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A MODIFIED PROJECTION METHOD FOR EQUATIONS OF THE SECOND KIND

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A modified projection method is suggested for the approximate solution of second kind equations and it is compared with other methods.

1. Introduction

Let T be a bounded operator on a Banach space X such that 1 is not in the spectrum of T, and (π_n) a sequence of finite rank projections on X such that $\lim_{n\to\infty} \pi_n x = x$ for all $x \in X$. The purpose of this note is to introduce a new approximation method, we call it *a modified projection method*, to obtain an approximation for the solution of the equation

$$x - Tx = y \,, \tag{1.1}$$

with the help of the projection π_n. It is to be noted that our assumption that 1 does not belong to the spectrum of T implies the existence of a unique solution for (1.1). We discuss the advantage of our method over the well-known approximation methods, namely, the Galerkin method, the iterated Galerkin method of Sloan and the Kantorovich method considered by Schock.

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The Galerkin approximation x_n^G is the solution of

$$x_n^G - \pi_n T x_n^G = \pi_n y \ . \tag{1.2}$$

The iterated Galerkin approximation (or Sloan approximation) x_n^S is defined by

$$x_n^S = y + T x_n^G \ . \tag{1.3}$$

The Kantorovich method considered by Schock [4,5,6] determines an approximation x_n^K as

$$x_n^K = y + z_n \ ,$$

where z_n is the solution of

$$z_n - \pi_n T z_n = \pi_n T y \ . \tag{1.4}$$

It can be verified that x_n^S and x_n^K satisfy the following equations:

$$x_n^S - T \pi_n x_n^S = y$$

$$x_n^K - \pi_n T x_n^K = y \ .$$

It is known that under certain conditions (for example, if T is a compact operator on a Hilbert space X and π_n is an orthogonal projection), the iterated Galerkin approximation x_n^S converges to the solution x of (1.1) faster than x_n^G (see Sloan [7] and also Sloan, Noussair and Burn [8]). Also, if T is an integral operator

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt \ , \ a \le s \le b$$

on a suitable function space, then, under some requlority conditions on y and the kernel k, x_n^S has better convergence rates than x_n^G (see Chandler [1]). It is observed in Schock [5] (see also [4]) that if T is a 'smoothing' operator then the Kantorovich approximation x_n^K is superior to both the Galerkin approximation x_n^G and the iterated Galerkin approximation x_n^S. Also, in a recent paper, Schock [6] showed that the Kantorovich method has a better global convergence behaviour than the

Galerkin and iterated Galerkin methods. But the disadvantage of the iterated Galerkin and the Kantorovich methods is that in one case we have to calculate Tx_n^G and in the other $\pi_n Ty$ - both these, in practice, would introduce some extra errors in the approximation when they are calculated by numerical methods. To overcome this difficulty we define a new (modified projection) approximation x_n^M as

$$x_n^M = x_n^G + (1 - \pi_n)y \ .$$

It should be noted that for the computation of x_n^G it is necessary to know $\pi_n y$ (see (1.2)). Thus x_n^M involves no more computation than x_n^G. The definition of x_n^M is motivated by the expression for x_n^S in (1.3): x_n^M is obtained from (1.3) by replacing Tx_n^G by $\pi_n Tx_n^G$. Using (1.2) we get

$$y + \pi_n Tx_n^G = y + (x_n^G - \pi_n y) = x_n^G + (1 - \pi_n)y \ .$$

Thus x_n^M is a modified form of both x_n^S and x_n^G. It can also be seen that x_n^M satisfies the equation

$$x_n^M - \pi_n T\pi_n x_n^M = y \ .$$

In the next Section we show that x_n^M converges to the solution x of (1.1) faster that x_n^G whenever x_n^S and x_n^K converge faster than x_n^G - we have already mentioned that there are cases when the latter situation occurs.

2. Main Results

We recall the definitions from Section 1:

$$x_n^S = y + Tx_n^G$$

$$x_n^K = y + z_n \ ,$$

where z_n is the solution of $z_n - \pi_n Tz_n = \pi_n Ty$, and

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$$x_n^M = x_n^G + (1 - \pi_n)y .$$

THEOREM 2.1. *For $y \in X$, let x be the solution of (1.1). Then*

$$\|x - x_n^M\| \le \|\pi_n\| \, \|x - x_n^S\| + \|1 - \pi_n\| \, \|x - x_n^K\| .$$

Proof. The crucial point of the proof is the observation

$$x_n^K = \pi_n x_n^K + (1 - \pi_n)y .$$

This follows from the definition of x_n^K :

$$(1 - \pi_n)x_n^K = (1 - \pi_n)(y + z_n) = (1 - \pi_n)y .$$

Then we have

$$\begin{aligned} x_n^K - x_n^M &= (\pi_n x_n^K + (1 - \pi_n)y) - (x_n^G + (1 - \pi_n)y) \\ &= \pi_n(x_n^K - x_n^S) , \end{aligned}$$

since $\pi_n x_n^S = x_n^G$. Thus,

$$\begin{aligned} x - x_n^M &= (x - x_n^K) + (x_n^K - x_n^M) \\ &= (x - x_n^K) + \pi_n(x_n^K - x_n^S) \\ &= (x - x_n^K) + \pi_n(x_n^K - x) + \pi_n(x - x_n^S) \\ &= \pi_n(x - x_n^S) + (1 - \pi_n)(x - x_n^K) , \end{aligned}$$

so that

$$\|x - x_n^M\| \le \|\pi_n\| \, \|x - x_n^S\| + \|1 - \pi_n\| \, \|x - x_n^K\| . \qquad \square$$

For the next result we introduce the following definition from Schock [6].

DEFINITION 2.1. *Suppose that x_n is an approximation to the solution $x = (1 - T)^{-1}y$ of (1.1) obtained by an approximation method with the help of the projection π_n . The method is said to be converging if $\lim_{n \to \infty} (x - x_n) = 0$ for all $y \in X$. A converging method is arbitrarily slowly converging if for each monotone decreasing positive null sequence (w_n) , that is $w_n \to 0$, there is a $y \in X$ such that $(\|x - x_n\|)$*

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converges more slowly than (w_n), *that is* $\lim_{n\to\infty} (\|x - x_n\| / w_n) = \infty$. *Otherwise the method is said to converge uniformly.*

THEOREM 2.2. *Let* X *be a Hilbert space and let* π_n*'s be orthogonal projections. Then we have the following:*

(1) $\|x - x_n^M\| \le 2 \max\{\|x - x_n^S\|, \|x - x_n^K\|\}$

(2) If T *is a compact operator, then the 'modified projection method' converges uniformly.*

(3) If T *is a compact self-adjoint operator, then the orders of convergence of* x_n^S, x_n^K *and* x_n^M *are the same.*

Proof. The result in (1) follows directly from Theorem 2.1 by noting that for an orthogonal projection π_n, $\|\pi_n\| = \|I - \pi_n\| = 1$.

(2) It is proved in Schock ([6], Theorem 3.1 (1)) that if T is a compact operator, then the iterated Galerkin method and the Kantorovich method converge uniformly. Hence by Theorem 2.1, the result follows.

(3) Considering the 'remainder operators' R_n^S, R_n^K as $R_n^S x = x - x_n^S$, $R_n^K x = x - x_n^K$, it is shown in Schock [6, Theorem 3.1 (2)] that, if T is compact and self-adjoint, then there exists a constant $c \ge 1$ such that $\frac{1}{c}\|R_n^S\| \le \|R_n^K\| \le c\|R_n^S\|$. From this relation and Theorem 2.1, we get

$$\|x - x_n^M\| = O(\|R_n^S\|) = \|x - x_n^S\|,$$

$$\|x - x_n^M\| = O(\|R_n^K\|) = \|x - x_n^K\|. \qquad \square$$

Computational remarks. Suppose that the range of the projection π_n has dimension n. Let $e_1, \ldots, e_n$ be a basis of range π_n and $e_1^*, \ldots, e_n^*$ in X^* an adjoint basis, that is $\langle e_i, e_j^* \rangle = \delta_{ij}$. Then π_n is given by

$$\pi_n x = \sum_{i=1}^{n} \langle x, e_i^* \rangle e_i, \qquad x \in X.$$

In the Galerkin method we are required to find $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ such

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that $x_n^G = \sum_{i=1}^{n} \alpha_i e_i$ satisfies the equation (1.2). This is equivalent to solving the system

(*) $$\alpha - A_n\alpha = \gamma$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)'$, where $A_n = (\langle Te_j, e_i^* \rangle)$ and $\gamma = (\langle y, e_1^* \rangle, \ldots, \langle y, e_n^* \rangle)'$. (Here α and γ are column vectors.) Once the equation (*) is solved, the iterated Galerkin approximation x_n^S is given by

$$x_n^S = y + \sum_{i=1}^{n} \alpha_i Te_i .$$

To obtain the Kantorovich approximation x_n^K, we have first to solve (1.4), that is, to find $(\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ such that $z_n = \sum_{i=1}^{n} \beta_i e_i$ satisfies (1.4). This is equivalent to solving the system

(**) $$\beta - A_n\beta = \delta ,$$

for $\beta = (\beta_1, \ldots, \beta_n)'$, where $\delta = (\langle Ty, e_1^* \rangle, \ldots, \langle Ty, e_n^* \rangle)$. Then

$$x_n^K = y + \sum_{i=1}^{n} \beta_i e_i .$$

Now, knowing the solution $\alpha = (\alpha_1, \ldots, \alpha_n)'$ of (*), the modified projection approximation x_n^M is given by

$$x_n^M = y + \sum_{i=1}^{n} \alpha_i e_i - \sum_{i=1}^{n} \langle y, e_i^* \rangle e_i .$$

Thus, the modified projection approximation x_n^M requires no more computation than x_n^G, except for some additions and subtractions; whereas for x_n^S and x_n^K we need to calculate Te_i and $\langle Ty, e_i^* \rangle$, respectively - these quantities often require further numerical approximations. For example suppose that T is a compact integral operator on $C[a,b]$ (the space of continuous functions on $[a,b]$ with supremum norm), given by

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$$(Tx)(s) = \int_a^b k(s,t)\, x(t)dt \ , \quad a \le s \le b \ .$$

If π_n is an interpolatory projection associated with a mesh $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$, then Te_i and $\langle Ty, e_i^* \rangle$ are the integrals.

$$\int_a^b k(s,t)e_i(t)dt \quad \text{and} \quad \int_a^b k(t_i,t)\, y(t)dt$$

respectively. If π_n is an orthogonal projection with $e_i^* = e_i$ then $\langle Ty, e_i^* \rangle$ is the double integral

$$\int_a^b \int_a^b k(s,t)\, y(t)\, e_i(s)dt\, ds \ .$$

All these above integrals are, often, computed by numerical quadrature rules, which introduce errors in the approximations x_n^S and x_n^K - these errors, in general, depend on the smoothness of the kernel $k(s,t)$ and the given function $y(t)$. Error analysis for the approximations x_n^G and x_n^S , when the integrals are approximated by quadrature rules, are considered by Joe in [2] and [3].

References

[1] G.A. Chandler, *Superconvergence of numerical solutions to second kind integral equations*. (Ph.D. Thesis, Australian National University, Canberra, 1979).

[2] S. Joe, "Discrete collocation methods for second kind Fredholm integral equations", *SIAM J. Numer. Anal.* 22 (1985), 1167-1177.

[3] S. Joe, "Discrete Galerkin methods for Fredholm integral equations of the second kind", *IMA J. Numer. Anal.* (to appear).

[4] E. Schock, "Uber die Konvergenzgeschwindigkeit projektiver Verfahren I, II " , *Math. Z.* 120 (1971) 148-156; 127 (1972), 191-198.

[5] E. Schock, "Galerkin-like methods for equations of the second kind", *J. Integral Equations* 4 (1982), 361-364.

[6] E. Schock, "Arbitrarily slowly convergence, uniform convergence and superconvergence of Galerkin-like methods", *IMA J. Numer. Anal.* 5 (1985), 153-160.

[7] I.H. Sloan, "Improvement by iteration for compact operator equations", *Math. Comp.* 30 (1976), 758-764.

[8] I.H. Sloan, E. Noussair and J. Burn, "Projection methods for equations of the second kind", *J. Math. Anal. Appl.* 69 (1979), 84-103.

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