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Homogeneous Lie algebras

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It is shown that the automorphism group of a real Lie algebra operates transitively on the set of its one-dimensional subspaces iff the Lie algebra is abelian, or isomorphic to the algebra of skew-symmetric 3×3 real matrices. This allows to conclude that R, $SO(2)$, $SO(3)$ and $Spin(3)$ are the only connected Lie groups such that:

(1) the conjugates of every connected set containing e cover a neighbourhood of e,

(2) every point sufficiently close to e lies on exactly one 1-parameter subgroup.

Let G be the group of 3×3 orthogonal matrices with positive determinant, that is, the rotations of R^3. It was observed by J. Mycielski that G has the following property

(*) if S is a non-trivial connected subset of G containing the identity, then $\bigcup_{x \in G} xSx^{-1}$ is a neighbourhood of the identity.

This can be seen as follows. Associate with each $z \in G$ its rotation angle $\varphi(z)$; $0 \leq \varphi(z) \leq \pi$. Since $\varphi : G \to R$ is continuous, the sets $V_\varepsilon = \{z \mid \varphi(z) < \varepsilon\}$ are open for every $\varepsilon > 0$. If $S \subset G$ is connected, non-trivial and $e \in S$, then $\varphi(S)$ contains an interval $\{t \mid 0 \leq t < \varepsilon\}$ for some $\varepsilon > 0$, whence $V_\varepsilon \subset \bigcup_{x \in G} xSx^{-1}$, since any two rotations by the same angle are conjugate.

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J. Mycielski posed the problem of determining all connected Lie groups G which have the property (*).

It is clear that (*) is possessed by the additive group of reals R and by the circle group S^1. Moreover, if (*) holds for G and there are arbitrary small neighbourhoods of the identity in G which are invariant under all inner automorphisms, then it is easily seen that (*) holds also for any group locally isomorphic to G. Thus (*) holds also for $Spin(3)$. Suppose a group G has the property

(**) there is a neighbourhood V of e such that through every $x \in V$ there passes at most one 1-parameter subgroup S of G.

If moreover (*) holds, then it is clear that the inner automorphisms of G operate transitively on the one-parameter subgroups. This leads to the following concept.

DEFINITION. Call a Lie algebra *homogeneous* if its automorphism group operates transitively on the set of its 1-dimensional subspaces.

Thus the Lie algebra of a group having properties (*) and (**) is homogeneous. The main purpose of this note is to prove the

THEOREM. *The only homogeneous Lie algebras are*

i) the abelian Lie algebras,

ii) the algebra A_1 *of skew-symmetric* 3×3 *real matrices.*

One concludes that the only connected Lie groups having properties (*) and (**) are R, S^1, $SO(3)$ and $Spin(3)$.

To prove the Theorem, assume that $\underline{g}$ is a non-abelian homogeneous Lie algebra. Since A_1 is known to be the only semi-simple, compact Lie algebra of rank 1 ([2], Chapter XI), the Theorem follows from the subsequent three Lemmas.

LEMMA 1. $\underline{g}$ *is semi-simple.*

Proof. Suppose the radical $\underline{r}$ of $\underline{g}$ is non-trivial. Since $\underline{r}$ is invariant under automorphisms of $\underline{g}$, $\underline{r}$ must coincide with $\underline{g}$, that is $\underline{g}$ is solvable. Let $\underline{g}^{(m)}$ be the last non-vanishing term of the derived series of $\underline{g}$. Then $\underline{g}^{(m)}$ is an abelian ideal which is invariant under

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all automorphisms of $\underline{g}$, hence $\underline{g}^{(m)} = \underline{g}$. It follows that $\underline{g}$ is abelian, a contradiction.

LEMMA 2. $\underline{g}$ *is compact.*

It will be shown that $\mathrm{Int}(\underline{g})$, the adjoint group of $\underline{g}$, is compact. This will suffice, since the adjoint representation $\underline{g} \to \mathrm{ad}(\underline{g})$ is an isomorphism and $\mathrm{ad}(\underline{g})$ is the Lie algebra of $\mathrm{Int}(\underline{g})$.

Let $\underline{g} = \underline{k} + \underline{p}$ be a Cartan decomposition, and let $\mathrm{ad}_{\underline{g}}(\underline{k})$ be the image of $\underline{k}$ under the adjoint representation $\mathrm{ad} : \underline{g} \to \mathrm{ad}(\underline{g})$. Denote by $\mathrm{Int}_{\underline{g}}(\underline{k})$ the subgroup of $\mathrm{Int}(\underline{g})$ corresponding to the subalgebra $\mathrm{ad}_{\underline{g}}(\underline{k})$ of $\mathrm{ad}(\underline{g})$. Then it is well known that $\mathrm{Int}_{\underline{g}}(\underline{k})$ is non-trivial and compact ([*1*], Chapter III, Proposition 7.4). Hence $\mathrm{Int}(\underline{g})$ contains a compact 1-parameter subgroup T .

Let $X \in \underline{g}$ be the vector such that T is tangent to $\mathrm{ad}X$ at 0 . Consider an automorphism σ of $\underline{g}$ and the corresponding 1-parameter subgroup $\sigma T \sigma^{-1}$ of $\mathrm{Int}(\underline{g})$. Then $\sigma T \sigma^{-1}$ is tangent to $\mathrm{ad}(\sigma X)$ at 0 , as can be seen from the identity

$$\sigma e^{t\mathrm{ad}X} \sigma^{-1} = e^{t\mathrm{ad}(\sigma X)} \quad \text{for all } t \in R .$$

By assumption, $\mathrm{ad}(\sigma X)$ runs over all of $\mathrm{ad}(\underline{g})$ when σ runs over the automorphism group of $\underline{g}$. It follows that every 1-parameter subgroup of $\mathrm{Int}(\underline{g})$ is compact, and thus $\mathrm{Int}(\underline{g})$ is compact ([*1*], Chapter I, Proposition 10.7).

LEMMA 3. $\underline{g}$ *is of rank* 1 .

Proof. Assume the notation of [*2*], Chapter XI. Let n be the rank of $\underline{g}$,

$$n = \min_{X \in \underline{g}} \dim \ker(\mathrm{ad}X : \underline{g} \to \underline{g}) .$$

An $X \in \underline{g}$ at which this minimum is attained is called regular. By the assumption of homogeneity every element of $\underline{g}$ is regular. Let $X_0 \in \underline{g}$ be arbitrary and denote $\underline{s} = \ker \mathrm{ad}X_0$. Then $\underline{s}$ is an abelian subalgebra of dimension n and there is a subset Σ of non-zero vectors in $\underline{s}$ (the

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root system of $\underline{g}$) with the following property: If $\underline{g}^C$ and $\underline{s}^C$ denote the complexifications of $\underline{g}$ and $\underline{s}$, then there is an injection $\Sigma \to \underline{g}^C$ given by $\alpha \mapsto r_\alpha$ for every $\alpha \in \Sigma$, such that,

a) the r_α ; $\alpha \in \Sigma$ are independent, and together with $\underline{s}^C$ they span $\underline{g}^C$,

b) $[X, r_\alpha] = -iB(\alpha, X)r_\alpha$ for every $\alpha \in \Sigma$, $X \in s$, where B is the Killing form of $\underline{g}$.

Now consider for any $X \in \underline{g}$ the characteristic polynomial $\chi(\lambda, X) = \det(\mathrm{ad}X - \lambda I)$ of $\mathrm{ad}X$. It is known that if X is regular then the multiplicity of the root $\lambda = 0$ equals the rank n of $\underline{g}$. Thus, in the present case, for every $X \in \underline{g}$

$$\chi(\lambda, X) = \lambda^r + \chi_1(X)\lambda^{r-1} + \ldots + \chi_{r-n}(X)\lambda^n \ ; \ (r = \dim \underline{g})$$

with $\chi_{r-n}(X) \neq 0$. Extend $\mathrm{ad}X : \underline{g} \to \underline{g}$ to the complexification $\underline{g}^C$. Denote the extended map also by $\mathrm{ad}X : \underline{g}^C \to \underline{g}^C$. In a basis common to both $\underline{g}$ and $\underline{g}^C$, both $\mathrm{ad}X : \underline{g} \to \underline{g}$ and $\mathrm{ad}X : \underline{g}^C \to \underline{g}^C$ have the same matrix. Hence $\chi(\lambda, X)$ is also the characteristic polynomial of $\mathrm{ad}X : \underline{g}^C \to \underline{g}^C$.

Now let $X \in \underline{s}$. Choosing a basis of $\underline{g}^C$ composed of a basis $S_1, \ldots, S_n$ of $\underline{s}$ and the vectors r_α ; $\alpha \in \Sigma$, one has by a), b) above that $\mathrm{ad}X : \underline{g}^C \to \underline{g}^C$ is represented by a diagonal matrix whose only non-zero terms are $-iB(\alpha, X)$; $\alpha \in \Sigma$. Thus,

$$\chi_{r-n}(X) = \prod_{\alpha \in \Sigma} (-i)^{r-n} B(\alpha, X) \ .$$

Since $\chi_{r-n}(X) \neq 0$ for every $X \in \underline{s}$, it follows that $B(\alpha, X) \neq 0$ for all $\alpha \in \Sigma$, $X \in \underline{s}$. This is possible only if $\dim \underline{s} = 1$, that is if $n = 1$.

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