

THE TRACING OF CONICS.

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It is hard to understand why the problem of tracing a conic from its equation is usually subordinated to if not identified with the problem, intrinsically more difficult, of calculating axes, or why students are not encouraged to bring their knowledge of geometrical properties of the curve to the drawing-board. The essence of tracing a curve is to render the sketching of it as accurate as may be desired, and I venture to assert that *for this purpose* the discovery of axes and vertices is not worth a tithe of the labour which it demands. From the advice in some of our text-books one would imagine that the ellipse could be drawn accurately from its four vertices alone, the hyperbola from its vertices and its asymptotes, and the parabola from its axis and vertex and one or two other points*. The truth is, that in any but expert hands these details are woefully insufficient; what is wanted is a multitude of points on the curve, and it matters little whether or not the vertices are included. The student finds satisfaction in drawing the curve from its equation and observing the symmetry that appears when the axes are inserted subsequently.

If θ is a given angle, the point $(x+r \cos \theta, y+r \sin \theta)$ is on the conic $ax^2+\dots=0$ if $a(x+r \cos \theta)^2+\dots=0$, and the midpoint of the chord through (x, y) with direction θ is (x, y) itself if the coefficient of r in the latter equation is zero, that is, if

$$(ax+hy+g) \cos \theta + (hx+by+f) \sin \theta = 0.$$

In other words, chords with direction θ have a diameter, and its equation is that just written. Neither this classical argument nor its appreciation depends on a knowledge of the shapes of conics.

Let us suppose the diameters

$$ax+hy+g=0, \quad hx+by+f=0,$$

which bisect horizontal and vertical chords, to be drawn on graph-paper, and let us call these lines h and v . Unless h is itself almost horizontal, the geometrical operation of passing from a given point P to the point P_h , which is such that PP_h is horizontal and is bisected by h , is both quick and accurate; the lines on the paper keep the direction true, and the distance may be measured by a ruler or stepped off by dividers or by means of marks on the edge of a loose sheet of paper. Similarly P_v , the point such that PP_v is vertical and is bisected by v , is readily constructed from P and v unless v is almost vertical. Thus, in general, if a single point P on the conic has been found, two sequences $P_h, P_{hh}, P_{hhh}, \dots$ and $P_v, P_{vv}, P_{vvv}, \dots$ of points, all of which are on the conic, can be extended to a considerable length with very little trouble and with a high degree of precision. As a rule, the sequences are not cyclic, and there is no theoretical limit to the number of points obtainable from one starting-point, and in practice, if the curve to be traced is in fact an ellipse, the points of one sequence are often seen to mark every part of the contour more and more distinctly.

But the sequences may be cyclic, and if the sequences are acyclic and the curve is not an ellipse, the sequences pass, and usually pass rapidly, beyond the boundary of the paper; divergent sequences may fail to indicate clearly the shape of the accessible parts of the conic, and cyclic sequences may provide fewer points than are desirable. In either event a simple and effective plan

* One of the best elementary books known to me contains the naïve admission that when the vertices and asymptotes of a hyperbola have been marked, "it is advisable by way of corroboration to find some points on the curve". It is not surprising that later the authors are content after a page of calculation on a particular parabola to assert that they can draw the curve "fairly accurately", and it must be confessed that their figure would not substantiate a less modest claim

is to insert the diameters of chords which bisect one of the angles between the axes of coordinates; these diameters are the lines

$$(a+h)x+(h+b)y+(g+f)=0, \quad (a-h)x+(h-b)y+(g-f)=0,$$

and we will call them f and g . To keep a ruler or the edge of a sheet of paper symmetrically across the axes of reference, under the guidance of the lines on the graph paper, is little more troublesome than to keep it horizontal or vertical, and so from a point P can be found points P_f and P_g . With the four lines h, v, f, g , one point P leads usually not to a finite number of sequences but to an involved ramification with points such as P_{fvgg} , and points sufficient for the accurate drawing of the curve are obtained without the steps in any one construction being too numerous for the result to be reliable. Examples do occur in which even with four diameters there are points from which the conic can not be traced, but in most of these cases not only is the curve exceptional in its relation to the axes of reference, but the ineffective starting-points are exceptional points of the curve.

For a starting-point the beginner naturally looks along one of the axes of coordinates or along a line parallel to one of them, but this habit, to the extent to which it is bad, is soon outgrown. In favour of looking for intersections of the curve by the diameters h, v , it is to be urged that these intersections correspond to extreme values of y and x , and indicate the scale on which the figure is to be drawn. Moreover, in the case of an ellipse or an undegenerate parabola intersections with h and v necessarily exist, and in the case of the parabola their calculation leads to an equation that is linear, not quadratic. For a hyperbola whose equation is not actually of the form $xy=k$, one of the four lines h, v, OX, OY does give a real point from which a start is possible.

In the case of a hyperbola, two points which it is easy to find geometrically are the points common to the line $2gx+2fy+c=0$ and the pair of lines $ax^2+2hxy+by^2=0$; the pair of lines, being parallel to the pair of asymptotes, is worth construction apart from this special use. For any value of k , this pair of lines joins the origin O to the intersections of $x=-2bk$ with the circle which has its centre at $\{(a-b)k, 2hk\}$ and passes through O , and also to the intersections of $y=-2ak$ with the circle which has its centre at $\{2hk, (b-a)k\}$ and passes through O ; the choice of k together with the choice between the two modes of construction assists to a combination of convenience with accuracy.

To mark one of the points $\{(a-b)k, 2hk\}, \{2hk, (b-a)k\}$ serves another purpose, for to say that the angles from OX to an axis of the curve satisfy the equation $\tan 2\theta=2h/(a-b)$ is to assert that the directions of the axes bisect the angles between OX and the radius to the first of these points or between OY and the radius to the second. It is a capital delusion to suppose that the directions of the axes are plotted best by finding 2θ from tables or by solving the equation for $\tan 2\theta$ as a quadratic equation in $\tan \theta$, and it is an error of the same kind to solve the pair of equations

$$ax+hy+g=0, \quad hx+by+f=0$$

algebraically in order to mark the centre of the curve. To bisect geometrically the angles between the x -axis and the line $y/x=6/13$ is manifestly quicker and is more accurate than to find that the bisectors are $y/x=(-13 \pm \sqrt{205})/6$ and to plot these lines from their equations, and the point of intersection of the lines

$$12x-y-8=0, \quad x+24y-29=0,$$

can not be identified as well from its coordinates, $13/17, 20/17$, as from the lines themselves.

It is partly because the coordinates of the centre are not assumed to be calculated that it is not for the actual asymptotes but for lines parallel to them through the origin that a geometrical construction is suggested, but the asymptotes should of course be drawn. The property of equal intercepts

can then be utilised for the rapid insertion of additional points on the curve, and an opportunity is given to emphasise that constructions which in theory are exact may in practice be unreliable.

The methods advocated have been tried successfully with students at different stages of development. Before discussing either the reduction of the general equation or any special forms of conic, I have left a class to discover* on the basis of the diametral property alone various forms which numerical examples do yield. At the other extreme, mature students find the careful plotting of six or eight members of a pencil $S_1 + \lambda S_2$ to be the work of only one afternoon in the drawing-school, provided that at least one common point is real and accessible. For different values of λ , the diameters $h_1 + \lambda h_2$ have a common point H , the diameters $v_1 + \lambda v_2$ a common point V , and so on, and these common points must be marked; then for a particular value of λ , one point H_λ other than H on $h_1 + \lambda h_2$ and one point V_λ other than V on $v_1 + \lambda v_2$ having been calculated, the diameters themselves join H_λ to H and V_λ to V , the intersection of these diameters is the centre C_λ , and $f_1 + \lambda f_2$ and $g_1 + \lambda g_2$ are the lines joining this centre to F and G : the conic of centres is traced incidentally. In an alternative process the conic of centres is first constructed with great care, and since in favourable cases thirteen † points of this conic are known in advance, this is not difficult; any point C_λ of this conic being taken as centre, the diameters $C_\lambda H$, $C_\lambda V$, $C_\lambda F$, $C_\lambda G$ can be drawn, and the conic found geometrically from one of the common points without reference to the value of λ which determines it algebraically. For a pencil with no real accessible common points, each conic requires its own starting-point, and the work is longer, but not prohibitive.

NOTES ON EXAMPLES.

(1) $9x^2 - 24xy + 16y^2 + 32x - 76y + 16 = 0$.

This is the parabola mentioned in the first footnote; the diameter f is useless, but h, v, g are efficient, and starts may be made either from points on the axes of reference or from the point where $3x - 4y = k$ cuts $100x = 4k^2 + 76k + 64$ for various values of k .

(2) $(x + 2y - 2)^2 + 4(2x - y + 1)^2 = 45$.

This seems ready-made, but although the axes and vertices can be plotted at once, there is no simpler way to mark additional points than by inserting the diameters h, v and developing sequences from the vertices; it is better to use the four vertices independently than to reflect in the axes of the curve.

(3) $4x^2 + 9y^2 = 36$.

As a rule a central conic referred to its own axes can be plotted from the vertices alone by means of the diameters f and g , but in this example $AfOg$ is so near to B that without other starting points only 3 intermediate points in each quadrant are obtained; however, from $(\pm 9/5, \pm 8/5)$ it is easy to find 28 other points well distributed on the contour.

(4) $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$.

Here h cuts the curve where $y = 3 \pm \frac{1}{2}\sqrt{142}$ and v where $x = 2 \pm \frac{1}{2}\sqrt{110}$, and the first six points in each of the sequences formed from these points by means of h and v combine to mark the curve well; almost the same sequences would be found from the points for which $x = 0$ and those for which $x = 1$.

(5) $6x^2 - xy - 12y^2 - 8x + 29y - 16 = 0$.

A hyperbola in which the sequences from h and v alone give as many points as can be desired; since the quadratic terms factorise, it is uncommonly easy to find the points in which $8x - 29y + 16 = 0$ cuts the curve.

(6) $6x^2 - 60xy - 19y^2 + 48x + 98y - 60 = 0$.

A typical hyperbola, in which h and v alone are almost useless and must be supplemented either by f and g or by the asymptotes.

(7) $(y - x)^2 = 4(y + x)$.

Even with the four lines h, v, f, g , the curve cannot be traced from any one starting-point, but the very simplicity to which this result is due renders an elaborate process unnecessary. We may trace the curve either by the intersection of $y - x = k$ with $y + x = \frac{1}{4}k^2$ for different values of k , or, to illustrate a different method, always available for a parabola, by observing that the level at which the horizontal distance to a point of the curve from the line $y - x = 0$ is k , is the level at which the horizontal distance from the line $y + x = 0$ to the line $y - x = 0$ is $\frac{1}{4}k^2 - k$.

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* Unfortunately the evidence of value in the experiment is always diminished by the students' possession of books.

† The usual nine points and the points H, V, F, G .