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ON THE RANK OF A p-GROUP OF CLASS 2

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ABSTRACT. Let $d(G)$ denote the minimal number of generators of the finite p-group G, $r(G)$ the maximum over all subgroups H of G of $d(H)$ and $r_a(G)$ the maximum over all abelian subgroups H of G of $d(H)$. If G is of class two it is clear that

$$d([G, G]) \leq r_a(G) \leq r(G) \leq d(G/[G, G]) + d([G, G]).$$

By considering properties of the stability number of graphs we construct examples which show that any value of $r(G)$ within these bounds can occur.

§1. If G is a group let $d(G)$ denote the minimal number of generators of G, and $r(G) = \max\{d(H) \mid H \leq G\}$, the rank of G. Let $r_a(G) = \max\{d(H) \mid H \leq G$ and H abelian$\}$. Patterson [3] has shown that if p is an odd prime then any finite p-group G has a subgroup N of nilpotency class at most two with $d(N) = r(G)$.

It is clear that in any group $r_a(G) \leq r(G)$; we shall look at groups in which

(I) $$r_a(G) = r(G)$$

that is groups with an abelian subgroup N with $d(N) = r(G)$. It was shown by Wehrfritz [4] that if G is the iterated wreath product of cyclic groups of order $p > 3$, then G has the stronger property that

(II) any subgroup H of G with $d(H) = r(G)$ must be abelian,

and one might ask whether this characterises such groups. We shall construct a family of p-groups of class two and exponent p which show among other things, that the answer is no.

If G is any nilpotent group of class two then it is clear that $d(G') \leq r_a(G) \leq r(G) \leq d(G/G') + d(G')$. We shall show that even among p-groups satisfying (I) it is possible for every value of $r(G)$ within these bounds to occur.

Our results follow from a construction which associates to any graph on n vertices and any odd prime p an n-generator group of class at most two and exponent p. We denote by $\mathcal{G}(n, p)$ the set of all such groups arising from graphs on n vertices.

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THEOREM. *Let p be an odd prime, and n an integer greater than two. There is a family $\mathcal{G}(n, p)$ of n generator groups of exponent p and nilpotency class at most two with the following properties.*

(i) *Every group in $\mathcal{G}(n, p)$ has property* I.

(ii) *If $n-1 \geq a \geq 2$ then $X_a = \{G \in \mathcal{G}(n, p) \mid r(G) = a + r(G')\}$ has unique elements G_1 and G_2 of largest and smallest order respectively, and G_1 has* II *and G_2 does not.*

(iii) *As*
$$n \to \infty, \qquad \frac{|X_{a_1}| + |X_{a_2}|}{|\mathcal{G}(n, p)|} \to 1$$
where $a_1 = \lfloor d \rfloor$, $a_2 = \lceil d \rceil$ and d is the real solution of
$$\binom{n}{d} = 2\binom{d}{2} \quad i.e. \quad d = \frac{2 \log n}{\log 2} + 0(\log \log n),$$

(iv) *X_n consists of the elementary abelian p-group p on n generators P; $r_a(P) = r(P) = n = n + r(P')$. X_1 consists of the free group Q of nilpotency class* 2 *and exponent p on r generators; $r_a(Q) = r(Q) = 1 + r(Q')$. Both P and Q have* II.

Part (iii) leads one to wonder whether a similar result is true if $\mathcal{G}(n, p)$ is replaced by the set of all n generator p-groups of exponent p, in other words whether almost all such groups would have ranks drawn from some finite set $d_1(n), \ldots, d_k(n)$.

It is perhaps worth noticing that Patterson's result does not hold for arbitrary nilpotent groups. For if $C = A \mathbin{\jmath} B$ where A is free abelian on three generators $\{a, b, c\}$ and B is cyclic generated by t, where $a^t = ab^3$, $b^t = bc^3$, $c^t = c$ then G has class 3 and $d(G) = 4$. If H is a subgroup of G with $d(H) = 4$ then $d(H) = 4 \leq d(H/H \cap A) + d(H \cap A) \leq 1 + 3$ so $d(H/H \cap A) = 1$ and $d(H \cap A) = 3$. But then H has the same class as G.

§2. **Some combinatorics.** In this section we describe the graphs needed for our construction in Section 3. Let Γ be a finite non-directed graph with $e(\Gamma)$ edges. We denote by K_r the complete graph on r vertices; then $e(K_r) = r(r-1)/2$. A subset T of the vertices of Γ involves a particular edge if both ends of it lie in T; in other words the edge lies in the subgraph spanned by T. The stability number of Γ (also called the vertex independence number in the literature) is $a(\Gamma) = \max\{m \mid \exists m$ vertices in Γ which involve no edge$\}$. It is clear that there is a unique graph Γ_1 on n vertices with stability number a and maximal number of edges: Γ_1 is the complement of the graph on n vertices consisting of a K_a and $(n-a)$ isolated vertices, and $e(\Gamma_1) = \binom{n}{2} - \binom{a}{2}$.

It is a famous result of Turan (in for example [1] Chapter 4, Section 2) that there is a unique graph Γ_2 on n vertices with stability number a and minimal number of edges. In fact if $n = ra + s$ with $0 \leq s < a$ then Γ_2 consists of s copies of K_{r+1} and $(a-s)$ copies of K_r, so
$$e(\Gamma_2) = s\binom{r+1}{2} + (a-s)\binom{r}{2} = \frac{1}{2}\left\lfloor\frac{n}{a}\right\rfloor\left\{2n - a\left(\left\lfloor\frac{n}{a}\right\rfloor + 1\right)\right\}.$$

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Thus if Γ has n vertices and stability number a then

$$e(\Gamma_2) \leq e(\Gamma) \leq \binom{n}{2} - \binom{a}{2},$$

and $e(\Gamma)$ can taken any value in this range.

It has been shown by Bollabas (see [1] Chapter 7, Section 4) that if A_n is the set of all graphs on n vertices, then the stability number of almost all graphs in A_n takes one of two possible values, d_1 and d_2. That is to say if $B_n = \{\Gamma \in A_n \mid a(\Gamma) = d_1 \text{ or } a(\Gamma) = d_2\}$, then $|B_n|/|A_n| \to 1$ as $n \to \infty$. Explicitly $d_1 = \lfloor d \rfloor$, $d_2 = \lceil d \rceil$, where d is the real solution to

$$\binom{n}{d} = 2^{\binom{d}{2}},$$

and $d = 2 \log n/\log 2 + 0(\log \log n)$.

If Γ has stability number a then any $(a+1)$ vertices must involve at least one edge. This generalises as follows.

LEMMA 1. *Let Γ be a graph on n vertices with stability number $1 \leq a \leq n$, and suppose any $(a+1)$ vertices involve at least $k \geq 1$ edges. Then any $(a+r)$ vertices involve at least $k+r-1$ edges, for $1 \leq r \leq n-a$.*

Proof. We use induction on r; the case $r=1$ is given. Suppose any $a+d$ vertices involve at least $k+d-1$ edges, $d \geq 1$. Let T be a set of $a+d-1$ vertices. Then for $d \geq 2$, T involves at least one edge. Removing one end of this from T leaves a set of $a+d-2$ vertices, which by induction involve at least $k+d-2$ edges, so T involves at least $k+d-1$ edges. □

§3. **The construction.** In this section we construct the elements of $\mathcal{G}(n, p)$. Let p be an odd prime, and n an integer greater than two.

Now given a graph Γ on n vertices $\{V_1, \ldots, V_n\}$ we construct a p-group G as follows. Let F be the free group of nilpotency class two and exponent p on n generators $\{X_1, \ldots, X_n\}$, so F/F' and $F' = Z(F)$ are elementary abelian on n and $\binom{n}{2}$ generators respectively, and F'**O** is generated by $\{[X_i, X_j] \mid i \neq j, 1 \leq i, j \leq n\}$. See for example [2]. Let X be the subgroup of F' generated by $\{[X_i, X_j] \mid V_i$ is not adjacent to V_j in $\Gamma\}$, so X is elementary abelian on $\binom{n}{2} - e(\Gamma)$ generators. As X is a subgroup of $Z(F)$, X is normal in F. We use bars to denote images modulo X; let $G = \bar{F}$. Then $d(G) = d(F) = n$, and $G' = \bar{F}'$ so $d(G') = d(\bar{F}') = d(F') - d(X) = e(\Gamma)$.

LEMMA 2. *Let Γ be a graph on n vertices with e edges and stability number a, and suppose any $(a+1)$ vertices of Γ involve at least $k \geq 1$ edges. Then if p is an odd prime, and L is any subgroup of the p-group G constructed as above, $d(L) \leq a+e$. If further $k \geq 2$ then $d(L) = a+e$ only if L is abelian.*

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Proof. As G' is central in G, $d(L)\le d(L, G')$, so we may assume $L\ge G'$. Let $L=\bar{H}$, and let $h=d(H/F')=d(L/G')$. If $h=1$ then L/G' is cyclic so L is abelian, and $d(L)=1+d(G')=1+e=h+e$. If $h>1$ pick a basis $\{u_1F',\ldots,u_hF'\}$ for H/F'. Let $D=\langle u_1,\ldots,u_h\rangle$. Then $D'=H'$, and D is free of exponent p on h generators, so $d(D')=d(H')=\binom{h}{2}$. Suppose that $u_1F'=X_1^{\alpha_{11}}\cdots X_n^{\alpha_{1n}}F',\ldots,u_hF'=X_1^{\alpha_{h1}}\cdots X_n^{\alpha_{hn}}F'$. Then by performing row operations and relabelling the X_i if necessary we may assume $u_i=X_in_i$, $1\le i\le h$, where $n_i\in\langle X_{h+1},\ldots,X_n\rangle$. If $i<j$, let $t_{ij}=[u_i,u_j]=[X_in_i,X_jn_j]=[X_i,X_j]m_{ij}$, where m_{ij} involves no $[X_r,X_s]$ with $1\le r,s\le h$. Then $\{t_{ij}\mid 1\le i<j\le h\}$ form a basis of H', and $[X_i,X_j]$ occurs in t_{ij} but in no other t_{rs} with $r\ne i$, $s\ne j$. Let T be the subgraph of Γ on vertices $\{V_1,\ldots,V_h\}$.

Suppose $d(H'\cap X)=x$; then $H'\cap X$ lies in the span of at least x of the t_{ij}. If some linear combination of the t_{ij} lies in $H'\cap X$ then each of the corresponding $[X_i,X_j]$ lies in X, so there is no edge joining V_i and V_j in Γ, and hence in T. Thus $x\le(\#\text{ of non edges of }T)=\binom{h}{2}-e(T)$.

It follows that

$$\begin{aligned}
d(L)&=d(L/L')\\
&=d(L/G')+d(G'/L')\\
&=h+d(G')-d(L')\\
&=h+e-d(L') &&(*)\\
&=h+e-(d(H')-d(H'\cap X)) &&(**)\\
&\le h+e-\binom{h}{2}+\left(\binom{h}{2}-e(T)\right)\\
&=h+e-e(T).
\end{aligned}$$

Now since any $(a+1)$ element subset of the vertices of Γ involves at least k edges, then by Lemma 1 any $(a+r)$ element subset involves at least $(k+r-1)$ edges. Thus if $h>a$, say $h=a+r$ with $r\ge1$, then $e(T)\ge r+k-1$ and $d(L)\le a+r+e-(r+k-1)=a+e+1-k\le a+e$, and if $k\ge2$, $d(L)<a+e$. If $h=a$ then $d(L)\le a+e-e(T)$, and if $d(L)=a+e$ then by $(*)$ $d(L')=0$, so L is abelian. If $h<a$ then $d(L)\le h+e+e(T)<a+e$. So in any case $d(L)\le a+e$, and if $k\ge2$ then $d(L)=a+e$ only if L is abelian. □

Now let A_n be the set of graphs on n vertices. If p is an odd prime and $\Gamma\in A_n$ let $G(\Gamma,p)$ be the group constructed above. Let $\mathcal{G}(n,p)=\{G(\Gamma,p)\mid\Gamma\in A_n\}$. Then each element of $\mathcal{G}(n,p)$ is nilpotent of class at most two and exponent p, and $|G(\Gamma,p)|=p^{n+e(\Gamma)}$.

Lemma 3. *If* $G=G(\Gamma,p)\in\mathcal{G}(n,p)$ *and* Γ *has stability number* a *then*
(i) $r_a(G)=r(G)=a+r(G')$ *and* G *satisfies* (I).

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(ii) *G satisfies* (II) *if and only if any $a+1$ vertices of Γ involve at least two edges.*

Proof. (i) As Γ has stability number a we can find a vertices $\{V_1, \ldots, V_a\}$ of Γ involving no edges: then let $A = \langle X_1, \ldots, X_a, F' \rangle \leq F$. It follows that $A' \leq X$ so $\bar{A}$ is abelian and $d(\bar{A}) = a+e$, so $r_a(G) \geq a+e$. But by Lemma 2 $r(G) \leq a+e$, so $r(G) = r_a(G) = a+e$.

(ii) If any $(a+1)$ vertices involve at least two edges then by Lemma 2 any subgroup L with $d(L) = a+e$ is abelian, so G satisfies (II). If there is a set of $(a+1)$ vertices $\{V_1, \ldots, V_{a+1}\}$ which involve only one edge let $N = \langle X_1, \ldots, X_{a+1}, F' \rangle \leq F$. Then by $(**)$

$$d(\bar{N}) = (a+1) + e - \binom{a+1}{2} + d(N' \cap X),$$

and here

$$d(N' \cap X) = \binom{a+1}{2} - e(T)$$

where T is the subgraph generated by $\{V_1, \ldots, V_{a+1}\}$, so $d(\bar{N}) = (a+1) + e - e(T) = a+e$. However as T has one edge, V_iV_j say, then $[\bar{X}_i, \bar{X}_j] \neq 1$ so $\bar{N}$ is not abelian. $\square$

This enables us to prove the Theorem.

Proof. Part (i) follows from Lemma 3. Part (ii) follows from our remarks in Section 3; set $G_1 = G(\Gamma_1, p)$ and $G_2 = G(\Gamma_2, p)$. Clearly any $(a+1)$ vertices of G_1 involve at least $a \geq 2$ edges so by Lemma 2 G_1 has property II. On the other hand we can find $(a+1)$ vertices in G_2 which involve only one edge (for since $a < n$, Γ_2 consists of a complete graphs at least one of which contains more than one vertex: select two vertices from this complete graph and one from each of the other complete graphs). Thus by Lemma 3, G_2 does not have property II. Part (iii) follows from the result of Bollobas quoted in Section 2. Part (iv) is clear from Lemma 2.

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