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SUBINVARIANCE IN SOLVABLE LIE ALGEBRAS

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In a recent paper, Wielandt has continued his investigation of subnormal subgroups. Since the analogous concept is also of interest in Lie algebras, this note considers the Lie algebra counterparts to Wielandt's results. Generally the results do not carry over to all Lie algebras, but do hold in the solvable case. In order to state the main results, several definitions are needed and consequently we begin by listing some of the consequences. All Lie algebras considered here are finite dimensional over a field.

THEOREM 1. *Let L be a solvable Lie algebra and A be a subalgebra of L. Then the following are equivalent:*

(1) *A is subinvariant in L.*

(2) *For each $x \in L$, A is subinvariant in $\langle x, A \rangle$, the subalgebra generated by x and A.*

(3) *For each $x \in L$, A is subinvariant in $\langle A, A \text{ ad } x \rangle$.*

(4) *If $x \in \langle A, A \text{ ad } x \rangle$, then $x \in A$.*

(5) *For each $a \in A$, $L = A + L_0(a)$ where $L_0(a)$ is the Fitting null component of* ad *a acting on L and the sum is a vector space sum.*

Note that the equivalence of (1) and (5) for the case where A is nilpotent is essentially shown in [**4**].

The following notation is used. $A \subseteq L (A \trianglelefteq L, A \trianglelefteq\trianglelefteq L)$ will stand for A is a subalgebra (ideal, subinvariant) in L. If $a \in A \subseteq L$, then $L_0(a)$ and $L_1(a)$ are the Fitting null and one components of ad a acting on L. $C_L(A)$ denotes the centralizer of A in L and $Z(A)$ stands for the center of A. A^L denotes the smallest ideal of L which contains A and is called the *normal closure* of A in L. A^ω will be the intersection of all members of the lower central series of A. Following Wielandt, by a property $\mathfrak{e}$ we mean a class of pairs (A, L) of Lie algebras where $A \subseteq L$ and the following are taken to be equivalent:

$(A, L) \in \mathfrak{e}$ and $A \mathrel{\mathfrak{e}} L$.

In particular $\trianglelefteq\trianglelefteq$ denotes all pairs (A, L) of Lie algebras such that $A \trianglelefteq\trianglelefteq L$. Evidently each of the conditions in Theorem 1 satisfies the following.

Definition 1. Let $\mathfrak{e}$ be a property on pairs of Lie algebras. Then $\mathfrak{e}$ satisfies I if whenever $A \mathrel{\mathfrak{e}} L$ and $A \subseteq B \subseteq L$, then $A \mathrel{\mathfrak{e}} B$.

To show that a property satisfying I implies subinvariance, we may assume a minimal counterexample and obtain the following situation.

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Definition 2. Define the property $\mathfrak{f}$ as follows: $A \,\mathfrak{f}\, L$ if $A \subseteq L$, A is not subinvariant in L but A is subinvariant in all B such that $A \subseteq B \subset L$.

Also related to various of the conditions in Theorem 1 is the following.

Definition 3. Define the property $\mathfrak{m}$ as follows: $A \,\mathfrak{m}\, L$ if $A \subseteq L$ and
(α) A is contained in a unique maximal subalgebra M of L;
(β) $A \trianglelefteq \trianglelefteq M$; and
(γ) for each $x \in L$ such that $x \notin M$, $\langle A, A \text{ ad } x\rangle = L$.

The concepts which have been introduced are related as follows.

THEOREM 2. *Let* $\mathfrak{e}$ *be a property satisfying I. In the class of solvable Lie algebras the following are equivalent:*
(1) $\mathfrak{e} \subseteq \trianglelefteq \trianglelefteq$
(2) $\mathfrak{e} \cap \mathfrak{m} = \emptyset$.

THEOREM 3. *In the class of solvable Lie algebras*, $\mathfrak{f} = \mathfrak{m}$.

First we note that these results do not hold for arbitrary Lie algebras as is seen in the 3-dimensional simple Lie algebra A_1 with basis e, h, g and multiplication $[e, h] = 2e$, $[g, h] = -2g$ and $[e, g] = h$. Now $\langle e\rangle$ is not subinvariant in A_1 but $L_0(e) = A_1$ and (5) of Theorem 1 holds. Furthermore (3) and (4) also hold. This same example shows that $\mathfrak{f} \neq \mathfrak{m}$ since $\langle e\rangle \mathfrak{f}\, A_1$ while in $M = \langle e, h\rangle$, $g \notin M$ but $\langle e, e \text{ ad } g\rangle \in M \neq A_1$. Hence $\mathfrak{f} \not\subseteq \mathfrak{m}$.
Before beginning the verification of the above results, we note that the difference in the Lie algebra and group cases appears in the proof of Theorem 3. Here we use a result (Theorem 4) which apparently has no counterpart in group theory. The remainder of the results follow by arguments like those in the group theory case.

Proof of Theorem 3. If $A \,\mathfrak{m}\, L$, let M be as in Definition 3. If $A \trianglelefteq \trianglelefteq L$, then $\langle A, A \text{ ad } x\rangle \subseteq A^L \subset L$ for each $x \in L$. This contradicts Definition 3. Suppose $A \subseteq B \subset L$. Then $B \subseteq M$ and $A \trianglelefteq \trianglelefteq B$. Hence $A \,\mathfrak{f}\, L$ and $\mathfrak{m} \subseteq \mathfrak{f}$. For the reverse inclusion we need the following.

THEOREM 4. *Let L be a solvable Lie algebra. Suppose that $A \,\mathfrak{f}\, L$ and that M is a maximal subalgebra of L, $A \subseteq M$. Then there exists $x \in A$ such that $A + L_0(x) = M$.*

Proof. Note that generally $A + L_0(x)$ is only a vector space sum. Also since $A \trianglelefteq \trianglelefteq M$, it follows that $A + L_0(x) \supseteq M$ for all $x \in A$. This holds since $M_1(x) \subseteq A$, hence $M = M_0(x) + M_1(x) \subseteq M_0(x) + A \subseteq L_0(x) + A$. Consequently we need only show that there exists an $x \in A$ such that $L_0(x) \subseteq M$.
Case I. Suppose that there exists a minimal ideal C of L such that $C \subseteq M$. Then $(A + C/C) \,\mathfrak{f}\, (L/C)$. By induction there exists $\bar{x} \in A + C/C$ such that $L_0(\bar{x}) \subseteq \bar{M}$. Then $L_0(x) \subseteq M$ since if $y \in L$ and $y \text{ ad}^n x = 0$, then

$(y + C)\mathrm{ad}^n(x + C) = (y \operatorname{ad}^n x) + C = C$ and $\bar{y} \in \bar{M}$. Hence $y \in M$ and $L_0(x) \subseteq M$.

Case II. No minimal ideal of L is contained in M. Then L is a solvable primitive algebra, and so has a unique minimal ideal D which is self-centralizing and complemented by M.

Suppose that A is nilpotent. Since $A \trianglelefteq \trianglelefteq M$, $L_0(x) \supseteq M$ for all $x \in A$. If $L_0(x) = L$ for all $x \in A$, then $A \trianglelefteq \trianglelefteq L$ by [**4**, Lemma 5] which contradicts $A\ \mathfrak{f}\ L$. Hence $L_0(x) \subset L$ for some $x \in A$ and since $L_0(x)$ is a subalgebra of L, $M = L_0(x)$ which completes the proof in this case.

Suppose that A is not nilpotent. Now $A^\omega \trianglelefteq M$ by [**3**, Theorem 3] and $A^\omega \neq 0$. Hence A^ω contains a minimal ideal B of M. Since M is solvable, B is abelian. Since $B \trianglelefteq M$, $L_0(x) \supseteq M$ for all $x \in B$. Hence either $L_0(x) = M$ for some $x \in B$ and we are done or $L_0(x) = L$ for all $x \in B$. Assume the second possibility holds and let $E = D + B$. Then E is nilpotent by Engel's Theorem and $E \trianglelefteq L$ since $E/D \simeq B \trianglelefteq M \simeq L/D$. Since $C_L(D) = D$, $C_E(D) = D$ follows. Therefore $Z(E) \subseteq D$. But $Z(E) \trianglelefteq L$ since $Z(E)$ is characteristic in E. By the minimality of D, either $Z(E) = 0$, which contradicts the nilpotency of E, or $Z(E) = D$, which yields $E = C_E(Z(E)) = C_E(D) = D$ and $B = 0$, another contradiction. Hence $L_0(x) \subseteq M$ for some $x \in A$ and $L_0(x) + A = M$. This completes the proof of Theorem 4.

We return to the proof of Theorem 3 and suppose that $A\ \mathfrak{f}\ L$. Let M be a maximal subalgebra of L which contains A and let $x \in A$ such that $M = A + L_0(x)$. If M_1 is a maximal subalgebra of L containing A, then $A \trianglelefteq \trianglelefteq M_1$ and $M_1 \subseteq A + L_0(x) = M$. Hence $M = M_1$ and (α) of Definition 3 is satisfied. Also (β) is clearly satisfied. Suppose that $y \in L$, $y \notin M$ and $\langle A, A \operatorname{ad} y\rangle \neq L$. Then $\langle A, A \operatorname{ad} y\rangle \subseteq M$ and $A \operatorname{ad} y \subseteq M$. Let $y = s + t$, $s \in L_0(x)$, $t \in L_1(x)$ and note that $t \notin M$ since $s \in L_0(x) \subseteq M$ and $y \notin M$. Then $[x, t] = [x, y] - [x, s] \in M$. On the other hand, ad x is non-singular on $L_1(x)$ and $M \cap L_1(x)$ is ad x invariant. Hence for any $z \in L_1(x)$, if $z \operatorname{ad} x \in M \cap L_1(x)$, then $z \in M \cap L_1(x)$. In particular, this holds for $z = t$ and $t \in M$, a contradiction. Hence (γ) of Definition 3 is satisfied and $A\ \mathfrak{m}\ L$. Therefore $\mathfrak{m} = \mathfrak{f}$ and Theorem 3 is shown.

We turn now to the proof of Theorem 2. If $\mathfrak{e} \subseteq \trianglelefteq \trianglelefteq$, then $\mathfrak{e} \cap \mathfrak{m} \subseteq (\trianglelefteq \trianglelefteq) \cap \mathfrak{m} = \emptyset$. If $\mathfrak{e} \not\subseteq \trianglelefteq \trianglelefteq$, then there exists $A \subseteq L$ such that $A\ \mathfrak{e}\ L$ and A is not subinvariant in L. Take a minimal such L. Hence $A\ \mathfrak{f}\ L$ and $A\ \mathfrak{m}\ L$ by Theorem 3. Since $A\ \mathfrak{e}\ L$, $A(\mathfrak{e} \cap \mathfrak{m})L$ and $\mathfrak{e} \cap \mathfrak{m} \neq \emptyset$. This completes the proof of Theorem 2.

Now for the proof of Theorem 1. The equivalence of (1) and (5) follows from Theorem 4. Clearly (1) implies (2) which in turn implies (3). To show that (3) implies (1) assume that (A, L) is a minimal counterexample. Then $A\ \mathfrak{f}\ L$ and $\langle A, A \operatorname{ad} x\rangle$ is properly contained in L for all $x \in L$. Then for each $a \in A$, there exists a positive integer n such that $L \operatorname{ad}^n a \in A$, hence $L_1(a) \subseteq A$

and $L_0(A) + A = L$. Hence (5) holds and (1), (2), (3) and (5) are equivalent. In order to show that (4) implies (1), let $\mathfrak{e}$ be the property: $A \mathfrak{e} L$ if when $x \in \langle A, A \text{ ad } x\rangle$, then $x \in A$ for all $x \in L$. Clearly $\mathfrak{e}$ satisfies I. Suppose $A \mathfrak{e} L$ and $A \mathfrak{m} L$. If $x \in L$ and $x \notin M$ where M is as in Definition 3, then $x \in \langle A, A \text{ ad } x\rangle = L$. Hence $x \in A \subseteq M$, a contradiction. Hence $\mathfrak{e} \cap \mathfrak{m} = \emptyset$ and $\mathfrak{e} \subseteq \trianglelefteq\trianglelefteq$ by Theorem 2. To show that $\trianglelefteq\trianglelefteq \subseteq \mathfrak{e}$, let $A \trianglelefteq\trianglelefteq L$ and take $x \in L$, $x \notin A$. Let N be the normal closure of A in $\langle A, x\rangle$. Since N is properly contained in $\langle A, x\rangle$, $x \notin N$. Also $\langle A, A \text{ ad } x\rangle \subseteq N$. Hence $x \notin \langle A, A \text{ ad } x\rangle$ and $\trianglelefteq\trianglelefteq \subseteq \mathfrak{e}$. Hence (1) and (4) are equivalent and this completes the proof of Theorem 1.

Wielandt uses his results to find conditions guaranteeing the nilpotency of the normal closure of certain subgroups. In the Lie algebra case we have

THEOREM 5. *Let L be a solvable Lie algebra. If L is of characteristic* 0, *then $A \subseteq N(L)$ if A is nilpotent and $x \in \langle A, A \text{ ad } x\rangle$ implies that $x \in A$. The result fails at characteristic p.*

Proof. $A \trianglelefteq\trianglelefteq L$ by Theorem 1 and A is nilpotent. Then for each $a \in A$, ad a is nilpotent acting on L and the enveloping associative algebra of $\text{ad}_L A$ is nilpotent by Engel's Theorem [**2**, p. 36] where $\text{ad}_L A$ is the Lie algebra of all ad a acting on L, $a \in A$. If L is of characteristic 0, then $\text{ad}_L A$ is contained in the radical of $(\text{ad } L)^*$, the enveloping associative algebra of ad L, by [**1**, Theorem 7.2]. Hence $A \subseteq N(L)$ by [**2**, p. 36, Theorem 3].

At characteristic p we consider the example in [**2**, p. 53]. Let L be the Lie algebra with basis $e_1, \ldots, e_p, E, G, H$ over a field of characteristic p with multiplication $[e_i, e_j] = 0$, $[e_i, E] = e_{i+1}$, $[e_i, H] = e_i$, $[e_i, G] = (i-1)e_{i-1}$, $[E, G] = H$ and $[E, H] = [G, H] = 0$ (everything mod p). Then $N(L) = \langle e_1, \ldots, e_p\rangle$ and $\langle G\rangle$ satisfies all the equivalent conditions of Theorem 1 since ad G is nilpotent. But $G \notin N(L)$ and the result fails.

A concept of interest in finite solvable groups is the *subnormalizer* $S(H)$ of the subgroup H of G, defined to be the largest subgroup of G in which H is subnormal, if such a subgroup exists. In solvable Lie algebras, the analogous concept always exists as follows from

THEOREM 6. *Let L be a solvable Lie algebra and A be a subalgebra of L. Suppose that B and C are subalgebras of L such that $A \trianglelefteq\trianglelefteq B$ and $A \trianglelefteq\trianglelefteq C$. Then $A \trianglelefteq\trianglelefteq \langle B, C\rangle$.*

Proof. We may assume that $L = \langle B, C\rangle$. Since $A^\omega \trianglelefteq B$, $A^\omega \trianglelefteq C$ by [**3**, Theorem 3], it follows that $A^\omega \trianglelefteq L$. We can work in L/A^ω, hence we may assume that $A^\omega = 0$. Thus A is nilpotent and $L_0(a) = \langle B, C\rangle$ for all $a \in A$. By [**4**, Lemma 5], $A \trianglelefteq\trianglelefteq L$.

In part (3) of Theorem 1, the conditions can be weakened.

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THEOREM 7. *Let L be a solvable Lie algebra and A be a subalgebra of L. Then the following are equivalent:*

(1) $A \trianglelefteq \trianglelefteq L$.

(2) $A \trianglelefteq \trianglelefteq \langle A, A \text{ ad } x\rangle$ *for all* $y \in L$ *and for all* $x \in A$ ad y.

Proof. By Theorem 1, (1) implies (2). Let $\mathfrak{e}$ be the property: $A \,\mathfrak{e}\, L$ if whenever $y \in L$ and $x \in A$ ad y, then $A \trianglelefteq \trianglelefteq \langle A, A \text{ ad } x\rangle$. We need to show that $\mathfrak{e} \subseteq \trianglelefteq \trianglelefteq$. Since $\mathfrak{e}$ satisfies I, by Theorem 2 it suffices to show that $\mathfrak{e} \cap \mathfrak{m} = \emptyset$. Suppose $A \,\mathfrak{e}\, L$ and $A \,\mathfrak{m}\, L$ and let M be the unique maximal subalgebra of L containing A. If there exists $y \in L$ such that A ad $y \nsubseteq M$, then there exists $x \in A$ ad y and $x \notin M$ which implies that $L = \langle A, A \text{ ad } x\rangle$ by (γ) of Definition 3. By assumption $A \trianglelefteq \trianglelefteq \langle A, A \text{ ad } x\rangle = L$. Since $A \,\mathfrak{m}\, L$ and $\mathfrak{m} = \mathfrak{f}$ this is a contradiction. Hence A ad $y \subseteq M$ for all $y \in L$. Hence L ad $a \subseteq [L, A] \subseteq M$ for all $a \in A$. Since $A \trianglelefteq \trianglelefteq M$, there exists a positive integer k such that $M \text{ ad}^k a \subseteq A$ for all $a \in A$. Hence $L \text{ ad}^{k+1} a \subseteq M \text{ ad}^k a \subseteq A$ for all $a \in A$. Hence $L_1(a) \subseteq A$ and $L = A + L_0(a)$ for all $a \in A$. By Theorem 1, $A \trianglelefteq \trianglelefteq L$ and this is a contradiction. Hence $\mathfrak{e} \cap \mathfrak{m} = \emptyset$ and $\mathfrak{e} \subseteq \trianglelefteq \trianglelefteq$.

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