels*, Differencial'nye Uravnenija **2**, (1965), 252–265. (Math. Reviews Vol. **32** (1966), Part 2).

4. T. Carleman, *Sur les équations intégrales singulières à noyau reél et symétrique*, Uppsala 1923.

5. ———, *La théorie des équations intégrales singulières et ses applications*, Ann. Inst. H. Poincaré, 1931.

6. M. F. Riesz, *Uber systeme integrierbarer Funktionen*, Math. Ann., 1910.

7. G. I. Targonski, *Seminar on functional operators and equations.* Lecture Notes in Mathematics No. **33**, Springer-Verlag, Berlin, 1962.

8. B. Misra, D. Speiser, and Gy. Targonski, Helv, Phys. Acta **36** (1963), 963–980.

McGill University,
Montreal, Quebec

https://doi.org/10.4153/CMB-1970-066-3 Published online by Cambridge University Press