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ω-limit sets for maps of the interval

LOUIS BLOCK AND ETHAN M. COVEN

Department of Mathematics, University of Florida, Gainesville, FL 32611, *USA*;
Department of Mathematics, Wesleyan University, Middletown, CT 06457, *USA*

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Abstract. Let f denote a continuous map of a compact interval to itself, $P(f)$ the set of periodic points of f, and $\Lambda(f)$ the set of ω-limit points of f. Sarkovskiĭ has shown that $\Lambda(f)$ is closed, and hence $\overline{P(f)} \subseteq \Lambda(f)$, and Nitecki has shown that if f is piecewise monotone, then $\Lambda(f) = \overline{P(f)}$. We prove that if $x \in \Lambda(f) - \overline{P(f)}$, then the set of ω-limit points of x is an infinite minimal set. This result provides the inspiration for the construction of a map f for which $\Lambda(f) \neq \overline{P(f)}$.

1. *Introduction*

Let f denote a continuous map of a compact interval I to itself, $P(f)$ the set of periodic points of f, and $\Lambda(f) = \bigcup_{x \in I} \omega(x)$, where $\omega(x)$ denotes the set of ω-limit points of x. A. N. Sarkovskiĭ [**S1**] has shown that $\Lambda(f)$ is closed, and hence that $\overline{P(f)} \subseteq \Lambda(f)$. Additional results concerning ω-limit sets for maps of the interval have been obtained by Sarkovskiĭ and H. K. Kenzegulov [**SK**].

In this paper, we consider the following problem: Is $\Lambda(f) = \overline{P(f)}$?

Z. Nitecki [**N1**] has shown that if f is piecewise monotone, then $\Lambda(f) \subseteq \overline{P(f)}$, so that in this case $\Lambda(f) = \overline{P(f)}$.

We attack the problem by assuming that $\Lambda(f) \neq \overline{P(f)}$ and seeing what follows. We obtain the following result.

THEOREM. *If* $x \in \Lambda(f) - \overline{P(f)}$, *then* $\omega(x)$ *is an infinite minimal set.*

This theorem and more specific results obtained in the course of its proof provide the inspiration for the construction of an example where $\Lambda(f) \neq \overline{P(f)}$. In this example, there is a point $x \in \Lambda(f) - \overline{P(f)}$ such that $f|\omega(x)$ is topologically conjugate to the group translation known as the 'adding machine.'

This example, incidentally, has the following properties:

(1) $P(f)$ consists of exactly one periodic orbit of period 2^n ($n = 0, 1, 2, \ldots$).

(2) $f|\Omega(f)$ is not one-to-one, where $\Omega(f)$ denotes the set of non-wandering points of f.

Thus the assertion of Sarkovskiĭ [**S2**, theorem **B**], that if all periodic points have period a power of two, then $f|\Omega(f)$ is a homeomorphism, is false.

In the example above, the restriction of f to $\omega(x)$ is topologically conjugate to a group translation. We give a second example to show that this need not always be the case. In fact, this example shows that $f|\omega(x)$ need not be one-to-one when $x \in \Lambda(f) - \overline{P(f)}$.

Finally, we give a third example to show that the theorem does not hold when $\Lambda(f)$ is replaced by $\Omega(f)$, i.e. if $x \in \Omega(f) - \overline{P(f)}$, then $\omega(x)$ need not be a minimal set. This example also yields an easy construction of a map f having a point $x \in \Omega(f) - \overline{P(f)}$ with an infinite orbit. An example of this phenomenon was first constructed by the second author and Nitecki [CN].

This work was done while the first author was J. H. Van Vleck Visiting Professor of Mathematics at Wesleyan University.

2. *The theorem and its proof*

We begin with some notation. Recall that f denotes a continuous map of a compact interval I to itself.

Let $x \in I$. Let orb (x) denote the orbit of x. By an R-neighbourhood of x we will mean an interval $[x, x+\varepsilon]$ where $\varepsilon > 0$, and by an L-neighbourhood of x we will mean an interval $[x-\varepsilon, x]$ where $\varepsilon > 0$. If p is a fixed point of f and S is R or L, we let $W^u(p, f, S)$ denote the set of $y \in I$ such that for any S-neighbourhood V of p, $y \in f^k(V)$ for some positive integer k.

LEMMA 1. *If $x \in \Lambda(f)$ and x has a finite orbit, then $x \in \overline{P(f)}$.*

Proof. Since $\Lambda(f) = \Lambda(f^n)$ and $P(f) = P(f^n)$, we may assume that $f(x)$ is a fixed point p of f. We may also assume that $x \neq p$, otherwise there is nothing to prove. Finally, without loss of generality, we may assume that $x < p$. Let $y \in I$ with $x \in \omega(y)$.

Case 1. There is a strictly increasing sequence (n_i) of positive integers such that the sequence $(f^{n_i}(y))$ is strictly decreasing and approaches x.

Let i be a positive integer. Then $f^{n_{i+1}}(y) < f^{n_i}(y)$, and hence $f^{n_{i+1}-n_i}(f^{n_i}(y)) < f^{n_i}(y)$. Since $f^{n_{i+1}-n_i}(x) = p > x$, this implies that there is a periodic point in $(x, f^{n_i}(y))$. It follows that $x \in \overline{P(f)}$.

Case 2. There is a strictly increasing sequence (n_i) of positive integers such that the sequence $(f^{n_i}(y))$ is strictly increasing and approaches x.

There is a side S of p such that a subsequence of $(f^{n_i+1}(y))$ approaches p from this side. It follows that $x \in W^u(p, f, S)$ and thus, since $W^u(p, f, S)$ is connected, that $[x, p] \subseteq W^u(p, f, S)$.

We claim that some element of the orbit of y is in $W^u(p, f, S)$. Since the claim follows immediately if $S = L$, we will assume in proving the claim that $S = R$. Also, if either x or p is an interior point of $W^u(p, f, R)$, then the claim follows. Thus to prove the claim, we may assume that $W^u(p, f, R) = [x, p]$.

Since f is uniformly continuous, there is a $\delta > 0$ such that if z_1 and z_2 are points in I with $|z_1 - z_2| < \delta$, then $|f(z_1) - f(z_2)| < p - x$. Let $v \in I$ with $p < v < p + \delta$. Since $v \notin W^u(p, f, R)$, there is a $\gamma > 0$ such that $v \notin \bigcup_{n=0}^{\infty} f^n([p, p+\gamma])$. Now, for some positive integer j, $f^j(y) \in (p, p+\gamma)$. On the other hand, for some integer $m > j$, $f^m(y) < x$. This implies that for some integer k with $j < k < m$, $f^k(y) \in (x, p)$. This establishes the claim.

Since $W^u(p, f, S)$ is invariant under f, it follows from the claim that x is in the interior of $W^u(p, f, S)$. Let K be an L-neighbourhood of x with $K \subseteq W^u(p, f, S)$.

Then $f(K)$ contains an S-neighbourhood of p, and hence $f^j(K) \supseteq K$ for some positive integer j. Thus $x \in \overline{P(f)}$. □

We now adopt some notation and make a standing hypothesis for the next three lemmas. Let J denote a component of $I - \overline{P(f)}$, c the left endpoint of J, and d the right endpoint of J. We also assume that there is a point $z \in J$ and a positive integer m such that $f^m(z) \in J$ and $f^m(z) > z$.

LEMMA 2 ([**C**], [**X**]). *If $y \in J$ and $f^n(y) \in J$ for some positive integer n, then $f^n(y) > y$.*

Proof. Let $g = f^m$, and observe that the interval $[z, g(z)]$ contains no periodic points of g. It follows easily by induction that $g^k(z) > z$ for every positive integer k. In particular, $f^{mn}(z) > z$. If $f^n(y) < y$, then a similar argument would show that $f^{mn}(y) < y$. Then f^{mn} would have a fixed point between y and z, a contradiction. Thus $f^n(y) > y$. □

LEMMA 3 ([**C**]). *If $x \in (c, d)$ is non-wandering and has an infinite orbit, then the sets H, $f(H)$, $f^2(H), \ldots$ are pairwise disjoint, where $H = [x, d]$.*

Proof. If the sets $f^k(H)$ are not pairwise disjoint, then $E = \bigcup_{k=0}^{\infty} f^k(H)$ has finitely many components. Since x has an infinite orbit, this implies that $f^N(x)$ is in the interior of E for some positive integer N. Since x is non-wandering, there are sequences of points $x_k \to x$ and positive integers $n_k \to \infty$ with $f^{n_k}(x_k) = x$ [**CN**, lemma 2.6]. Now, for k sufficiently large, $n_k > N$ and $f^N(x_k) \in E$. Thus $x = f^{n_k - N}(f^N(x_k)) = f^j(y)$ for some $y \in H$ and some positive integer j. Hence $f^j(v) < v$ and $f^j(v) \in J$ for some $v \in (x, d)$. This contradicts lemma 2. □

LEMMA 4. *If $x \in (c, d) \cap \Lambda(f)$, then $d \in \omega(x)$.*

Proof. Let z_f denote the greatest lower bound of $\{y \in \text{orb}\,(x): y > x\}$. It follows from the standing hypothesis that this set is non-empty. By lemma 3, $z_f \geq d$ and d is not an element of orb (x). Thus to prove the lemma, it suffices to show that $z_f = d$. We will assume that $z_f > d$ and obtain a contradiction.

There is a periodic point $p \in [d, z_f]$. Let n be the period of p, and let $g = f^n$. Note that since $\Lambda(f) = \Lambda(g)$ and $\overline{P(f)} = \overline{P(g)}$, $x \in \Lambda(g)$ and J is a component of $I - \overline{P(g)}$. It follows from lemma 2 and the fact that $x \in \Lambda(g)$ that the standing hypothesis holds for g.

For $y \in I$, let orb$'\,(y)$ denote the orbit of y under g and $\omega'(y)$ the set of ω-limit points of y under g. Let z_g denote the greatest lower bound of $\{y \in \text{orb}'\,(x): y > x\}$. Then $z_g \geq z_f$.

Let $W^u(x, g)$ denote the set of $y \in I$ such that for every neighbourhood V of x, $y \in g^k(V)$ for some positive integer k. Then $f(W^u(x, g)) \subseteq W^u(x, g)$ and by lemma 2, $(c, x) \cap W^u(x, g) = \emptyset$. Now $x \in \omega'(y)$ for some $y \in I$. Again by lemma 2, orb$'\,(y) \cap (x, d) = \emptyset$. Thus $[x, z_g] \subseteq W^u(x, g)$.

Let A be the component of $W^u(x, g)$ which contains x. Then A is an interval with left endpoint x, $p \in A$, and $g(A) \subseteq A$. Hence there is a positive integer $j \leq 2$ such that $g^j(x)$ is in the interior of A. Let K be a neighbourhood of $g^j(x)$ with $K \subseteq A$, and let V be a neighbourhood of x with $V \subseteq J$, $g(V) \cap V = \emptyset$, and $g^j(V) \subseteq K$.

For any $v \in V$ and any positive integer k, $g^k(v) \geq x$. It follows from this and lemma 3 that $x \notin \omega'(y)$, which is a contradiction. □

THEOREM. *If $x \in \Lambda(f) - \overline{P(f)}$, then $\omega(x)$ is an infinite minimal set.*

Proof. Let J be the component of $I - \overline{P(f)}$ which contains x, and let c and d denote the left and right endpoints of J. By [**CN**, lemma 2.7], x is not an endpoint of the ambient interval I; hence $x \in (c, d)$.

Since $x \in \Lambda(f)$, there is a point $z \in J$ and a positive integer m such that $f^m(z) \in J$. Without loss of generality we may assume that $f^m(z) > z$. By lemma 1, x has an infinite orbit, and by lemma 4, $d \in \omega(x)$. It follows from lemma 3 that d has an infinite orbit and that $\omega(x) = \omega(d)$. Since $\omega(x)$ is a closed invariant set, it follows that $\omega(x) = \overline{\text{orb}\,(d)}$. Thus to prove the theorem, it suffices to show that $\overline{\text{orb}\,(d)}$ is a minimal set.

A classical result of G. D. Birkhoff [**B**, p. 199] states that $\overline{\text{orb}\,(d)}$ is a minimal set if and only if d is almost periodic, i.e. for every neighbourhood V of d, the orbit of d returns to V with bounded gaps. We complete the proof by proving the stronger statement:

(*) *For every $\varepsilon > 0$, there is a positive integer n such that $f^{jn}(d) \in (d, d+\varepsilon)$ for every positive integer j.*

Let $\varepsilon > 0$. For some $y \in (c, d)$, $x \in \omega(y)$. It follows from lemmas 2 and 3 that $\text{orb}\,(y) \cap [x, d] = \emptyset$. Since $d \in \omega(x)$, $f^M(y) \in (d, d+\varepsilon)$ for some positive integer M. Also, since $x \in \omega(y)$, $f^K(y) \in (c, x)$ for some integer $K > M$.

Since $f^{K+i}(y) \in (c, x)$ for some positive integer i, it follows from lemmas 2 and 3 that $f^i([f^K(y), x])$ contains an interval $[d, d+\gamma]$, where $\gamma > 0$ and $d + \gamma < f^M(y)$. There is a periodic point $p \in (d, d+\gamma)$. Let n be the period of p.

Let j be any positive integer. We must show that $f^{jn}(d) \in (d, d+\varepsilon)$.

First, suppose that $f^{jn}(d) \geq d + \varepsilon$. Then some power of f maps $[d, p]$ to an interval containing $f^M(y)$. Hence some (other) power of f maps $[f^K(y), x]$ to an interval containing $f^K(y)$. This contradicts lemma 2.

Finally, suppose that $f^{jn}(d) \leq d$. By lemmas 2 and 3 and the continuity of the powers of f, $f^{jn}(d) \leq c$. Hence $f^{jn}([d, p]) \supseteq [c, p]$, and again some power of f maps $[f^K(y), x]$ to an interval containing $f^K(y)$, a contradiction.

This proves (*) and hence the theorem. □

3. *Examples*

Example 1. We construct a continuous map f of the interval with a point $x \in \Lambda(f) - \overline{P(f)}$. According to the theorem, $\omega(x)$ must be an infinite minimal set. In this example, the restriction of f to $\omega(x)$ is topologically conjugate to the 'adding machine', i.e. translation by $+1$ on the group of 2-adic integers. We briefly review the construction of the adding machine and its realization on the interval.

The 2-adic integers A is the topological group (based on the Cantor set) consisting of all formal expressions $\sum_{i=0}^{\infty} a_i 2^i$ where a_i is 0 or 1. We may think of the non-negative integers as a subset of A by associating with each non-negative integer its base 2 expansion. Addition in A is the extension of the usual addition on the non-negative integers to A, i.e. with carrying. The adding machine is the map $p: A \to A$ defined by $p(a) = a + 1$. It is well-known that A is a minimal set.

We may identify A with the Cantor middle third set C by identifying $\sum_{i=0}^{\infty} a_i 2^i$ with the real number $\sum_{i=0}^{\infty} 2a_i 3^{-i-1}$. Notice that

$$C \subseteq [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, \tfrac{25}{27}] \cup \cdots \cup \{1\}.$$

We define a map f on the set on the right as follows. Let f map $[0, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, \frac{25}{27}], \ldots$ by translation onto $[\frac{2}{3}, 1]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{27}, \frac{1}{9}], \ldots$ and let $f(1) = 0$. It can be easily verified that $f(C) = C$ and that $f|C$ is topologically conjugate to the adding machine.

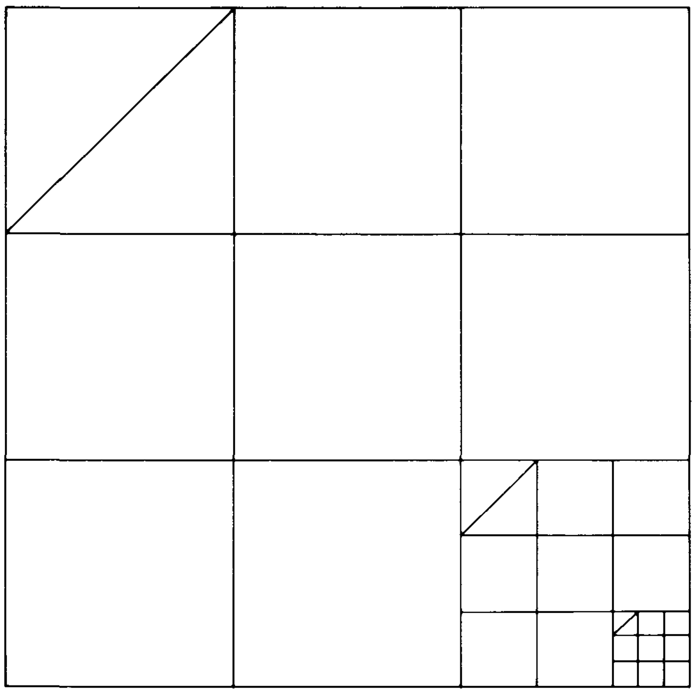

FIGURE 1

We remark that to this point our construction is the same as that used by J. P. Delahaye **[D]** to construct a map of the interval having exactly one periodic orbit of period 2^n $(n = 0, 1, 2, \ldots)$ and no other periodic orbits. Notice that no matter how we extend f to a map of an interval $I \supseteq [0, 1]$, $f|C$ will be topologically conjugate to the adding machine and $\omega(0)$ will be C.

We will extend f to the interval $[-1, 2]$ in such a way that:

(1) $-\frac{2}{3} \in \Lambda(f) - P(f)$.

(2) $f(-\frac{2}{3}) = f(0)$, and hence $\omega(-\frac{2}{3}) = C$.

To extend f to the interval $[-1, 2]$, it suffices to define f on the intervals $[-1, 0]$, $[1, 2]$ and $I_1 = [\frac{1}{3}, \frac{2}{3}]$, $I_2 = [\frac{7}{9}, \frac{8}{9}]$, $I_3 = [\frac{25}{27}, \frac{26}{27}], \ldots$.

First, we let $f(x) = 1 - x$ on the interval $[1, 2]$. Then f maps $[1, 2]$ linearly onto $[-1, 0]$, reversing order. Let $t_0 = 2$, $x_0 = \frac{5}{3}$ (the unique point in $[1, 2]$ satisfying $f(x_0) = -\frac{2}{3}$) and $s_0 = (\frac{2}{3})x_0 + (\frac{1}{3})t_0 = \frac{16}{9}$.

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Next, we define f inductively on the intervals $I_n = [l_n, r_n]$. f will have exactly one turning point in each I_n, the midpoint $t_n = 1 - 1/2 \cdot 3^{n-1}$, and will be linear on $[l_n, t_n]$ and $[t_n, r_n]$. Since f is already defined on l_n and r_n, it suffices to define f on t_n.

Let $f(t_1) = s_0$, and (having extended f to $[l_1, r_1]$) let x_1 be the unique point in $[l_1, t_1]$ satisfying $f(x_1) = x_0$. Let $s_1 = (\frac{2}{3})x_1 + (\frac{1}{3})t_1$. We repeat the procedure on I_2. Let $f(t_2) = s_1$, let $x_2 \in [l_2, t_2]$ satisfy $f(x_2) = x_1$, and let $s_2 = (\frac{2}{3})x_2 + (\frac{1}{3})t_2$.

For the inductive step, we use the auxiliary intervals $[l'_n, t'_n] = [1/3^n, 1/2 \cdot 3^{n-1}]$ $(n = 2, 3, \dots)$. Note that $f^{2^{n-1}-1}$ is already defined on $[l'_n, t'_n]$ - in fact $[l'_n, t'_n]$, $f([l'_n, t'_n]), \dots, f^{2^{n-1}-2}([l'_n, t'_n])$ all lie in $[0, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup \cdots$ - and that $f^{2^{n-1}-1}$ maps $[l'_n, t'_n]$ by translation onto $[l_n, t_n]$.

Assuming that $f(t_n)$, x_n, and s_n have already been defined, let x'_n and s'_n be the unique points in $[l'_n, t'_n]$ such that $f^{2^{n-1}-1}(x'_n) = x_n$ and $f^{2^{n-1}-1}(s'_n) = s_n$. Let $f(t_{n+1}) = s'_n$, let x_{n+1} be the unique point in $[l_{n+1}, t_{n+1}]$ satisfying $f(x_{n+1}) = x'_n$, and let $s_{n+1} = (\frac{2}{3})x_{n+1} + (\frac{1}{3})t_{n+1}$.

The construction is depicted in figure 2.

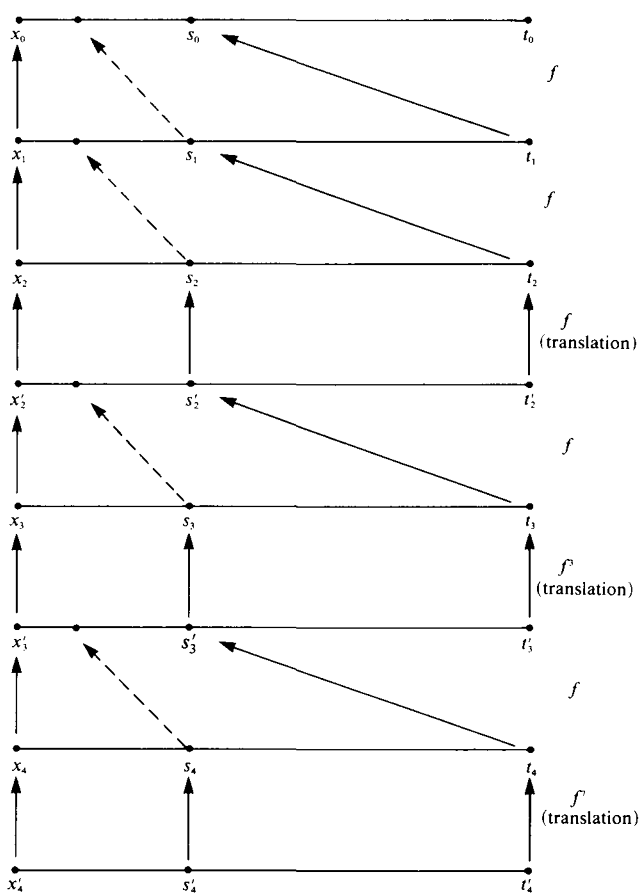

FIGURE 2

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The last step in the construction is to define f on the interval $[-1, 0]$. Let f have the constant value $\frac{2}{3}$ on the interval $[-\frac{2}{3}, 0]$ and let $f(-1)$ be the unique point in $[\frac{1}{3}, \frac{2}{3}] \cap f^{-1}(\frac{1}{3})$. (In case anyone cares, $f(-1) = \frac{55}{84}$.) Finally, let f be linear on the interval $[-1, -\frac{2}{3}]$.

The graph of f is displayed in figure 3.

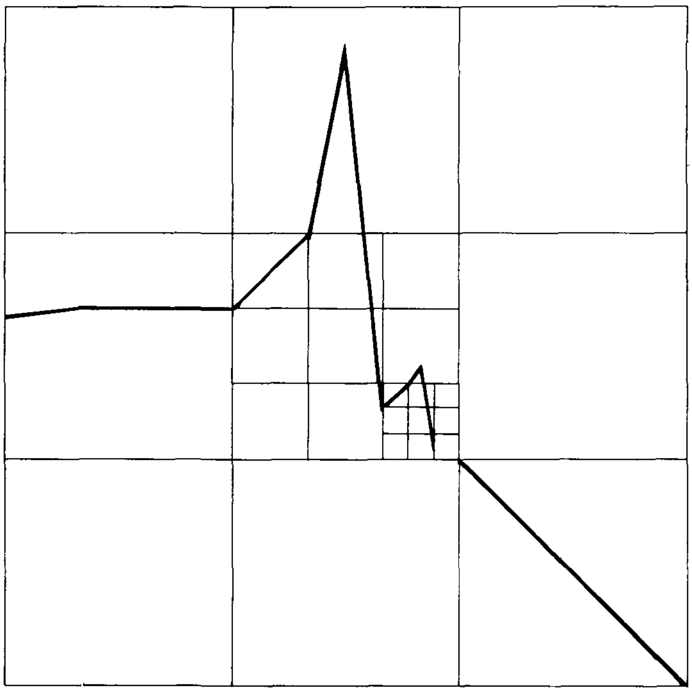

FIGURE 3

We will verify that $-\frac{2}{3} \in \Lambda(f) - \overline{P(f)}$. Consider the Cantor middle third set constructed on the interval $J = [-1, -\frac{2}{3}]$. By a deleted interval in J we will mean a component of the complement of this set in J. We make the following observations.

(1) If K is a deleted interval in J of length $1/3^n$, then $K \subseteq (-1, -\frac{2}{3} - 1/3^{n+1})$.

(2) If $x \in J$ is not in a deleted interval, then $f^k(x) \geq 0$ for every positive integer k.

(3) If $x \in K$, for K as in (1), then $f^k(x) \geq -\frac{2}{3} - 1/3^{n+2}$ for every positive integer k.

Note that (1) is obvious and that (2) follows immediately from the fact that f^2 maps J linearly onto the interval $[\frac{2}{9}, \frac{1}{3}]$. (3) also follows from the construction of f, although not quite immediately. For example, suppose that $x \in K$, where K is the deleted middle third of J, i.e. $K = (-\frac{8}{9}, -\frac{7}{9})$. Then $f^2(x) \in (\frac{7}{27}, \frac{8}{27})$, the deleted middle third of $[\frac{2}{9}, \frac{1}{3}]$, and hence $f^3(x) \in (\frac{25}{27}, \frac{26}{27})$. From this interval different orbits may take different paths to return to J. However, any such path must pass through I_1, and the only way a point can move from a deleted interval in $[0, 1]$ of length $1/3^n$ to one of larger length is to pass through I_n and then through $(l'_{n-1}, s'_{n-1}]$. Thus, in order for

x to return to J, we must have for some positive integer k

$$f^k(x) \in (\tfrac{25}{27}, \tfrac{26}{27})$$
$$f^{k+1}(x) \in (\tfrac{1}{9}, s_2']$$
$$f^{k+2}(x) \in (\tfrac{7}{9}, s_2]$$
$$f^{k+3}(x) \in (\tfrac{1}{3}, (\tfrac{2}{3})x_1 + (\tfrac{1}{3})s_1]$$
$$f^{k+4}(x) \in (1, (\tfrac{8}{9})x_0 + (\tfrac{1}{9})s_0] = (1, \tfrac{5}{3} + \tfrac{1}{81}]$$

and finally

$$f^{k+5}(x) \in [-\tfrac{2}{3} - \tfrac{1}{81}, 0).$$

By following the orbit of $(-\frac{8}{9}, -\frac{7}{9})$ as above, we see that

$$f^8(-\tfrac{8}{9}, -\tfrac{2}{3}) \supseteq f^8(-\tfrac{8}{9}, -\tfrac{7}{9}) \supseteq (-\tfrac{2}{3} - \tfrac{1}{81}, -\tfrac{2}{3}) \supseteq (-\tfrac{2}{3} - \tfrac{2}{243}, -\tfrac{2}{3}).$$

In the same way we obtain

(4) For every integer $i \geq 2$, there is a positive integer n_i such that

$$f^{n_i}(-\tfrac{2}{3} - 2/3^i, -\tfrac{2}{3}) \supseteq (-\tfrac{2}{3} - 2/3^{i+3}, -\tfrac{2}{3}).$$

PROPOSITION. $-\frac{2}{3} \in \Lambda(f) - \overline{P(f)}$.

Proof. It follows from (1), (2), and (3) that $[-1, 0] \cap P(f) = \emptyset$, and hence $-\frac{2}{3} \notin \overline{P(f)}$.

To see that $-\frac{2}{3} \in \Lambda(f)$, notice that by (4) there are closed intervals K_i and positive integers n_i $(i = 1, 2, 3, \dots)$ such that $f^{n_i}(K_i) \supseteq K_{i+1}$ for each positive integer i and $\bigcap_{i=1}^{\infty} K_i = \{-\frac{2}{3}\}$. Let $J_1 = K_1$, $J_2 = \{x \in J_1 : f^{n_1}(x) \in K_2\}$, $J_3 = \{x \in J_2 : f^{n_1+n_2}(x) \in K_3\}$, etc. There is a point $y \in \bigcap_{n=1}^{\infty} J_n$, and $-\frac{2}{3} \in \omega(y)$. □

The reader can verify that f has exactly one periodic orbit of period 2^n $(n = 0, 1, 2, \dots)$ and no other periodic orbits. (Each interval I_n contains a repelling periodic orbit of period 2^{n-1}.) Since $f(-\frac{2}{3}) = f(0)$ and both points are non-wandering, this map provides a counterexample to the assertion of Sarkovskiĭ [**S2**, theorem B] that if the only periods of periodic points are powers of two, then $f|\Omega(f)$ is a homeomorphism.

In example 1, $\omega(-\frac{2}{3})$ is a topological group and f acts on it by translation. In particular, f is one-to-one on $\omega(-\frac{2}{3})$. This need not be the case. In the next example, we construct (following Nitecki [**N2**, 4.18]) a map f' with a point $x' \in \Lambda(f') - \overline{P(f')}$ such that $f'|\omega(x')$ is not one-to-one.

Example 2. We modify the map f of example 1 by replacing every point in a backward orbit by an interval.

First, pick a point $z \in C$ (for example, $z = \frac{1}{4}$) such that every R-neighbourhood of z and every L-neighbourhood of z contains points of C other than z. Next, let $[0', 1']$ be a copy of $[0, 1]$, and let $C' \subseteq [0', 1']$ be a perfect set whose components are in one-to-one order-preserving correspondence with those of C, such that for each integer $k \geq 0$, the component corresponding to the unique point in $f^{-k}(z) \cap C$ is a non-degenerate closed interval J_k', and all other components are points.

Let f' be the continuous map of C' to C' defined as follows.

(1) If $x \in C$ and $x \notin \bigcup_{k=0}^{\infty} f^{-k}(z)$, then $f'(x') = (f(x))'$.

(2) For each integer $k \geq 1$, f' maps J_k' linearly onto J_{k-1}', preserving order.

(3) f' collapses J_0' to the point $(f(z))'$.

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Next, extend f' to the intervals $[0', (\frac{1}{3})']$, $[(\frac{2}{3})', (\frac{7}{9})'], \ldots$ so that f' is linear on each deleted interval of C' which lies in any of these intervals. Notice that if M' denotes the set of boundary points of C', i.e. the points in C' not interior to any of the J'_k, then M' is a minimal set and f' is not one-to-one on M'. (f' identifies the endpoints of J'_0.)

Finally, extend f' to $[(-1)', 2']$ in the same way that f was extended to $[-1, 2]$. Then $(-\frac{2}{3})' \in \Lambda(f') - \overline{P(f')}$ and $\omega((-\frac{2}{3})') = M'$.

Our third and final example shows that the theorem does not hold if we replace $\Lambda(f) - \overline{P(f)}$ by $\Omega(f) - \overline{P(f)}$. We construct a map $g: [-1, 1] \to [-1, 1]$ with a point $x \in \Omega(g) - \overline{P(g)}$ such that $\omega(x)$ is not a minimal set.

Example 3. Let $g(x) = 3x$ on the interval $[0, \frac{1}{3}]$ and $g(x) = 3x - 2$ on the interval $[\frac{1}{2}, 1]$. If C again denotes the Cantor middle third set, then $g(C) = C$ and $g|C$ is topologically conjugate in a natural way to the full one-sided shift of two symbols. In particular, for some $y \in C$, $\omega(y) = C$. (Any $y \in C$ whose ternary expansion contains every finite string of zeros and twos will do.)

Pick such a point y and extend g to $[-1, 1]$ as follows. Let $g(-1) = 0$, $g(x) = y$ for $x \in [-\frac{1}{2}, -\frac{1}{4}]$, and let g be linear on the intervals $[-1, -\frac{1}{2}]$, $[-\frac{1}{4}, 0]$, and $[\frac{1}{3}, \frac{1}{2}]$. The graph of g is shown in figure 4.

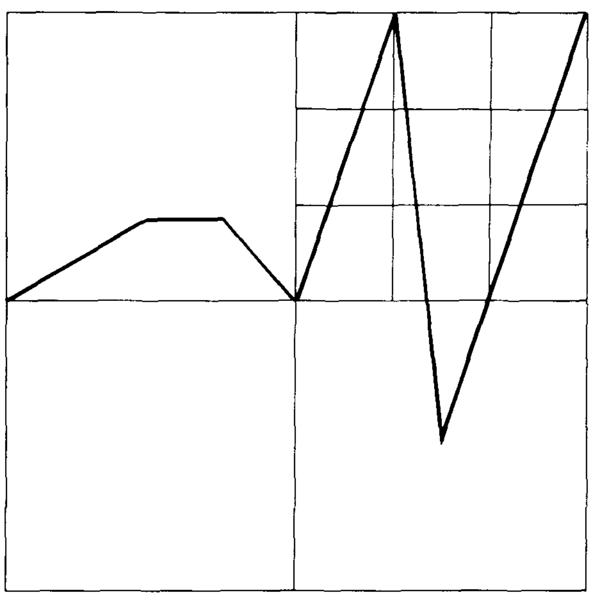

FIGURE 4

It is easy to check that $-\frac{1}{2} \in \Omega(g) - \overline{P(g)}$ and that $\omega(-\frac{1}{2}) = \omega(y) = C$, and hence that $\omega(-\frac{1}{2})$ is not a minimal set.

Notice that g is a particularly simple, in fact piecewise linear, example of a map of the interval with a point in $\Omega - \bar{P}$ which has an infinite orbit. A more complicated example was constructed in [CN, § 4].

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