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NONLINEAR FILTERING OF A SYSTEM OF LOGISTIC EQUATIONS

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This paper is concerned with the filtering problem for a nonlinear stochastic system of prey-predator logistic equations. Based on the innovations approach, we establish the Zakai equation for the unnormalised conditional distribution and the adjoint Zakai equation for the unnormalised conditional density of the nonlinear filter. Using a perturbation technique, we obtain the appropriate expressions for the unnormalised conditional distribution and density of stochastic integrals with respect to the observation processes.

1. Introduction

In studies of population dynamics, the state of a system is commonly described by a logistic-type equation:

$$dN_t = AN_t(1 - BN_t)dt. \tag{1.1}$$

Here, N_t denotes the population density and A the initial per capita growth rate, while B is the reciprocal of the environmental carrying capacity. A more realistic model to describe the state of the system is to include a stochastic element in (1.1) so that the system under investigation is given by the following stochastic differential equation:

$$dN_t = AN_t(1 - BN_t)dt + \sigma dW_1(t), \tag{1.2}$$

where $W_1(t)$ is a Brownian motion.

It is a common situation in practice that the signal process $\{N_t\}$ cannot be observed directly, but must be estimated from observation on a related process y_t which can be assumed to have the form

$$dy_t = kN_t dt + \sigma_l dW_2(t), \tag{1.3}$$

where $W_2(t)$ is another Brownian motion independent of $W_1(t)$. The objective then is to compute a least squares estimate of N_t given the observations $\{y_s, 0 \leqslant s \leqslant t\}$. In

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addition, it is required that this computation be done recursively. This is the classical filtering problem.

In this paper, we consider the filtering of a nonlinear system of prey-predator competition in a stochastic environment given by:

$$\begin{aligned} dN_t^{(1)} &= N_t^{(1)}\Big(A - BN_t^{(1)} - CN_t^{(2)}\Big)dt + \varepsilon_1 dW_t^1, \\ dN_t^{(2)} &= N_1^2\Big(-D + EN_t^{(1)}\Big)dt + \varepsilon_2 dW_t^2, \end{aligned} \tag{1.3}$$

where N_t^1 = prey population, N_1^2 = predator population, and the observation processes are given by:

$$\begin{aligned} dy_t^{(1)} &= k_1 N_t^{(1)} dt + dB_t^{(1)}, \\ dy_t^{(2)} &= k_2 N_t^{(2)} dt + dB_t^{(2)}, \end{aligned} \tag{1.4}$$

where W_t^1, W_t^2, B_t^1 and B_t^2 are independent Brownian motions on a probability space $(\Omega, \mathfrak{F}, P)$ with a filtration $\{\mathfrak{F}_t\}_{t\geqslant 0}$.

The filtering problem for the logistic equation of a single species (system (1.1) and (1.2)) is solved in [4]. In this paper, we extend the results of [4] to the prey-predator system (1.3) and (1.4). As described in [3] and [4], nonlinear filtering can be approached via the innovations technique. We shall collect some key results of this technique in Section 2. Using the innovations technique, we obtain in Section 3 the Kushner equation for the conditional joint distribution of $N_t^{(1)}$, $N_t^{(2)}$ given the past history of y_t. The Kushner equation still poses two difficulties: It contains nonlinear terms and is driven by the innovations process. To overcome these difficulties, we obtain in Section 4 the Zakai equation for the unnormalised conditional distribution. The Zakai equation for our model consists of two stochastic integrals with respect to the observation processes. Using a pertubation technique of Kunita [6], we establish in Section 5 an expression for the unnormalised conditional distribution involving no stochastic integrals. Finally, in Section 6, we derive the adjoint Zakai equation for the unnormalised conditional density, which again can be transformed into an expression free of stochastic integrals by appealing to Kunita's technique.

The Zakai equation for the unnormalised conditional distribution extends the result of Elliott [4] in the single species case; while the multidimensional adjoint Zakai equation for the unnormalised conditional density of the system (1.3) and (1.4) is new. These results solve the nonlinear filtering problem for the system (1.3) and (1.4) of prey-predator logistic equations completely.

2. The innovations method to nonlinear filtering

The basic references for this section are Davis and Marcus [2], Elliott [3, 4] and Kunita [6]. All stochastic processes will be defined on a fixed probability space $(\Omega, \mathfrak{F}, P)$

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and a finite time interval $[0, T]$, on which there is defined an increasing family of σ-fields $\{\mathfrak{F}_t, 0 \leqslant t \leqslant T\}$. All processes considered will be $\mathfrak{F}_t$-adapted.

Consider a real-valued Markov process ξ_t and the related observation process y_t given by

$$dy_t = h(\xi_t)dt + dB_t, \tag{2.1}$$

where B_t is Brownian motion. We shall assume that ξ_t is independent of the increments of B_t. Define $\mathfrak{Y}_t = \sigma\{y_s, 0 \leqslant s \leqslant t\}$ as the σ-field generated by the given observation process y_t. The objective is to compute in a recursive form an expression for the best estimate of ξ_t given the history of the observations to time t, $\mathfrak{Y}_t$. That is, we want to obtain an expression for

$$\widehat{\xi}_t = E[\xi_t \mid \mathfrak{Y}_t]. \tag{2.2}$$

Now, let us introduce the innovations process

$$\nu_t = y_t - \int_0^t \widehat{h}(\xi_s)ds. \tag{2.3}$$

The incremental innovations $\nu_{t+h} - \nu_t$ represents the "new information" concerning the process $\{Z_t = h(\xi_t)\}$ available from the observations between t and $t+h$, in the sense that $\nu_{t+h} - \nu_t$ is independent of $\mathfrak{Y}_t$.

The process ν_t has the following properties:

(i) The process $(\nu_t, \mathfrak{Y}_t)$ is a standard Brownian motion, that is ν_t is a $\mathfrak{Y}_t$-martingale and $\langle \nu \rangle_t = t$, where $\langle \nu \rangle$ is the unique predictable process such that $\nu^2 - \langle \nu \rangle$ is a martingale.

(ii) Every square integrable martingale $(m_t, \mathfrak{Y}_t)$ with respect to the observation σ-field $\mathfrak{Y}_t$ is sample continuous and has the representation

$$m_t = \int_0^t g_s d\nu_s, \tag{2.4}$$

where the process g is $\mathfrak{Y}_t$-predictable. In other words, m_t can be written as a stochastic integral with respect to the innovations process.

Suppose the signal process ξ_t is a real-valued $\mathfrak{F}_t$-semimartingale of the form

$$\xi_t = \xi_0 + \int_0^t \alpha_s ds + \eta_t, \tag{2.5}$$

where α is an $\mathfrak{F}$-adapted process such that

$$E\int_0^T \alpha_s^2 ds < \infty,$$

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ξ_0 is an $\mathfrak{F}_0$-measurable random variable with $E\xi_0^2 < \infty$, and η_t is a square integrable $\mathfrak{F}_t$-martingale. There is a unique predictable process $\langle \eta, B\rangle$ such that $(\eta_t B_t - \langle \eta, B\rangle_t, \mathfrak{F}_t)$ is a martingale. Let us assume that this process is of the form $\langle \eta, B\rangle_t = \int_0^t \beta_s ds$.

THEOREM 2.1. *Write $\widehat{\xi} = E[\xi_t|\mathfrak{Y}_t]$ for the filtered estimate of ξ_t, given $\mathfrak{Y}_t$. Then $\{\widehat{\xi}_t\}$ satisfies the stochastic differential equation*

$$\widehat{\xi}_t = \widehat{\xi}_0 + \int_0^t \widehat{\alpha}_s ds + \int_0^t \left(\widehat{\xi_s h}(x_s) - \widehat{\xi}_s \widehat{h}(x_s) + \widehat{\beta}_s\right) d\nu_s. \tag{2.6}$$

PROOF: See Elliott [4]. □

Formula (2.6) is not a recursive equation for $\widehat{\xi}_t$ and hence is not very useful as it stands, but can be used to obtain an infinite-dimensional recursive equation for the filtering problem.

COROLLARY 2.2. *Let $\{x_t\}$ be the solution of the stochastic differential equation*

$$dx_t = f(t, x_t)dt + \sigma(x_t)dW_t; \tag{2.7}$$

that is,

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t \sigma(x_s)dW_s, \tag{2.8}$$

and let Y_t be given by (2.1) with $\{W_t\}$ and $\{B_t\}$ being independent. Then for any function $F \in C^2$, $\Pi_t(F) = E[f(x_t) \mid \mathfrak{Y}_t]$ satisfies

$$\Pi_t(F) = \int_0^t \Pi_s(LF)ds + \int_0^t \left[\Pi_s(Fh) - \Pi_s(F)\Pi_s(h)\right] d\nu_s, \tag{2.9}$$

where $$LF(x) = f(s, x)F_x(x) + \frac{1}{2}\sigma(x)^2 F_{xx}(x).$$

PROOF: The Ito differential rule yields

(2.10)
$$F(x_t) = F(x_0) + \int_0^t F_x(x_s)f(s, x_s)ds + \int_0^t F_x(x_s)\sigma(x_s)dW_s + \frac{1}{2}\int_0^t F_{xx}(x_s)\sigma(x_s)^2 ds.$$

Now applying Theorem 2.1 to the semimartingale $f(x_t)$ gives

$$\Pi_t(F) = \Pi_0(F) + \int_0^t \Pi_s(LF)ds + \int_0^t \left[\Pi_s(Fh) - \Pi_s(F)\Pi_s(h)\right] d\nu_s.$$

□

Since $\{\Pi_t(F) : F \in C^2\}$ determines a measure-valued stochastic process Π_t, (2.6) can be regarded as a recursive (infinite-dimensional) stochastic differential equation for the conditional measure Π_t of ξ_t given $\mathfrak{Y}_t$, and $\Pi_t(F)$ is a conditional statistic computed from Π_t in a memoryless fashion.

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REMARK. Elliott [4] solved the filtering problem for the single species system (1.1) and (1.2) using these methods together with the perturbation technique of Kunita [6]. Whereas these sophisticated methods are appropriate for an n-dimensional system, $n \geqslant 2$, as can be seen in the next sections, the Kalman-Bucy filter for linear systems is adequate for the one-dimensional problem, if the observation process is formulated in an appropriate way. In fact, let us consider the following observation process:

$$dY_t = k\left(\frac{1}{BN_t} - 1\right)dt + \sigma_2 dW_2(t) \tag{2.11}$$

(see Antonelli [1]). Using the substitution

$$N^* = \frac{1}{BN_t} - 1, \tag{2.12}$$

we obtain the following linear estimation problem:

$$dN_t^* = -AN_t^* dt + \sigma_1 dW_1(t), \tag{2.13}$$

$$dY_t = -kN_t^* dt + \sigma_2 dW_2(t). \tag{2.14}$$

Now, to (2.13) and (2.14) we apply the Kalman-Bucy filtering method (see Lipster and Shiryayev [8]; Oksendal [9]) to obtain $\widehat{N}_t^*$, the conditional expectation of N_t^* given the observations up to time t on $\mathfrak{Y}_t$, and S_t the mean square error estimate. Suppose that the initial distribution $\mathfrak{N}_0$ is Gaussian with zero mean and variance S_0. The Kalman-Bucy [5] equations are

$$d\widehat{N}_t^* = \left(-A - \frac{k^2 S(t)}{\sigma_2^2}\right)\widehat{N}_t^* dt + \frac{kS(t)}{\sigma_2^2} dy_t, \tag{2.15}$$

where $S(t) = E\left[\left(N_t^* - \widehat{N}_t^*\right)^2 \mid \mathfrak{Y}_t\right]$ satisfies the (deterministic) Riccati equation

$$\frac{dS}{dt} = -2AS(t)\frac{k^2}{\sigma_2^2}S^2(t) + \sigma_1^2,\ S(0) = \alpha^2, \tag{2.16}$$

which has the solution

$$S(t) = \frac{A_1 - M.A_2 \exp\left(\frac{(A_2 - A_1)k^2 t}{\sigma_2^2}\right)}{1 - M.\exp\frac{(A_2 - A_1)k^2 t}{\sigma_2^2}},$$

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where

$$A_1 = \frac{1}{k_2}\left(-A\sigma_2^2 - \sigma_2\sqrt{A^2\sigma_2^2 + k^2\sigma_1^2}\right),$$

$$A_2 = k^2\left(-A\sigma_2^2 - \sigma_2\sqrt{A^2\sigma_2^2 + k^2\sigma_1^2}\right),$$

and

$$M = \frac{\alpha^2 - A_1}{\alpha^2 - A_2}.$$

Hence we obtain the solution for $\widehat{N}_t^*$ in the form

$$\widehat{N}_t^* \exp\left(\int_0^t G(s)ds\right)\widehat{N}_0^* + \frac{k^2}{\sigma_2^2}\int_0^t \exp\left(\int_s^t G(u)du\right)S(s)dy_s,$$

where

$$G(s) = -A - \frac{k^2}{\sigma_2^2}S(s).$$

3. Nonlinear filtering of a prey-predator system

Consider the system (1.3) and(1.4). Let $\Pi_t(N) = E[N_t \mid \mathfrak{Y}_t]$ where $\mathfrak{Y}_t = \sigma\left\{y^{(1)}, y_s^{(2)}, x \leqslant t\right\}$. The innovations processes are

$$\nu_t^1 = y_t^1 - \int_0^t k_1\Pi_s\left(N^{(1)}\right)ds,$$

$$\nu_t^2 = y_t^2 - \int_0^t k_2\Pi_s\left(N^{(2)}\right)ds.$$

Let $F : R^2 \to R^1$ be any function in C^2. Using Ito's differential rule, we have

$$F\left(N_t^{(1)}, N_t^{(2)}\right) = F\left(N_0^{(1)}N_0^{(2)}\right) + \int_0^t \frac{\partial F}{\partial N^{(1)}}dN_s^{(1)} + \int_0^t \frac{\partial F}{\partial N^{(2)}}dN_s^{(2)} + \frac{1}{2}\int_s^t \varepsilon_1^2\frac{\partial^2 F}{\partial {N^{(1)}}^2}ds + \frac{1}{2}\int_0^t \varepsilon_2^2\frac{\partial^2 F}{\partial {N^{(2)}}^2}ds,$$

that is,

$$F\left(N_t^{(1)}, N_t^{(2)}\right) = F\left(N^{(1)}, N_0^{(2)}\right) + \int_0^t \frac{\partial F}{\partial N^{(1)}}\left(\left(A - BN_s^{(1)} - CN_s^{(2)}\right)N_s^{(1)}\right)ds + \int_0^t \frac{\partial F}{\partial N^{(2)}}\left(\left(-D + EN_s^{(1)}\right)N_s^{(2)}\right)ds$$
$$+ \int_0^t \varepsilon_1\frac{\partial F}{\partial N_s^{(1)}}dW_t^1 + \int_0^t \varepsilon_2\frac{\partial F}{\partial N_s^{(2)}}dW_t^2 + \frac{1}{2}\int_0^t\left(\varepsilon_1^2\frac{\partial^2 F}{\partial N^{(1)2}} + \varepsilon_2^2\frac{\partial^2 F}{\partial N^{(2)2}}\right)ds.$$

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Let us use the notation

$$\Pi_t(F) = E\left[F\left(N_t^{(1)}, N_t^{(2)}\right) \,\middle|\, \mathfrak{Y}_t\right].$$

Then Π_t can be thought of as the conditional joint distribution of $N_t^{(1)}$, $N_t^{(2)}$ given $\mathfrak{Y}_t$. Since W_t^1, W_t^2, B_t^1, B_t^2 are independent, applying Theorem 2.1 to the semimartingale $F\left(N_t^{(1)}, N_t^{(2)}\right)$ gives

$$\begin{aligned}
\Pi_t\left(F\left(N_t^{(1)}, N_t^{(2)}\right)\right) &= \Pi_0\left(F\left(N_0^{(1)}, N_0^{(2)}\right)\right) \\
&\quad + \int_0^t \Pi_s\left(\frac{\partial F}{\partial N^{(1)}}\left(A - BN_s^{(1)} - CN_s^{(2)}\right)N_s^{(1)}\right) ds \\
&\quad + \int_0^t \Pi_s\left(\frac{\partial F}{\partial N^{(2)}}\left(-D + EN_s^{(1)}\right)N_s^{(2)}\right) ds \\
&\quad + k_1 \int_0^t \left(\Pi_s\left(FN^{(1)}\right) - \Pi_s(F)\Pi_s\left(N_s^{(1)}\right)\right) d\nu_s^1 \\
&\quad + k_2 \int_0^t \left(\Pi_s\left(FN_s^{(2)}\right) - \Pi_s(F)\Pi_s\left(N_s^{(2)}\right)\right) d\nu_s^2 .
\end{aligned} \tag{3.1}$$

4. The unnormalised filtering equation

There are two difficulties with equation(3.1). Firstly, it contains the nonlinear terms $\Pi_s(F)\Pi_s(N^{(1)})$, $\Pi_s(F)\Pi_s(N^{(2)})$ and secondly, it is driven by the innovations processes ν_t^1, ν_t^2. Let the process be defined on $[0,T]$. The first step is to define a new measure $\overline{P}$ on the measurable space $(\Omega, \mathfrak{F})$ by

$$\overline{P}(A) = \int \frac{d\overline{P}}{dP}(\omega)P(d\omega)$$

for all $A \in \mathfrak{F}$, where

$$\left.\frac{d\overline{P}}{dP}\right|_{\mathfrak{F}_1} = \Lambda_t^{-1},$$

$$\mathfrak{F}_t = \sigma\{y_s^1, y_s^2, N_s^{(1)}, N_s^{(2)}, s \leqslant t\},$$

and

$$\Lambda_t = \exp\left[\left(\int_0^t k_1 N_s^{(1)} dy_s^1 + \int_0^t k_2 N_s^{(2)} dy_s^2\right) - \frac{1}{2}\int_0^t \left(\left(k_1 N_s^{(1)}\right)^2 + \left(k_2 N_s^{(2)}\right)^2\right) ds\right].$$

Girsanov's Theorem. *Suppose we start with a filtered probability space* $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \overline{P})$ *such that*

(i) y_t^1, y_t^2 *are independent Brownian motions;*

(ii) N_t^1, N_t^2 *satisfy Equation 1.3.*

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We can define P *by setting* $\frac{dP}{d\overline{P}} \Big| \mathfrak{F}_t = \Lambda_t$.

Under $P : y_t^1 y_t^2$ *are not Brownian motions; in fact*

$$y_t^1 - \int_0^t k_s^1 N_s^1 ds = B_t^1 \text{ is a Brownian motion under } P,$$

$$y_t^2 - \int_0^t k_s^2 N_s^2 ds = B_t^2 \text{ is a Brownian motion under } P.$$

Consider any function $F \in C_0^2(R^2)$ with compact support. Using the Ito differential rule, we have

$$F\left(N_t^{(1)}, N_t^{(2)}\right) = F\left(N_0^{(1)}, N_0^{(2)}\right) + \int_0^t \frac{\partial F}{\partial N^{(1)}} dN_s^{(1)} + \int_0^t \frac{\partial F}{\partial F^{(2)}} dN_s^{(2)} + \frac{1}{2}\int_0^t \varepsilon_1^2 \frac{\partial^2 F}{\partial N^{(1)2}} ds + \frac{1}{2}\int_0^t \varepsilon_2^2 \frac{\partial^2 F}{\partial N^{(2)2}} ds;$$

that is

(4.1)

$$\begin{aligned} F\left(N_t^{(1)}, N_t^{(2)}\right) = F\left(N_0^{(1)}, N_0^{(2)}\right) &+ \int_0^t \frac{\partial F}{\partial N^{(1)}} \left(\left(A - BN_s^{(1)} - CN_s^{(2)}\right) N_s^{(1)}\right) ds \\ &+ \int_0^t \frac{\partial F}{\partial N^{(2)}} \left(\left(-D + EN_s^{(1)}\right) N_s^{(2)}\right) ds + \int_0^t \varepsilon_1 \frac{\partial F}{\partial N_1} dW_s^1 \\ &+ \int_0^t \varepsilon_2 \frac{\partial F}{\partial N_2} dW_s^2 + \frac{1}{2}\int_0^t \left(\varepsilon_1^2 \frac{\partial^2 F}{\partial N^{(1)2}}\right) ds + \frac{1}{2}\int_0^t \left[\varepsilon_2^2 \frac{\partial^2 F}{\partial N^{(2)2}}\right] ds. \end{aligned}$$

Let $\mathfrak{Y}_t = \sigma\{y_s^1, y_s^2, s \leqslant t\}$. Then using a Bayes-type theorem [**11**],

$$\Pi_t\left(F\left(N_t^{(1)}, N_t^{(2)}\right)\right) \stackrel{d}{=} E\left[F\left(N_t^{(1)}, N_t^{(2)}\right) \Big| \mathfrak{Y}_t\right] = \frac{\overline{E}\left[\Lambda_t F\left(N_t^{(1)}, N_t^{(2)}\right) \mid \mathfrak{Y}_t\right]}{\overline{E}[\Lambda_t \mid \mathfrak{Y}_t]}$$

where $\overline{E}$ is the expectation under $\overline{P}$. Consider the numerator

$$\overline{E}\left[\Lambda_t F\left(N_t^{(1)}, N_t^{(2)} \mid \mathfrak{Y}_t\right)\right] \stackrel{d}{=} \sigma_t\left(F\left(N_t^{(1)}, N_t^{(2)}\right)\right),$$

where $\sigma_t\left(F\left(N_t^{(1)}, N_t^{(2)}\right)\right)$ is an unnormalised conditional distribution and is a measure-valued process. Further, if $F\left(N_t^{(1)}, N_t^{(2)}\right) = 1$, then

$$\overline{E}(\Lambda_t \mid \mathfrak{Y}_t) = \sigma_t(1).$$

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Hence

$$\Pi_t(F) = \frac{\sigma_t(F)}{\sigma_t(1)}. \tag{4.2}$$

We first obtain a semimartingale expression for $\sigma_t(1)$. Using the Ito differential rule, we get

$$d\Lambda_t = \Lambda_t\Big(k_1 N_t^{(1)} dy_t^1 + k_2 N_t^{(2)} dy_t^2\Big). \tag{4.3}$$

That is,

$$\Lambda_t = 1 + \int_0^t k_1 N_s^{(1)} \Lambda_s dy_s^1 + \int_0^t k_2 N^{(2)} \Lambda_2 dy_s^2, \tag{4.4}$$

so that Λ_t is a $(\mathfrak{F}_t, \overline{P})$-martingale. Consequently, as in Theorem 2.1, $\widehat{\Lambda}_t = \overline{E}[\Lambda_t \mid \mathfrak{Y}_t]$ is a $\mathfrak{Y}_t$-martingale. Since the process $\{y_t^1\}$ and $\{y_t^2\}$ are Brownian motions under $\overline{P}$, there must exist $\mathfrak{Y}_t$-adapted processes $\{\eta_t^1\}$ and $\{\eta_t^2\}$ such that

$$\widehat{\Lambda}_t = 1 + \int_0^t \eta_s^1 ds + \int_0^t \eta_s^2 ds. \tag{4.5}$$

To determine η_s^1, consider, using equation (4.3),

$$y_t^1 \Lambda_t = \int_0^t \Lambda_s dy_s^1 + \int_0^t y_s^1 k_1 N_s^1 \Lambda_s dy_s^1 + \int_0^t \Lambda_s k_1 N_s^1 ds, \tag{4.6}$$

$$y_1^2 \Lambda_t = \int_0^t \Lambda_s dy_s^2 + \int_0^t y_s^2 k_2 N_s^2 \Lambda_s dy_s^2 + \int_0^t \Lambda_s k_2 N_s^2 ds. \tag{4.7}$$

Conditioning on $\mathfrak{Y}_t$ under measure $\overline{P}$,

$$\overline{E}\big[y_t^1 \Lambda_t \mid \mathfrak{Y}_t\big] = y_t^1 \widehat{\Lambda}_t = \int_0^t k_1 \Lambda_s N_s^1 ds + M_t^1, \tag{4.8}$$

$$\overline{E}\big[y_t^2 \Lambda_t \mid \mathfrak{Y}_t\big] = y_t^2 \widehat{\Lambda}_t = \int_0^t k_2 \Lambda_s N_s^2 ds + M_t^2, \tag{4.9}$$

where M_t^1 and M_t^2 are $(\mathfrak{Y}_t, \overline{P})$-martingales. However, from equation (4.5) and Ito's rule,

$$y_t^1 \widehat{\Lambda}_t = \int_0^t \eta_s^1 ds + R_t^1, \tag{4.10}$$

$$y_1^2 \widehat{\Lambda}_t = \int_0^t \eta_s^2 ds + R_t^2. \tag{4.11}$$

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The decompositions (4.8) and (4.10) must be the same, so too (4.9) and (4.11). Hence

$$M_t^1 = R_t^1 \quad \text{and} \quad M_t^2 = R_t^2,$$

which implies that

$$\eta_s^1 = k_1 \Lambda_s N_s^{(1)} = k_1 \overline{E}\left[\Lambda_s N_s^{(1)} \mid \mathfrak{Y}_s\right],$$
$$\eta_s^2 = k_2 \Lambda_s N_s^{(2)} = k_2 \overline{E}\left[\Lambda_s N_s^{(2)} \mid \mathfrak{Y}_s\right].$$

Using Bayes' rule again, we see that

$$\eta_s^1 = k_1 \widehat{\Lambda}_s \Pi_s(N_s^1),$$
$$\eta_s^2 = k_2 \widehat{\Lambda}_s \Pi_s(N_s^2),$$

Substituting for η_s^1 and η_s^2 in equation (4.5) we get

$$\widehat{\Lambda}_t = 1 + \int_0^t k_1 \widehat{\Lambda}_s \Pi_s\left(N_s^{(1)}\right) ds + \int_0^t k_2 \widehat{\Lambda}_s \Pi_s\left(N_s^{(2)}\right) ds. \tag{4.12}$$

However, equation (4.12) has the unique solution

$$\begin{aligned}\widehat{\Lambda}_t &= \exp\Big(\int_0^t k_1 \Pi_s\left(N_s^{(1)}\right) dy_s^1 + \int_0^t k_2 \Pi_s\left(N_s^{(2)}\right) dy_s^2 \\ &\quad - \frac{1}{2}\int_0^t \left(k_1^2 \Pi_s\left(N_s^{(1)}\right)^2 + k_2^2 \Pi_2\left(N_s^{(2)}\right)^2\right) ds\Big) \\ &= \sigma_t(1).\end{aligned} \tag{4.13}$$

(Note that $\widehat{\ }$ denotes conditioning under measure $\overline{P}$, while Π denotes conditioning under the original measure P.)

From the equations (4.1) and (4.2), using Ito's rule, we get

(4.14)

$$\begin{aligned}F\left(N_t^{(1)}, N_t^{(2)}\right)\Lambda_t &= F\left(N_0^{(1)}, N_0^{(2)}\right) + \int_0^t \Lambda_s \frac{\partial F}{\partial N_s^{(1)}}\left(\left(A - BN_s^{(1)} - CN_s^{(2)}\right)N_s^{(1)}\right) ds \\ &\quad + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(1)}} \varepsilon_1 dW_s^1 + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(2)}}\left(\left(-D + EN_s^{(1)}\right)N_s^{(2)}\right) ds + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(2)}} \varepsilon_2 dW_s^2 \\ &\quad + \frac{1}{2}\int_0^t \Lambda_s\left(\varepsilon_1^2 \frac{\partial^2 F}{\partial N^{(1)^2}} + \varepsilon_2^2 \frac{\partial^2 F}{\partial N^{(2)^2}}\right) ds + \int_0^t \Lambda_s F_s\left(k_1 N_s^{(1)} dy_s^1 + k_2 N_s^{(2)} dy_s^2\right).\end{aligned}$$

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Conditioning each side on $\mathfrak{Y}_t$ under measure $\overline{P}$, we have

(4.15)

$$\begin{aligned}
&\overline{E}[\Lambda_t F\Big(N_t^{(1)}, N_t^{(2)}\Big) \mid \mathfrak{Y}_t] = \overline{E}\Big[F\Big(N_0^{(1)}, N_0^{(2)}\Big) \mid \mathfrak{Y}_t\Big] \\
&\quad + \overline{E}\Big[\Big(\int_0^t \Lambda_s \frac{\partial F}{\partial N^{(1)}}\Big(\Big(A - BN_s^{(1)} - CN_s^{(2)}\Big)N_s^{(1)}\Big)ds \\
&\quad + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(1)}}\varepsilon_1 dW_s^1 + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(2)}}\Big(\Big(-D + EN_s^{(1)}\Big)N_s^{(2)}\Big)ds + \int_0^t \Lambda_s \frac{\partial F}{\partial N^{(2)}}\varepsilon_2 dW_s^2 \\
&\quad + \frac{1}{2}\int_0^t \Lambda_s\Big(\varepsilon_1^2 \frac{\partial^2 F}{\partial N^{(1)^2}} + \varepsilon_2^2 \frac{\partial^2 F}{\partial N^{(2)^2}}\Big)ds + \int_0^t \Lambda_s F_s\big(k_1 N_s^1 dy_s^1 + k_2 N_s^2 dy_s^2\big)\Big) \mid \mathfrak{Y}_t\Big] \\
&= \overline{E}\Big[F\Big(N_0^{(1)}, N_0^{(2)}\Big) \mid \mathfrak{Y}_1\Big] + \int_0^t \overline{E}\Big[\Lambda_s \frac{\partial F}{\partial N^{(1)}}\Big(\Big(A - BN_s^{(1)} - CN_s^{(2)}\Big)N_s^{(1)}\Big) \mid \mathfrak{Y}_s\Big]ds \\
&\quad + \int_0^t \overline{E}\Big[\Lambda_s \frac{\partial F}{\partial F^{(2)}}\Big(\Big(-D + EN_s^{(1)}\Big)N_s^{(2)}\Big) \mid \mathfrak{Y}_t\Big]ds \\
&\quad + \frac{1}{2}\int_0^t \overline{E}\left[\Lambda_s\left(\varepsilon_1^2 \frac{\partial^2 F}{\partial N_s^{(1)^2}} + \varepsilon_2^1 \frac{\partial^2 F}{\partial N^{(2)^2}}\right) \mid \mathfrak{Y}_s\right]ds \\
&\quad + \int_0^t \overline{E}[\Lambda_s F_s k_1 N_s^{(1)} \mid \mathfrak{Y}_s]dy_s^1 + \int_0^t \overline{E}[\Lambda_s F_s k_1 N_s^{(2)} \mid \mathfrak{Y}_s]dy_s^2.
\end{aligned}$$

As a result, we have the *Zakai equation for the unnormalised conditional distribution*:

$$\begin{aligned}
\sigma_t\Big(F\Big(N_t^{(1)}, N_t^{(2)}\Big)\Big) &= \sigma_0\Big(F\Big(N_0^{(1)}, N_0^{(2)}\Big)\Big) \\
&\quad + \int_0^t \sigma_s\left(\frac{\partial F}{\partial N^{(1)}}\Big(N_s^{(1)}\Big(A - BN_s^{(1)} - CN_s^{(2)}\Big)\Big)\right)ds \\
&\quad + \int_0^t \sigma_s\left(\frac{\partial F}{\partial N^{(2)}}\Big(N_s^{(2)}\Big(-D + EN_s^{(1)}\Big)\Big)\right)ds \qquad (4.16)\\
&\quad + \frac{\varepsilon_1^2}{2}\int_0^t \sigma_s\left(\frac{\partial^2 F}{\partial N^{(1)^2}}\right)ds + \frac{\varepsilon_2^2}{2}\int_0^t \sigma_s\left(\frac{\partial^2 F}{\partial N^{(2)^2}}\right)ds \\
&\quad + k_1\int_0^t \sigma_s\Big(F_s N_s^{(1)}\Big)dy_s^1 + k_2\int_0^t \sigma_s\Big(F_s N_s^{(2)}\Big)dy_s^2.
\end{aligned}$$

5. Decomposition of solution

The equation (4.16) contains stochastic integrals. We now apply the decomposition techniques of Kunita [**6**, Section 5.2] to obtain an expression that involves no stochastic integrals.

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In terms of Stratonovich integrals [10], Equation (4.16) becomes

$$\begin{aligned}\sigma_t\Big(F\big(N_t^{(1)}, N_t^{(2)}\big)\Big) &= \sigma_0\Big(F\big(N_0^{(1)}, N_0^{(2)}\big)\Big)\\ &+ \int_0^t \sigma_s\Big(AF\big(N_s^{(1)}, N_s^{(2)}\big) - \frac{k_1^2}{2}\big(N_s^{(1)}\big)^2 F\big(N_s^{(1)}, N_s^{(2)}\big)\\ &- \frac{k_2^2}{2}\big(N_s^{(2)}\big)^2 F\big(N_s^{(1)}, N_s^{(2)}\big)\Big)ds,\\ &+ k_1\int_0^t \sigma_s\Big(N_s^{(1)}F\big(N_s^{(1)}\big)\Big)\circ dy_s^1 + k_2\int_0^t \sigma_s\Big(N_s^{(2)}F\big(N_s^{(2)}\big)\Big)\circ dy_s^2.\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}AF\big(N_s^{(1)}, N_s^{(2)}\big) &= \Big(N_s^{(1)}\big(A - BN_s^{(1)} - CN_s^{(2)}\big)\Big)\frac{\partial F}{\partial N^{(1)}}\big(N_s^{(1)}, N_s^{(2)}\big)\\ &+ \Big(N_s^{(2)}\big(-D + EN_s^{(1)}\big)\Big)\frac{\partial F}{\partial N^{(2)}}\big(N_s^{(1)}, N_s^{(2)}\big) + \frac{\varepsilon_1^2}{2}\frac{\partial^2 F}{\partial N^{(1)^2}} + \frac{\varepsilon_2^2}{2}\frac{\partial^2 F}{\partial N^{(2)}}.\end{aligned}$$

Consider the following operators defined on the functons $F\big(N_1^{(1)}, N^{(2)}\big)$:

(5.2)

$$\begin{aligned}L(t)F\big(N^{(1)}, N^{(2)}\big) &= AF\big(N^{(1)}, N^{(2)}\big) - \frac{k_1^2}{2}\big(N^{(1)}\big)^2 F\big(N^{(1)}, N^{(2)}\big)\\ &- \frac{k_2^2}{2}\big(N^{(2)}\big)^2 F\big(N^{(1)}, N^{(2)}\big),\\ \mu_t F\big(N^{(1)}, N^{(2)}\big) &= F\big(N^{(1)}, N^{(2)}\big)\exp\Big(-\sum_{i=1}^{2} k_i N^{(i)} y_t^i\Big),\end{aligned}$$

$$\mu_t^{-1} F\big(N^{(1)}, N^{(2)}\big) = F\big(N^{(1)}, N^{(2)}\big)\exp\Big(-\sum_{i=1}^{2} k_i N^{(i)} y_t^{(i)}\Big). \tag{5.3}$$

Denoting

$$\begin{aligned}\Delta\big(N^{(1)}\big) &= \big(A - BN^{(1)} - CN^{(2)}\big)N^{(1)},\\ \Delta\big(N^{(2)}\big) &= \big(-D + EN^{(1)}\big)N^{(2)},\end{aligned}$$

we have

$$\begin{aligned}\big[\mu_t L(t)\mu_t^{-1}\big]F\big(N^{(1)}, N^{(2)}\big) &= \sum_{i=1}^{2}\Big\{\frac{\varepsilon_1^2}{2}\frac{\partial^2 F}{\partial N^{(1)^2}} + \Big(\big(\Delta(N^{(i)}) - k_i\varepsilon_i^2 y_t^i\big)\frac{\partial F}{\partial N^{(i)}}\Big)\\ &+ \Big(\frac{k_i^2\varepsilon_i^2\big(y_t^i\big)^2}{2} - \Delta\big(N^{(i)}\big)y_t^i - \frac{k_i^2\big(N^{(i)}\big)^2}{2}\Big)F\big(N^{(1)}, N^{(2)}\big)\Big\}.\end{aligned} \tag{5.4}$$

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Let $\left(\widetilde{N}_t^{(1)}, \widetilde{N}_t^{(2)}\right)$ be the solution of the system

$$(5.5) \qquad \widetilde{N}_t^{(i)} = \widetilde{N}_0^{(i)} + \int_0^t \left(\Delta\left(N_s^{(i)} - k_i\varepsilon_i^2 y_s^i\right)ds + \varepsilon_i W_t^i\right), \quad i = 1, 2.$$

Considering the expectation, given $\mathfrak{Y}_t$, of

$$(5.6) \qquad F\left(\widetilde{N}_t^{(1)}, \widetilde{N}_t^{(2)}\right)\exp\left\{\int_0^t \sum_{i=1}^2 \left(\frac{\left(\varepsilon_i y^i k_i\right)^2}{2} - \Delta\left(\widetilde{N}_s^{(i)}\right)y_s^i - \frac{k_i^2}{2}\left(\widetilde{N}_s^{(i)}\right)^2\right)ds\right\}$$

and letting

$$(5.7) \quad \nu_t(F) = E\Bigg(F\left(\widetilde{N}_t^{(1)}, \widetilde{N}_t^{(2)}\right) \times \exp\left\{\int_0^t \sum_{i=1}^2 \left(\frac{\left(\varepsilon_i k_i y_s^i\right)^2}{2} - \Delta\left(\widehat{N}_s^{(i)}\right)y_s^{(i)} - \frac{k_1^2}{2}\left(\widehat{N}_s^{(i)}\right)^2\right)ds\right\} \Bigg| \mathfrak{Y}_t\Bigg),$$

where the expectation is taken over the forward sample paths, we have

$$(5.8) \qquad \nu_t(F) = V_0(F) + \int_0^t \nu_s(\mu_s L(s)\mu_s^{-1})F\left(N_s^{(1)}, N_s^{(2)}\right)ds,$$

that is, $\nu_t(F)$ solves equation (5.4). Applying Kunita's Theorem, the solution of the Zakai equation is given by

$$\sigma_t(F) = \nu_t(\mu_t F)(x).$$

Evaluating $\nu_t(\mu_t F)$, we have

$$(5.9) \quad \nu_t(\mu_t F) = \nu_0(F) + \int_0^t \nu_s(\mu_s L(s)\mu_s^{-1})(\mu_s F)ds$$
$$+ k_1 \int_0^t \nu_0\left(\mu_s\left(N_s^{(1)} F\left(N_s^{(1)}, N_s^{(2)}\right)\right)\right) \circ dy_s^1 + k_2 \int_0^t \nu_s\left(\mu_s\left(N_s^{(2)} F\left(N_s^{(1)}, N_s^{(2)}\right)\right)\right) \circ dy_s^2.$$

Hence, *the solution of the Zakai equation for the unnormalised conditional distribution* is given by

$$(5.10) \quad \sigma_t(F) = \nu_t(\mu_s F) = E\Bigg\{\left(\exp\sum_{i=1}^2 k_i \widehat{N}^{(i)} y_t^i\right) F\left(\widetilde{N}_t^{(1)}, N_t^{(2)}\right) \times \exp\left\{\int_0^t \sum_{i=1}^2 \left(\frac{\left(\varepsilon_i k_i y_i\right)^2}{2} \Delta\left(\widetilde{N}^{(i)}\right)y_s^i - \frac{k_i^2}{2}\left(\widetilde{N}^{(i)}\right)^2\right)ds\right\} \Bigg| \mathfrak{Y}_t\Bigg\}.$$

The advantage of this expression (5.10) over (4.16) is that it involves no stochastic integrals. The observation trajectories y_t^i, $i = 1, 2$, appear just as parameters. Further, the operator $\mu_t L(t)\mu_t^{-1}$ differs from $L(t)$ only by terms of less than second order.

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6. Zakai equation for unnormalised conditional density

Now, suppose that σ_t has a density $q_t\left(N^{(1)}, N^{(2)}\right)$. We then have

(6.1)

$$\sigma_t\left(F\left(N_t^{(1)}, N_t^{(2)}\right)\right) = \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) q_t\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$

$$= \int_{\Re_2} F\left(N^{(1)}, N^{(2)}\right) q_0\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$

$$+ \int_0^t \left(\int_{\Re^2} \frac{\partial F}{\partial N^{(1)}}\left(N^{(1)}, N^{(2)}\right)\left(N^{(1)}\left(A - BN^{(1)} - CN^{(2)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}\right) ds$$

$$+ \int_0^t \left(\int_{\Re^2} \frac{\partial F}{\partial N^{(2)}}\left(N^{(1)}, N^{(2)}\right)\left(N^{(2)}\left(-D + EN^{(1)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}\right) ds$$

$$+ \frac{\varepsilon_1^2}{2} \int_0^t \left(\int_{\Re^2} \left(\frac{\partial^2 F}{\partial N^{(1)^2}}\left(N^{(1)}, N^{(2)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}\right) ds$$

$$+ \frac{\varepsilon_2^2}{2} \int_0^t \left(\int_{\Re^2} \left(\frac{\partial^2 F}{\partial N^{(2)^2}}\left(N^{(1)}, N^{(2)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}\right) ds.$$

Since $F \in C_0^2(\Re^2)$ with compact support, integrating by parts yields

$$\int_{\Re^2} \frac{\partial F}{\partial N^{(1)}}\left(N^{(1)}, N^{(2)}\right)\left(N^{(1)}\left(A - BN^{(1)} - CN^{(2)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$

$$= F\left(N^{(1)}, N^{(2)}\right)\left(N^{(1)}\left(A - BN^{(1)} - CN^{(2)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) \Big|_{-\infty}^{\infty}$$

$$- \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \frac{\partial}{\partial N^{(1)}}\left(N^{(1)}\left(A - BN^{(1)} - CN^{(2)}\right) q_s\left(N^{(1)}, N^{(2)}\right)\right) dN^{(1)} dN^{(2)}$$

$$= - \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right)\Big\{\left(A - 2BN^{(1)} - CN^{(2)}\right) q_s\left(N^{(1)}, N^{(2)}\right)$$

$$+ \left(N^{(1)}\left(A - BN^{(1)} - CN^{(2)}\right)\right) \frac{\partial q_s}{\partial N^{(1)}}\left(N^{(1)}, N^{(2)}\right)\Big\} dN^{(1)} dN^{(2)},$$

$$\int_{\Re^2} \frac{\partial F}{\partial N^{(2)}}\left(N^{(1)}, N^{(2)}\right)\left(N^{(2)}\left(-D + EN^{(1)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$

$$= F\left(N^{(1)}, N^{(2)}\right)\left(N^{(2)}\left(-D + EN^{(1)}\right)\right) q_s\left(N^{(1)}, N^{(2)}\right) \Big|_{-\infty}^{\infty}$$

$$- \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right)\Big\{\left(-D + EN^{(1)}\right) q_s\left(N^{(1)}, N^{(2)}\right)$$

$$+ \left(N^{(2)}\left(-D + EN^{(1)}\right)\right) \frac{\partial q_s}{\partial N^{(2)}}\left(N^{(1)}, N^{(2)}\right)\Big\} dN^{(1)} dN^{(2)}$$

$$= - \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right)\Big\{\left(-D + EN^{(1)}\right) q_s\left(N^{(1)}, N^{(2)}\right)$$

$$+ \left(N^{(2)}\left(-D + EN^{(1)}\right)\right) \frac{\partial q_s}{\partial N^{(2)}}\left(N^{(1)}, N^{(2)}\right)\Big\} dN^{(1)} dN^{(2)},$$

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$$\int_{\Re^2} \frac{\partial^2 F}{\partial N^{(1)^2}} \left(N^{(1)}, N^{(2)}\right) q_s \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$= \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \frac{\partial^2 q_s}{\partial N^{(1)^2}} \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$\int_{\Re^2} \frac{\partial^2 F}{\partial N^{(2)^2}} \left(N^{(1)}, N^{(2)}\right) q_s \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$= \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \frac{\partial^2 q_s}{\partial N^{(2)^2}} \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}.$$

Substituting in equation (6.1) gives

$$\int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) q_t \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$= \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) q_0 \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$+ \int_0^t \left(\int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \left(\left(-A + 2BN^{(1)} + CN^{(2)}\right) q_s \left(N^{(1)}, n^{(2)}\right) \right.\right.$$
$$\left.\left. + \left(N^{(1)}\left(-A + BN^{(1)} + CN^{(2)}\right)\right) \frac{\partial q_s}{\partial N^{(1)}} \left(N^{(1)}, N^{(2)}\right) \right) dN^{(1)} dN^{(2)} \right) ds$$
$$+ \int_0^t \left\{ \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \left(\left(D - EN^{(1)}\right) q_s \left(N^{(1)}, N^{(2)}\right) \right.\right.$$
$$\left.\left. + \left(N^{(2)}\left(D - EN^{(1)}\right)\right) \frac{\partial q_s}{\partial N^{(2)}} \left(N^{(1)}, N^{(2)}\right) \right) \right\} \times \left(dN^{(1)} DN^{(2)}\right) ds$$
$$+ \frac{\alpha_1^2}{2} \int_0^t \left(\int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \frac{\partial^2 q_s}{\partial N^{(1)^2}} dN^{(1)} dN^{(2)} \right) ds$$
$$+ \frac{\epsilon_2^2}{2} \int_0^t \left(\int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \frac{\partial^2 q_s}{\partial N^{(2)^2}} dN^{(1)} dn^{(2)} \right) ds$$
$$+ k_1 \int_0^t \left(\int_{\Re^2} \left(F\left(N^{(1)}, N^{(2)}\right) N^{(1)}\right) q_s \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)} \right) dy_s^1$$
$$+ k_2 \int_0^t \left(\int_{\Re^2} \left(F\left(N^{(1)}, N^{(2)}\right) N^{(2)}\right) q_s \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)} \right) dy_s^2.$$

Hence

$$\sigma^t \left(F\left(N_t^{(1)}, N_t^{(2)}\right) \right) = \overline{E}\left[\Lambda_t F\left(N_t^{(1)}, N_t^{(2)}\right) \right] = \langle F, q_t \rangle$$
$$+ \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) q_t \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$= \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) q_0 \left(N^{(1)}, N^{(2)}\right) dN^{(1)} dN^{(2)}$$
$$+ \int_{\Re^2} F\left(N^{(1)}, N^{(2)}\right) \left\{ \int_0^t \left(\left(-A + 2BN^{(1)} + CN^{(2)}\right) q_s \left(N^{(1)}, N^{(2)}\right) \right.\right.$$

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$$+\left(N^{(1)}\left(-A+BN^{(1)}+CN^{(2)}\right)\right)\frac{\partial q_s}{\partial N^{(1)}}\left(N^{(1)},\,N^{(2)}\right)\Big)ds\Big\}dN^{(1)}dN^{(2)}$$

$$+\int_{\Re^2}F\left(N^{(1)},\,N^{(2)}\right)\Big\{\int_0^t\Big(\left(D-EN^{(1)}\right)q_s\left(N^{(1)},\,N^{(2)}\right)$$

$$+\left(N^{(2)}\left(D-EN^{(1)}\right)\right)\frac{\partial q_s}{\partial N^{(2)}}\left(N^{(1)},\,N^{(2)}\right)\Big)ds\Big\}\times dN^{(1)}dN^{(2)}$$

$$+\int_{\Re^2}F\left(N^{(1)},\,N^{(2)}\right)\left(\frac{\varepsilon_1^2}{2}\int_0^t\frac{\partial^2 q_s ds}{\partial N^{(1)^2}}\right)dN^{(1)}dN^{(2)}$$

$$+\int_{\Re^2}F\left(N^{(1)},\,N^{(2)}\right)\left(\frac{\varepsilon_2^2}{2}\int_0^t\frac{\partial^2 q_s ds}{\partial N^{(2)^2}}\right)dN^{(1)}dN^{(2)}$$

$$+\int_{\Re^2}F\left(N^{(1)},\,N^{(2)}\right)\left(k_1\int_0^t N^{(1)}q_s\left(N^{(1)},\,N^{(2)}\right)dy_s^1\right)dN^{(1)}dN^{(2)}$$

$$+\int_{\Re^2}F\left(N^{(1)},\,N^{(2)}\right)\left(k_2\int_0^t N^{(2)}q_s\left(N^{(1)},\,N^{(2)}\right)dy_s^2\right)dN^{(1)}dN^{(2)}.$$

Hence, *the adjoint Zakai equation for the unnormalised conditional density* $q_t\left(N^{(1)},\,N^{(2)}\right)$ *given* $\mathfrak{Y}_t=\sigma\left\{y_s^1,\,y_s^2;\;s\leqslant t\right\}$ is given by

(6.2)

$$q_t\left(N^{(1)},\,N^{(2)}\right)=q_0\left(N^{(1)},\,N^{(2)}\right)+\int_0^t\Big[\left(-A+2BN^{(1)}+CN^{(2)}\right)q_s\left(N^{(1)},\,N^{(2)}\right)$$
$$+\left(N^{(1)}\left(-A+BN^{(1)}+CN^{(2)}\right)\right)\frac{\partial q_s}{\partial N^{(1)}}\left(N^{(1)},\,N^{(2)}\right)\Big]ds$$
$$+\Big[\int_0^t\left(D-EN^{(1)}\right)q_s\left(N^{(1)},\,N^{(2)}\right)+\left(N^{(2)}\left(D-EN^{(1)}\right)\right)\frac{\partial q_s}{\partial N^{(2)}}\left(N^{(1)},\,N^{(2)}\right)\Big]ds$$
$$+\frac{\varepsilon_1^2}{s}\int_0^t\frac{\partial^2 q_s}{\partial N^{(1)^2}}ds+\frac{\varepsilon_2^2}{2}\int_0^t\frac{\partial^2 q_s}{\partial N^{(2)^2}}ds$$
$$+k_1\int_0^t N^{(1)}q_s\left(N^{(1)},\,N^{(2)}\right)dy^1+k_2\int_0^t N^{(2)}q_s\left(N^{(1)},\,N^{(2)}\right)dy^2.$$

The above expression again contains stochastic integrals. We once more apply the decomposition techniques of Kunita [**6**, Section 5.2] to enable us to obtain a density free of such integrals. The approach is similar to that adopted for the Zakai equation. Let

$$(6.9)\quad A^*q\left(N^{(1)},\,N^{(2)}\right)=-\frac{\partial}{\partial N^{(1)}}\Big[\left(A-BN^{(2)}-CN^{(2)}\right)N^{(1)}q\left(N^{(1)},\,N^{(2)}\right)\Big]$$
$$-\frac{\partial}{\partial N^{(2)}}\Big[\left(-D+EN^{(1)}\right)N^{(2)}q\left(N^{(1)},\,N^{(2)}\right)\Big]+\frac{\varepsilon_1^2}{2}\frac{\partial^2 q}{\partial N^{(1)^2}}+\frac{\varepsilon_2^2}{2}\frac{\partial^2 q}{\partial N^{(2)^2}}.$$

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The equation (6.2) takes the form

(6.10)

$$q_t\left(N^{(1)}, N^{(2)}\right) = q_0\left(N^{(1)}, N^{(2)}\right) + \int_0^t A^* q_s\left(N^{(1)}, N^{(2)}\right) ds$$
$$+ k_1 \int_0^t N^{(1)} q_s\left(N^{(1)}, N^{(2)}\right) dy_s^1 + k_2 \int_0^t N^{(2)} q_s\left(N^{(1)}, N^{(2)}\right) dy_s^2,$$

where A^* is the adjoint of A. In terms of Stratonovich integrals, this becomes

$$(6.11)\quad q_t\left(N^{(1)}, N^{(2)}\right) = q_0\left(N^{(1)}, N^{(2)}\right) + \int_0^t \left(A^* - \frac{k_1^2}{x}\left(N^{(1)}\right)^2 - \frac{k_2^2}{2}\left(N^{(2)}\right)^2\right)$$
$$\times q_s\left(N^{(1)}, N^{(2)}\right) ds + k_1 \int_0^t N^{(1)} q_s\left(N^{(1)}, N^{(2)}\right) \circ dy_s^1 + k_2 \int_0^t N^{(2)} q_s\left(N^{(1)}, N^{(2)}\right) \circ dy_s^2.$$

Consider the following operators defined on $q\left(N^{(1)}, N^{(2)}\right)$:

$$L^*(t) q\left(N^{(1)}, N^{(2)}\right) = A^* q\left(N^{(1)}, N^{(2)}\right) - \frac{k_1^2}{2}\left(N^{(1)}\right)^2 q\left(N^{(1)}, N^{(2)}\right)$$
$$(6.12)\qquad - \frac{k_2^2}{2}\left(N^{(2)}\right)^2 q\left(N^{(1)}, N^{(2)}\right),$$

$$(6.13)\qquad \mu_t q\left(N^{(1)}, N^{(2)}\right) = q\left(N^{(1)}, N^{(2)}\right).\exp\left(\sum_{i=1}^{2} k_i N^{(i)} y_t^i\right),$$
$$\mu_t^{-1} q\left(N^{(1)}, N^{(2)}\right) = q\left(N^{(1)}, N^{(2)}\right).\exp\left(-\sum_{i=1}^{2} k_i N^{(i)} y_t^i\right).$$

We then obtain

(6.14)

$$\left(\mu_t^{-1} L^*(t) \mu_t\right) q\left(N^{(1)}, N^{(2)}\right) = \frac{1}{2}\varepsilon_1^2 \frac{\partial^2 q}{\partial N^{(1)^2}} + \frac{1}{2}\varepsilon_2^2 \frac{\partial^2 q}{\partial N^{(2)^2}}$$
$$+ \left(\frac{1}{2} k_1 \varepsilon_1^2 y_t^1 - \Delta\left(N^{(1)}\right)\right) \frac{\partial q}{\partial N^{(1)}} + \left(\frac{1}{2} k_2 \varepsilon_2^2 y_1^2 - \Delta\left(N^{(2)}\right)\right) \frac{\partial q}{\partial N^{(2)}}$$
$$+ \left(\frac{1}{2} k_1^2 \varepsilon_1^2 \left(y_t^1\right)^2 - k_1 y_t^1 \Delta\left(N^{(1)}\right) - \frac{1}{2} k_1^2 \left(N_t^{(1)}\right)^2\right.$$
$$\left. + \left(A - 2BN^{(1)} - CN^{(2)}\right)\right) q\left(N^{(1)}, N^{(2)}\right)$$
$$+ \left(\frac{1}{2} k_2^2 \varepsilon_2^2 \left(y_1^2\right)^2 - k_2 y_t^2 \Delta\left(N^{(2)}\right) - \frac{1}{2} k_2^2 \left(N_t^{(2)}\right)^2 + \left(-D + EN^{(1)}\right)\right) q\left(N^{(1)}, N^{(2)}\right),$$

where

$$\Delta\left(N^{(1)}\right) = \left(A - BN^{(1)} - CN^{(2)}\right) N^{(1)},$$
$$\Delta\left(N^{(2)}\right) = \left(-D + EN^{(1)}\right) N^{(2)}.$$

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Let $\left(\widetilde{N}_t^{(1)}, \widetilde{N}_t^{(2)}\right)$ be the solution of the system

$$\widetilde{N}_t^{(i)} = \widetilde{N}_0^{(i)} + \int_0^t \left(\frac{1}{2}k_i\varepsilon_i^2 y_t^i - \Delta\left(\widetilde{N}^i\right)\right)ds + \varepsilon_i d\widetilde{W}_t^i. \tag{6.15}$$

Let us now consider the expectation, given $\mathfrak{Y}_t$, of

$$F\left(\widetilde{N}_t^{(1)}, N_t^{(2)}\right)\exp\left\{\int_0^t \sum_{i=1}^2 \left(\left(\varepsilon_i y^i k_i\right)^2 - k_i y^i \Delta\left(\widetilde{N}^{(i)}\right) - \frac{1}{2}k_t^2\left(\widetilde{N}^{(i)}\right)^2 + \frac{\partial}{\partial N^{(i)}}\left(\Delta\left(\widetilde{N}^{(i)}\right)\right)\right)ds\right\}. \tag{6.16}$$

Put

$$\nu_t(F) = E\Bigg(F\left(\widetilde{N}_t^{(1)}, \widetilde{N}_t^{(2)}\right) \times \exp\left\{\int_0^t \sum_{i=1}^n \left(\left(\varepsilon_i y^i k_i\right)^2 - k_i y^i \Delta\left(\widetilde{N}^{(i)}\right) - \frac{1}{2}k_i^2\left(\widetilde{N}^{(i)}\right)^2 + \frac{\partial}{\partial N^{(i)}}\left(\Delta\left(\widetilde{N}^{(i)}\right)\right)\right)ds\right\} \Bigg| \mathfrak{Y}_t\Bigg), \tag{6.17}$$

where

$$\frac{\partial}{\partial N^{(1)}}\left(\Delta\left(N^{(1)}\right)\right) = A - 2BN^{(1)} - CN^{(2)},$$
$$\frac{\partial}{\partial N^{(2)}}\left(\Delta\left(N^{(2)}\right)\right) = \left(D + EN^{(1)}\right),$$

and the expectation is taken over the backward sample paths. We then have

$$\nu_t(F) = \nu_0(F) + \int_0^t \nu_s(\mu_s^{-1}L^*(s)\mu_s)F\left(N^{(1)}, N^{(2)}\right)ds; \tag{6.18}$$

that is, $\nu_t(F)$ solves equation (6.14). Applying Kunita's theorem, the solution of the adjoint Zakai equation is given by

$$q_t(F) = \mu_t(\nu_t F)(x).$$

Evaluating $\mu_t(\nu_t F)$, we have

(6.19)

$$\mu_t(\nu_t F) = \mu_0(F) + \int_0^t L^*(s)\mu_s(\nu_s(F))ds + k_1\int_0^t N_s^{(1)}\mu_s(\nu_s(F)) \circ dW_s^1 + k_2\int_0^t N_s^{(2)}\mu_s(\nu_s(F)) \circ dW_s^2.$$

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Hence, *the solution of the adjoint Zakai equation for the unnormalised conditional density* is given by

(6.20)

$$q_t\left(N_t^{(1)}, N_t^{(2)}\right) = E\left[\exp \sum_{i=1}^{2} k_i y^i \widetilde{N}^{(i)}.F\left(N_t^{(1)}, N_t^{(2)}\right)\right.$$
$$\times \exp\left\{\int_0^t \left(\sum_{i=1}^{2} \frac{1}{2} k_i^2 \varepsilon_i^2 \left(y_s^i\right)^2 - k^i y_s^i \Delta\left(\widetilde{N}_s^{(i)}\right) - \frac{k_i^2}{2}\left(\widetilde{N}_s^i\right)^2\right.\right.$$
$$\left.\left.\left. -k_i y_s^i \frac{\partial}{\partial N^{(i)}}\left(\Delta\left(\widetilde{N}^{(i)}\right)\right)\right) ds\right\} \,\middle|\, \mathfrak{Y}_t\right].$$

Note the appearance of the divergence $\partial/(\partial N^{(1)})((a - BN^{(1)} - CN^{(2)})N^{(1)})$ and $\partial/(\partial N^{(2)})((-D + EN^{(1)})N^{(2)})$ in formula (6.20).

This formula is compatible with that obtained for the unnormalised conditional distribution where the expectation is over the forward sample paths and the divergence terms do not appear because there we worked with the Zakai equation itself.

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