AFFINE SUBPLANES OF FINITE PROJECTIVE PLANES

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Introduction. Let π be a finite projective plane of order n containing a finite projective subplane π^* of order $u < n$. Bruck has shown (**1**, p. 398) that if π contains a point that does not lie on any line of π^*, then $n \geqslant u^2 + u$, while if every point of π lies on a line of π^* then $n = u^2$.

Let π be a finite projective plane of order n containing a finite *affine* subplane π_0 of order $m < n$. Ostrom and Sherk have shown (**5**, p. 551) that if π contains a point that does not lie on any line of π_0, then $n \geqslant m^2 - 1$, while if every point of π lies on a line of π_0, then $m^2 - 1 \geqslant n \geqslant m^2 - m + 1$, except for the special case $m = 3$, $n = 4$.

In this paper we deal only with the case in which *every point of* π *lies on a line of* π_0, except in § 6. If we write $n = m^2 - 1 - k$, the above result states that $0 \leqslant k \leqslant m - 2$, except when $m = 3$, $n = 4$. We prove here that

$$k + 1 \leqslant \tfrac{1}{2}(m + 1) \quad \text{except when} \quad m = 3,\ k = 4,\ n = 4,$$

and

$$k + 1 \geqslant (m + 1)^{\frac{1}{2}} \quad \text{except when} \quad m = 2,\ k = 0,\ n = 3.$$

We also slightly improve this second inequality in certain cases, after a deeper investigation of the structure of π (cf. § 1).

Examples of planes of this type are known to exist when $m = 3$, $n = 4$ and when $m = 3$, $n = 7$ (§ 1, and **5**, p. 556), also in the trivial case $m = 2$, $n = 3$. We find no new examples of such planes in this paper. However, the above inequalities show that n cannot be a square and the results of § 6 (quoted in the next paragraph), show that π cannot be Desarguesian, except in the examples already known. This restricts the choice of π in any search for new examples.

In § 6 we assume that π is Desarguesian (finite or infinite) and we drop the restriction that every point of π lies on a line of π_0. We show that if π_0 has order greater than 3, then π_0 also is Desarguesian, that the lines of a parallel-class in π_0 all meet at a common vertex in π, that the vertices of all parallel-classes are collinear in π, and that in the finite case n is a power of m. Ostrom and Sherk have shown (**5**, p. 556) that these results are not always true if $m = 3$ (except that π_0 must of course be Desarguesian).

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For ease of reference, results will be numbered consecutively, irrespective of whether they are called theorems or lemmas.

I should like to thank F. A. Sherk and the referee for their helpful suggestions.

1. Basic definitions and theorems. A *projective plane* is a system of elements called *points* and *lines*, together with a relation of *incidence*, satisfying the following axioms:

(i) Any two distinct points are incident with just one line.
(ii) Any two distinct lines are incident with just one point.
(iii) There exist four points, no three of which are incident with the same line.

An *affine plane* is a system of elements called *points* and *lines*, together with a relation of *incidence*, satisfying the following axioms:

(i) Any two distinct points are incident with just one line.
(ii) Given a line l and a point P not incident with l, there exists exactly one line l' incident with P which does not meet l (two lines *meet* if they are incident with the same point).
(iii) There exist three points not all incident with the same line.

We shall use the usual terminology of incidence, namely "lies on," "passes through," "collinear," "concurrent," etc. With this terminology the axioms assume a more familiar look.

Two lines of an affine plane that do not meet are called *parallel.*

A projective or affine plane is finite if it contains only a finite number of points and lines.

Axiom (iii) in each case is used to exclude trivial uninteresting planes, such as that represented diagrammatically in Figure 1, which satisfies axioms (i) and (ii) for projective planes.

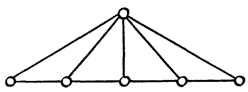

FIGURE 1

THEOREM 1.1 (**1**, p. 348).

(a) *If one line of a finite projective plane* π *contains* $n + 1$ *points, then* $n \geqslant 2$ *and:*

Every line of π *contains* $n + 1$ *points.*

Through every point of π *there pass* $n + 1$ *lines.*

The plane π *contains* $n^2 + n + 1$ *points and* $n^2 + n + 1$ *lines.*

(b) *If one line of a finite affine plane* π_0 *contains* m *points, then* $m \geqslant 2$ *and:*

Every line of π_0 *contains* m *points.*

Through every point of π_0 *there pass* $m + 1$ *lines.*

The plane π_0 *contains* m^2 *points and* $m^2 + m$ *lines.*

The lines of π_0 *can be divided into* $m + 1$ *mutually exclusive parallel-classes containing* m *lines each, two lines belonging to the same parallel-class if and only if they are parallel.*

Only (a) is proved in **(1)**, but (b) can be proved similarly.

If n and m are defined as in the above theorem, we call π a projective plane *of order* n and π_0 an affine plane of *order* m. In an affine plane, a parallel class consists of a set of lines, every pair of which are parallel. Lines from distinct parallel-classes intersect.

Theorem 1.1 plays a fundamental part in any discussion of finite planes. It will not be quoted explicitly each time it is used.

An important question about finite planes is this. Given an integer n, how many types of projective or affine planes of order n exist (if any)? All the planes known at present have prime-power order, but the only restriction on the order up to now is given by

THEOREM 1.2 (The Bruck–Ryser Theorem, **1**, p. 394). *If there exists a projective or affine plane of order* n, *and if* $n \equiv 1$ *or* $2 \pmod 4$, *then* n *is expressible as the sum of the squares of two integers.*

It follows that there are no finite planes of order 6, but it is not known whether or not a finite plane of order 10 exists ($10 \equiv 2 \pmod 4$, but $10 = 1^2 + 3^2$).

One method of trying to construct planes is to investigate *subplanes*. A *subplane* (projective or affine) of a plane π is a system consisting of a subset of the points of π and a subset of the lines of π which itself forms a projective or affine plane with respect to the incidence already defined in π.

It is well known that if we take a projective plane π of order n and remove a single line l and all the points on it, then the resulting system π_0 is an affine plane of order n, a subplane of π. Lines of π (other than l) concurrent in a point of l form a parallel class in π_0.

Conversely, if we take an affine plane π_0 of order n and add to it $n + 1$ new points, each new point being incident with every line of a given parallel-class and with no other line of π_0, distinct new points being incident with distinct parallel-classes, and if we also add one new line incident with all the new points but with no point of π_0, then the resulting system is a projective plane π of order n; π_0 is a subplane of π. Let us call this plane π the *projective extension of* π_0. We shall use this term in § 6.

The following result is due to R. H. Bruck (**1**, p. 398):

THEOREM 1.3. *Let* π *be a projective plane of order* n, *containing a projective subplane* π^* *of order* $u < n$. *If* π *contains a point that does not lie on any line of* π^*, *then* $n \geqslant u^2 + u$. *If every point of* π *lies on a line of* π^*, *then* $n = u^2$.

Bruck also raised the question of what can be said in the case of *affine* subplanes. This question was first considered by Ostrom and Sherk, but before quoting their results we shall consider some examples. These examples will help to clarify the situation we shall be considering, and will give some indication of the type of diagram to be used later on.

Table I gives an affine plane of order 3 consisting of the nine points $A, B, C, D, E, F, G, H, K$. The twelve lines have not been labelled, but the table signifies that there is a line containing just the three points A, B, C, etc. The lines have been divided into the four parallel-classes.

TABLE I

ABC
DEF
GHK
ADG
BEH
CFK
BDK
CEG
AFH
BFG
AEK
CDH

TABLE II

$ABCXY$	$AMRVZ$
$DEFXZ$	$BLQUZ$
$GHKYZ$	$CNPWZ$
$ADGPQ$	$DNRUY$
$BEHPR$	$EMQWY$
$CFKQR$	$FLPVY$
$BDKVW$	$GLRWX$
$CEGUV$	$HNQVX$
$AFHUW$	$KMPUX$
$BFGMN$	
$AEKLN$	
$CDHLM$	

Figure 2 gives an incomplete representation of this plane. It is impossible to give a complete representation of the abstract points and lines of the plane by Euclidean points and lines. In fact A lies on FH, C on DH, G on BF, and K on BD. The other eight lines are completely represented. Do not be misled by the diagram. For instance, AE and BD meet at K, not at a non-existent point "inside the square $ABED$."

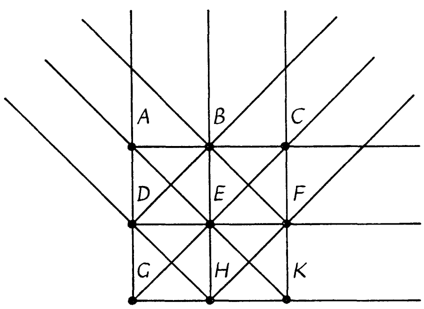

FIGURE 2

We can embed this affine plane of order 3 in a projective plane of order 3 as described above. The result is illustrated in Figure 3, where the four new points are denoted by J_1, J_2, J_3, J_4. Note that A still lies on the line HFJ_3, etc.

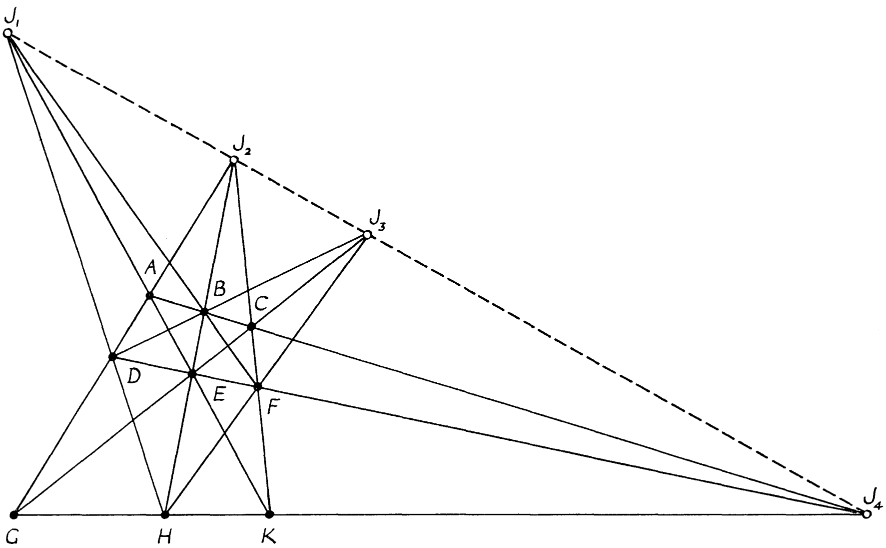

FIGURE 3

Table II gives a projective plane of order 4, consisting of $4^2 + 4 + 1 = 21$ points and 21 lines. If we consider only the nine points A, B, C, D, E, F, G, H, K and the first twelve lines of the plane, we obtain Table I. Thus our affine plane of order 3 is a subplane of our projective plane of order 4. This is illustrated in Figure 4. All 21 points are shown, but we have given up

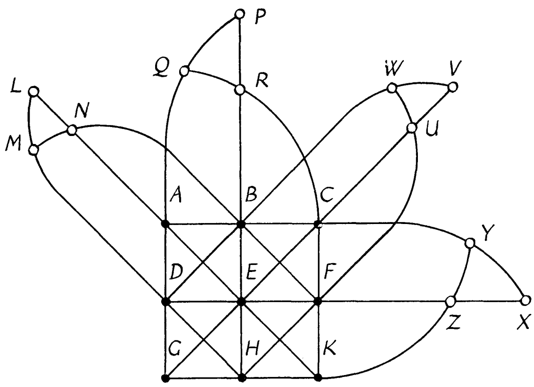

FIGURE 4

using straight Euclidean lines, and no attempt has been made to show the last nine lines of Table II. Note that A still lies on $HFUW$, etc.

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Suppose now that the projective plane π of order n contains an affine subplane π_0 of order m. If $m = n$, then π can be obtained from π_0 only by the method already described. Clearly we cannot have $m > n$. Hence we shall assume from now on that $m < n$. Ostrom and Sherk (**5**, p. 551) have proved

THEOREM 1.4. *If π contains a point that does not lie on any line of π_0, then $n \geqslant m^2 - 1$. If every point of π lies on a line of π_0, then either $m = 3$ and $n = 4$ (the example considered above) or $m^2 - 1 \geqslant n \geqslant m^2 - m + 1$.*

As we remarked in the Introduction, in this paper we deal only with the case in which *every point of π lies on a line of π_0*, except in § 6.

We shall write $n = m^2 - 1 - k$. Theorem 1.4 states that either $m = 3$, $k = 4$, $n = 4$ or $0 \leqslant k \leqslant m - 2$. We shall prove the following results:

Either $m = 3$, $k = 4$, $n = 4$ *or* $k + 1 \leqslant \frac{1}{2}(m + 1)$, in § 2.
Either $m = 2$, $k = 0$, $n = 3$ *or* $k + 1 \geqslant (m + 1)^{\frac{1}{2}}$, in § 3.

After further investigations of the structure of π in § 4, we improve the results of § 3 in § 5, showing that:

Either $m = 2$, $k = 0$, $n = 3$, *or* $m = 3$, $k = 1$, $n = 7$, *or* $7 \leqslant m \leqslant 12$, $k \geqslant (m - 3)^{\frac{1}{2}}$, *or* $m \geqslant 13$, $k \geqslant (m - 4)^{\frac{1}{2}}$.

In § 6 we prove the result about Desarguesian planes mentioned in the first paragraph of the paper. The results of § 5 serve to increase by 1 the lower bound obtained for k in § 3, for most but not all values of m. One is tempted to say that the extra information obtained does not justify the amount of extra calculation used to obtain it, but these calculations do show, in the absence of examples for $m > 3$, that if any significant improvement is possible in the lower bound for k, it must be obtained by using much stronger inequalities than we have used here.

It is useful to bear in mind that when $n = 2, 3, 4, 5, 7, 8$ there is just one type of projective (or affine) plane of order n, to within isomorphism, namely that which can be co-ordinatized by using the Galois field GF(n) of n elements (**2**; **3**; **4**). There is no projective or affine plane of order 6 (by the Bruck–Ryser theorem, 1.2) or of order 1.

The symbol $\pi - \pi_0$ denotes the set of those points and lines of π that are not points or lines of π_0.

2. Initial results.

If $m = 2$, then $n = 3$ by 1.4. When $m = 2$, π_0 consists of a quadrangle and its six sides. This configuration is contained in every projective plane, so the case $m = 2$, $n = 3$ certainly exists. We shall assume henceforth that $m > 2$.

We can draw a diagram showing the m^2 points of π_0 arranged in a square

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and the $m^2 + m$ lines of π_0 in $m + 1$ parallel-classes of m lines each. The remaining points of π all lie on the lines of π_0 (since this is our assumption throughout §§ 2–5). Each line of π_0 contains $n + 1$ points of π, and so contains $n + 1 - m$ points of $\pi - \pi_0$. *The total number of points of* $\pi - \pi_0$ *lying on the lines of a particular parallel-class depends on how these lines intersect in* π. (Parallel lines of π_0 must meet in π, since π is a *projective* plane.) The m lines of a parallel-class may all meet in a single point of π, as in Figure 3, in which case we shall say that they form a *pencil* in π, or they may meet by twos in $\frac{1}{2}m(m - 1)$ points of π, as in Figure 4, or an intermediate situation may occur. Various possibilities are shown in Figure 5 with $m = 5$. Only three of the six parallel-classes are shown there. Figure 5 and some of the subsequent figures are intended only as helpful illustrations of various situations and should not be taken too literally. We shall show that $m = 5$ is impossible in planes of the type under discussion, and in Figure 9, for example, which shows $m = 5$, we must in fact have $m \geqslant 11$.

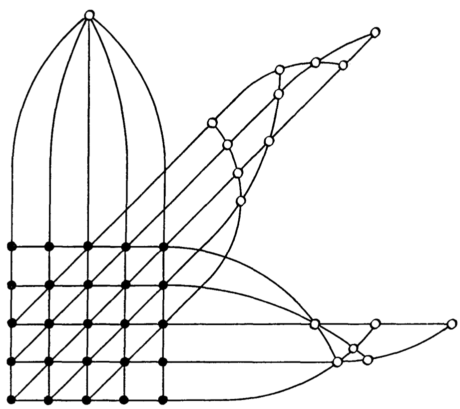

FIGURE 5

LEMMA 2.1. $n \leqslant m^2 - 1$, *with equality if and only if each parallel-class of* π_0 *forms a pencil in* π.

Proof. Consider the m lines of a parallel-class. The first contains $n + 1 - m$ points of $\pi - \pi_0$. The second (which must intersect the first in a point P of $\pi - \pi_0$) contains $(n + 1 - m) - 1 = n - m$ new points of $\pi - \pi_0$. The third (which must intersect the first two) contains at most $n - m$ new points, and contains exactly $n - m$ if and only if it passes through P. This last statement is also true for the remaining lines of the parallel-class. Hence the number of points of $\pi - \pi_0$ in a parallel-class is less than or equal to $m(n - m) + 1$, with equality if and only if the parallel-class forms a pencil in π.

Lines of distinct parallel-classes have no point of $\pi - \pi_0$ in common. Hence the number of points of π is less than or equal to

$$(m+1)[m(n-m)+1]+m^2,$$

with equality if and only if every parallel-class forms a pencil. But the number of points of π is n^2+n+1. Hence

$$n^2+n+1 \leqslant (m+1)[m(n-m)+1]+m^2,$$

which reduces to

$$(n-m)[n-(m^2-1)] \leqslant 0.$$

Since $n > m$, this gives $n \leqslant m^2 - 1$, with equality if and only if each parallel-class forms a pencil.

LEMMA 2.2. *If $m > 2$, not every parallel-class can form a pencil; hence $n \neq m^2 - 1$.*

Proof. Suppose the parallel-classes all form pencils, with vertices P_0, $P_1, \ldots, P_m$, say. Suppose there exists a line of π through P_0 not containing any point of π_0 and not passing through $P_1, P_2, \ldots,$ or P_m. This line will meet the m^2 lines of π_0 through $P_1, P_2, \ldots, P_m$ in m^2 distinct points of π, all distinct from P_0. The line will thus contain at least m^2+1 points, that is, $n+2$ points (by 2.1). This is impossible since a line of π contains just $n+1$ points. Thus every line of π through P_0 must be one of the m lines of π_0 through P_0, or must pass through $P_1, P_2, \ldots,$ or P_m. Thus there are at most $m+m$ lines of π through P_0. But there are just $n+1$ lines of π through P_0. Hence by 2.1

$$2m \geqslant n+1 = m^2, \qquad \text{so } m \leqslant 2.$$

The result now follows by 2.1.

Note. The other case, besides $m = 2$, $n = 3$, when every parallel-class can and does form a pencil is the trivial case $m = n$ mentioned in § 1.

COROLLARY. *If $m > 2$, then $n \leqslant m^2 - 2$ (i.e., $k \geqslant 1$).*

THEOREM 2.3. *Either $m = 3$, $n = 4$, or $k + 1 \leqslant \frac{1}{2}(m+1)$ so that*

$$n \geqslant m^2 - \tfrac{1}{2}m - \tfrac{1}{2}.$$

Proof. (Cf. the proof of 2.1) Consider the m lines of a parallel-class. The first contains $n+1-m$ points of $\pi - \pi_0$. The second (which must intersect the first in a point of $\pi - \pi_0$) contains $(n+1-m)-1$ new points of $\pi - \pi_0$. The third contains at least $(n+1-m)-2$ new points (since it has at most two points in common with the previous lines). The rth line ($r = 4, 5, \ldots, m$) contains at least $(n+1-m)-(r-1)$ new points. Hence the m lines contain at least

$$m(n+1-m) - \tfrac{1}{2}m(m-1)$$

points of $\pi - \pi_0$. Hence π contains at least

$$(m + 1)[m(n + 1 - m) - \tfrac{1}{2}m(m - 1)] + m^2$$

points. But π contains $n^2 + n + 1$ points. Hence

$$n^2 + n + 1 \geqslant (m + 1)[m(n + 1 - m) - \tfrac{1}{2}m(m - 1)] + m^2,$$

which reduces to

(1) $$n^2 - (m^2 + m - 1)n + \tfrac{1}{2}(m^2 - 1)(3m - 2) \geqslant 0.$$

Hence

(2) $$n \leqslant \tfrac{1}{2}\{(m^2 + m - 1) - \sqrt{[(m^2 - 2m - \tfrac{1}{2})^2 + (2m - \tfrac{13}{4})]}\}$$

or

(3) $$n \geqslant \tfrac{1}{2}\{(m^2 + m - 1) + \sqrt{[(m^2 - 2m - \tfrac{1}{2})^2 + (2m - \tfrac{13}{4})]}\}.$$

We are assuming that $m > 2$, so

$$m^2 - 2m - \tfrac{1}{2} > 0 \quad \text{and} \quad 2m - \tfrac{13}{4} > 0.$$

Hence from (2)

$$n < \tfrac{1}{2}\{(m^2 + m - 1) - (m^2 - 2m - \tfrac{1}{2})\} = \tfrac{3}{2}m - \tfrac{1}{4},$$

which is impossible since we have $n \geqslant m^2 - m + 1$ (by 1.4), unless $m = 3$, $n = 4$.

Alternatively, from (3)

$$n > \tfrac{1}{2}(m^2 + m - 1) + (m^2 - 2m - \tfrac{1}{2}) = m^2 - \tfrac{1}{2}m - \tfrac{3}{4}.$$

Since n is an integer, we deduce that

$$n \geqslant m^2 - \tfrac{1}{2}m - \tfrac{1}{2}.$$

Note. Although we have neglected the term $2m - \frac{13}{4}$ in (2) and (3) it is easy to check that the final inequality is the best that can be obtained from (1).

3. The structure of parallel-classes. A point of π where just $s + 1$ lines of π_0 meet will be called a point of *valency* s or an *s-point*. (The standard name here would be a point of valency $s + 1$, but I feel that the gain in compactness produced in many of the subsequent expressions justifies this departure from the standard name.) We see that the points of π_0 are m-points, the vertex of a pencil of parallels is an $(m - 1)$-point, and a point lying on just one line of π_0 is a 0-point.

We define the *valency* of a line as the sum of the valencies of the points of that line.

LEMMA 3.1. *Every line of $\pi - \pi_0$ has valency $m + k$.*

Proof. Consider a line l of π which is not a line of π_0. Each of the $n+1$ points of l lies on a line of π_0. Each of the m^2+m lines of π_0 meets l, and $s+1$ lines of π_0 pass through an s-point on l. Let A_s denote the number of s-points on l. Then

$$\sum (s+1)A_s = m^2+m,$$

and

$$\sum A_s = n+1 = m^2-k.$$

Hence $\sum sA_s = m+k$.

Note. A better geometrical insight can perhaps be obtained by presenting the above proof in a more informal way. If l met all the m^2+m lines of π_0 in distinct points, then l would contain m^2+m points. But l contains only $n+1 = m^2-k$ points, so somehow we have to "lose" $m+k$ points. We lose s points whenever l passes through an s-point, since the $s+1$ lines meet l in one point instead of $s+1$ points. Thus the total number of points lost is the valency of l. Hence the valency of l is $m+k$.

This result does *not* apply to lines of π_0, which are easily seen to have valency m^2+m-1.

LEMMA 3.2. *If $m > 3$, then at most one parallel-class of π_0 can be a pencil in π.*

Proof. Suppose we have two pencils with vertices P_0 and P_1, each of valency $m-1$. The line P_0P_1 has valency at least $2m-2$. But P_0P_1 is clearly not a line of π. Hence $2m-2 \leqslant m+k$ (by 3.1). Thus $k \geqslant m-2$. But $k \leqslant \frac{1}{2}m - \frac{1}{2}$ (by 2.3), so $\frac{1}{2}m - \frac{1}{2} \geqslant m-2$. Thus $m \leqslant 3$.

LEMMA 3.3. *A parallel-class of π_0 that is not a pencil in π cannot contain points of $\pi - \pi_0$ of valency greater than k (i.e., if the parallel-class contains an s-point of $\pi - \pi_0$, then $s \leqslant k$).*

Proof. Let S be an s-point of $\pi - \pi_0$ belonging to a parallel-class that is not a pencil. Then there is a point Q of π_0 not lying on any of the $s+1$ lines of π_0 through S. (See Figure 6.) The valencies of S and Q are s and m respec-

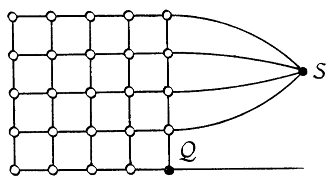

FIGURE 6

tively, so SQ has valency $s+m$ at least. But SQ is clearly not a line of π_0. Hence $s+m \leqslant m+k$ (by 3.1). Hence $s \leqslant k$.

Notes. (a) Since, in a parallel-class that is not a pencil, there must exist an s-point with $s \geqslant 1$, 3.3 shows that $k \geqslant 1$, so we have an alternative proof of the corollary to 2.2, when $m \geqslant 3$ (by 3.1). But we still need 2.2 to show that when $m = 3$ not every parallel-class is a pencil.

(b) Unless $m = 3$, $n = 4$, we have $k < m - 1$ (by 2.3) so a k-point cannot be the vertex of a pencil.

LEMMA 3.4. *If the points of intersection (in $\pi - \pi_0$) of the m lines of a parallel-class consist of B_s s-points ($s = 1, 2, \ldots, m - 1$), then*

$$\sum s(s + 1)B_s = m(m - 1).$$

Proof. If the m lines met by twos, they would have $\frac{1}{2}m(m - 1)$ points of intersection, but if $s + 1$ lines meet together (at an s-point), then $\frac{1}{2}s(s + 1)$ of these points of intersection are absorbed into a single point. Thus

$$\sum \tfrac{1}{2}s(s + 1)B_s = \tfrac{1}{2}m(m - 1).$$

Note. This formula is true for any m lines, each of which meets every other.

For a given value of k, the number of points of $\pi - \pi_0$ in a parallel-class depends on how the lines of the parallel-class intersect, as we remarked in § 2. From the results of § 2 it seems reasonable to suppose that the lower the valencies of the intersections, the fewer the number of points in the parallel-class. This turns out to be so.

In the proof of 2.1 we found an upper bound for the number of points of $\pi - \pi_0$ in a parallel-class. We shall now decrease this upper bound (except for pencils) by using 3.3. In § 5 we shall decrease the upper bound still further.

LEMMA 3.5. *The lines of a parallel-class that is not a pencil contain at most*

$$m(m^2 - m - k) - m(m - 1)/(k + 1)$$

points of $\pi - \pi_0$.

Proof. Each line of such a parallel-class contains $n + 1 - m = m^2 - m - k$ points of $\pi - \pi_0$. Using two different ways to count the number of point-line pairs (P, l), where P is a point of $\pi - \pi_0$ lying on a line l of the parallel-class, we have (with the notation of 3.4)

$$m(m^2 - m - k) = \sum (s + 1)B_s. \tag{4}$$

Hence the number of points in the parallel-class is

$$\sum B_s = m(m^2 - m - k) - \sum sB_s \tag{5}$$

where the summation goes from 0 to k only (by 3.3). (As in 3.1, we can obtain this expression in a more informal way. If the lines of the parallel-class did not meet, they would contain a total of $m(m^2 - m - k)$ points of $\pi - \pi_0$; but for each s-point we must subtract s from this total. Hence the parallel-class contains

$$m(m^2 - m - k) - \sum sB_s$$

points of $\pi - \pi_0$.)

Now

$$m(m-1) = \sum_{s=0}^{k} s(s+1)B_s \qquad \text{(by 3.4 and 3.3)}$$

$$\leqslant \sum_{s=0}^{k} s(k+1)B_s = (k+1)\sum_{s=0}^{k} sB_s.$$

Hence

$$\sum_{s=0}^{k} sB_s \geqslant m(m-1)/(k+1).$$

The result now follows from (5).

Note. This inequality is the best possible one at present, since for suitable values of m and k (e.g., $m = 7$, $k = 2$) there seems to be no reason why the lines of a parallel-class should not meet entirely in k-points. If this occurs, the above inequality becomes an equality. We shall show in 4.1, however, that when $m > 3$ we cannot have equality occurring in every parallel-class that is not a pencil.

THEOREM 3.6. *Either* $m = 2$, $k = 0$, $n = 3$, *or* $k + 1 \geqslant (m+1)^{\frac{1}{2}}$ *so that*

$$n \leqslant m^2 - (m+1)^{\frac{1}{2}}.$$

Proof. We have already dealt with $m = 2$. When $m = 3$ the result follows from the corollary to 2.2. Assume then that $m > 3$.

By 3.2, m of the parallel-classes are not pencils. By 3.5, these contain at most $m[m(m^2 - m - k) - m(m-1)/(k+1)]$ points of $\pi - \pi_0$. The remaining parallel-class, which may be a pencil, contains at most

$$m(n-m) + 1 = m(m^2 - 1 - k - m) + 1$$

points of $\pi - \pi_0$. (See proof of 2.1.)

Since π_0 contains m^2 points and π contains $n^2 + n + 1$ points, we deduce that

$$\text{(6)} \quad \begin{aligned} n^2 + n + 1 &\leqslant m^2 + m[m(m^2 - m - k) - m(m-1)/(k+1)] \\ &\qquad + m(m^2 - 1 - k - m) + 1 \\ &= (m+1)m(m^2 - m - k) + m^2 - m + 1 - m^2(m-1)/(k+1). \end{aligned}$$

Putting $n = m^2 - 1 - k$, we obtain

$$\text{(7)} \qquad m^3 - m^2(k^2 + 2k + 2) + m(k+1)^2 + k(k+1)^2 \leqslant 0.$$

The last two terms on the left-hand side are strictly positive (since $k \geqslant 1$ by the corollary to 2.2), so $m^3 - m^2(k^2 + 2k + 2) < 0$. Hence $k + 1 > (m-1)^{\frac{1}{2}}$.

There is no integer strictly between $(m-1)^{\frac{1}{2}}$ and $m^{\frac{1}{2}}$, and $k + 1$ is an integer, so $k + 1 \geqslant m^{\frac{1}{2}}$.

If we put $k + 1 = m^{\frac{1}{2}}$ on the left-hand side of (7) we obtain

$$m^3 - m^2(m+1) + m^2 + m(m^{\frac{1}{2}} - 1) = m(m^{\frac{1}{2}} - 1),$$

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which is greater than zero, so the inequality is not satisfied. Hence

$$k+1>m^{\frac{1}{2}}.$$

Again, there is no integer strictly between $m^{\frac{1}{2}}$ and $(m+1)^{\frac{1}{2}}$, so

$$k+1\geqslant(m+1)^{\frac{1}{2}}.$$

This is the best we can do, since (7) is satisfied if $k+1=(m+1)^{\frac{1}{2}}$.

Note. It is worth considering whether we can improve this inequality if none of the parallel-classes forms a pencil. In this case, instead of (6) we obviously get

$$n^2+n+1\leqslant m^2+(m+1)[m(m^2-m-k)-m(m-1)/(k+1)].$$

We deal with this as with (6), but it turns out that we can still obtain nothing better than $k+1\geqslant(m+1)^{\frac{1}{2}}$.

LEMMA 3.7. *If $m>3$, then $m\geqslant 7$.*

Proof. If $m=4$, then (by 2.3 and 3.6), $5/2\geqslant k+1\geqslant\sqrt{5}$, which is impossible since k must be an integer. If $m=5$, then $3\geqslant k+1\geqslant\sqrt{6}$, so $k=2$. This gives $n=22$, which is impossible by the Bruck–Ryser Theorem (1.2). Also $m=6$ is impossible by the Bruck–Ryser Theorem.

4. Further structure of parallel-classes. Let C_s denote the number of s-points in the whole plane π. Then $C_m=m^2$ (the points of π_0 are the only m-points) and $C_{m-1}=0$ or 1 (by 3.2). Apart from these two cases, $C_s=0$ if $s>k$ (by 3.3). We can easily show that 0-points (points lying on only one line of π_0) must exist on every line of π_0, except when $m=3$, $n=4$.

By 2.3 and 3.6 we see that if $m=3$ then $n=4$ or 7. Ostrom and Sherk **(5)** have shown that both these cases exist. *We shall assume from now on that $m\geqslant 7$ (using 3.7). It follows by 3.6 that $k\geqslant 2$.*

LEMMA 4.1. *There exists a point of positive valency less than k (i.e., there exists an s, $0<s<k$, such that $C_s>0$). Furthermore, if there exists a pencil (i.e., if $C_{m-1}=1$) and if there exists a k-point, then there exists a 1-point.*

Proof. We deal first with the last part. Suppose π contains P, the vertex of a pencil, of valency $m-1$, and K, a k-point, of valency k. (See Figure 7.) Then P and K, which are distinct since $m-1\neq k$ (by 2.3), contribute $m+k-1$ to the valency of PK. But PK is clearly not a line of π_0, so its valency is $m+k$ (by 3.1). Hence PK must contain a 1-point, to contribute the extra 1 to the total valency.

We have now only to exclude the case when there is no pencil and where the lines of every parallel-class meet entirely in k-points. In this case (see 3.5 and the notes following 3.5 and 3.6) we have

$$n^2+n+1=m^2+(m+1)[m(m^2-m-k)-m(m-1)/(k+1)],$$

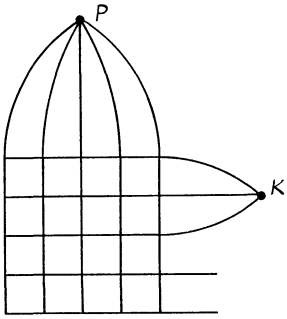

FIGURE 7

which reduces to $f(k + 1) = 0$, where

$$f(x) = x^3 - (m^2 - m + 1)x^2 - (m - 1)x + (m^3 - m).$$

Now $f(0) > 0$,

$$f((m+1)^{\frac{1}{2}}) = (m+1)^{\frac{1}{2}}[2 - (m+1)^{\frac{1}{2}}] < 0 \qquad \text{since } m \geqslant 7,$$
$$f(\tfrac{1}{2}(m+1)) = -\tfrac{1}{8}(m+1)(m-1)(2m^2 - 7m + 3) < 0 \qquad \text{since } m \geqslant 7,$$
$$f(x) \to \pm\infty \qquad \text{as } x \to \pm\infty.$$

Hence $f(x) = 0$ has three roots, one negative, one between 0 and $(m + 1)^{\frac{1}{2}}$, and one greater than $\frac{1}{2}(m + 1)$. Hence it has no root between $(m + 1)^{\frac{1}{2}}$ and $\frac{1}{2}(m + 1)$ or equal to either, so $f(k + 1)$ is never zero in the possible range of values for $k + 1$. This excludes the case under discussion.

Let g denote the least positive value of s for which $C_s > 0$. Thus π contains no points with valency between 0 and g. We can restate 4.1 as follows:

LEMMA 4.1*. *$g < k$, and if $C_{m-1} = 1$ and $C_k > 0$, then $g = 1$.*

LEMMA 4.2. *$g \leqslant \frac{1}{2}k$.*

Proof. Let S be a g-point, and let Q be a point of π_0 not lying on any of the $g + 1$ lines of π_0 through S. (See Figure 6.) The line SQ is clearly not a line of π_0, so its valency is $m + k$ (by 3.1). The points S and Q contribute $g + m$ to this valency. The remaining contribution to the valency, namely $k - g$, must come from the other points of SQ. Now $k - g > 0$ (by 4.1*) and no point can have positive valency less than g; hence $k - g \geqslant g$.

LEMMA 4.3. *There exist no s-points if $k > s > k - g$ (i.e., $C_s = 0$ for values of s in this range).*

Proof. Let S be an s-point, where $s < k$, and let Q be a point of π_0 not lying on any of the $s + 1$ lines of π_0 through S. (See Figure 6.) The line SQ is not a line of π_0, so its valency is $m + k$ (by 3.1). The points S and Q contribute $s + m$ to this valency. The remaining contribution to the valency, namely $k - s$, must come from the other points of SQ. Now $k - s > 0$ (by our assumption) and no point can have positive valency less than g; hence $k - s \geqslant g$. Hence if $s < k$, then $s \leqslant k - g$.

Note. We can use 4.3 to prove 4.2, putting $s = g$.

This last lemma does not imply the existence of $(k - g)$-points.

Denote by $k - z$ the greatest value of $s(< k)$ for which there actually exists an s-point.

LEMMA 4.4. $g \leqslant z \leqslant k - g$.

Proof. The first inequality is simply a restatement of 4.3. For the second, observe that $k - z \geqslant g$ by the definition of g.

LEMMA 4.5. *If* $g > \frac{1}{3}k$, *then* $z = g$.

Proof. Using the notation of 4.2, the points of SQ other than S and Q must have total valency $k - g$. If $z > g$, there are no $(k - g)$-points, so this valency cannot come from a single $(k - g)$-point. Hence it must come from at least two points, each of valency greater than or equal to g. Hence $k - g \geqslant 2g$, so $g \leqslant \frac{1}{3}k$. Hence if $g > \frac{1}{3}g$, we must have $z = g$.

The situation now is this. If $k - z \geqslant s \geqslant g$, then s-points can exist, and there exist at least one g-point and at least one $(k - z)$-point. Apart from such points, π contains only 0-points, k-points (perhaps), at most one $(m - 1)$-point, and m^2 m-points. Conditions and inequalities satisfied by g and z are given by 4.1, 4.2, 4.4, and 4.5.

LEMMA 4.6. $g \neq \frac{1}{2}k$, unless $g = 1$, $k = 2$.

Proof. If $k = 2g$, then π contains only 0-points, g-points, $2g$-points, at most one $(m - 1)$-point, and m-points.

If there exists a pencil and if there exists a $2g$-point (i.e., a k-point), then $g = 1$ by 4.1*.

If there exists a pencil with vertex P, but no $2g$-point, let l be any line of $\pi - \pi_0$ through P. Apart from P, l can contain only g-points and 0-points. The total valency of points of l other than P is

$$(m + k) - (m - 1) = 2g + 1 \qquad \text{(by 3.1)}.$$

But we cannot obtain a total valency $2g + 1$ from g-points if $g > 1$. Hence $g = 1$.

If there is no pencil, the lines of every parallel-class contain

$$m(m^2 - m - 2g) - gB_g - 2gB_{2g}$$

points of $\pi - \pi_0$, using equation (5) of 3.5 and the notation of 3.4. Summing this expression over the $m + 1$ parallel-classes and adding the m^2 points of π_0 to obtain the number of points of π, we obtain

$$n^2 + n + 1 = (m + 1)m(m^2 - m - 2g) - gC_g - 2gC_{2g} + m^2.$$

Putting $n = m^2 - 1 - 2g$ and simplifying, we obtain

(8) $$gC_g + 2gC_{2g} = 2gm^2 - 2gm - 4g^2 - 2g + m^2 - 1.$$

Also in a parallel-class we have (by 3.4)

(9) $$g(g+1)B_g + 2g(2g+1)B_{2g} = m(m-1).$$

Summing over the $m + 1$ parallel-classes, we obtain

(10) $$g(g+1)C_g + 2g(2g+1)C_{2g} = (m+1)m(m-1).$$

Now any line of π_0 is met by the remaining $m - 1$ lines of the same parallel-class in g-points or $2g$-points. Hence g divides $m - 1$. Write $m - 1 = rg$. Dividing (8) and (10) by g, we have

$$C_g + 2C_{2g} = 2m^2 - 2m - 4g - 2 + (m+1)r$$

and

$$(g+1)C_g + 2(2g+1)C_{2g} = m(m+1)r.$$

Converting these to congruences modulo g, we have, since $m \equiv 1$,

$$C_g + 2C_{2g} \equiv -2 + 2r \quad \text{and} \quad C_g + 2C_{2g} \equiv 2r.$$

Hence $0 \equiv 2$. So g divides 2, giving $g = 2$ or $g = 1$.

If $g = 2$, then (8) and (10) become

$$2C_2 + 4C_4 = 5m^2 - 4m - 21 \quad \text{and} \quad 6C_2 + 20C_4 = m^3 - m.$$

Hence $8C_4 = m^3 - 15m^2 + 11m + 63$, so

$$0 \equiv m^3 + m^2 + 3m - 1 \pmod 8.$$

Hence m must be odd; $m = 2p + 1$ say. But

$$\begin{aligned}(2p+1)^3 + (2p+1)^2 + 3(2p+1) - 1 &= 8p^3 + 16p^2 + 16p + 4 \\ &\equiv 4 \pmod 8.\end{aligned}$$

Hence $0 \equiv 4 \pmod 8$, which is impossible. So $g \neq 2$ and hence $g = 1$.

This exhausts all possible cases, leaving us with $g = 1$ each time.

LEMMA 4.7. *We cannot have* $g = 1$, $k = 2$.

Proof. Suppose $g = 1$, $k = 2$. From the inequalities

$$(m+1)^{\frac{1}{2}} \leqslant k + 1 \leqslant \tfrac{1}{2}(m+1)$$

we see that $5 \leqslant m \leqslant 8$, so that $m = 7$ or 8 (by 3.7). When $m = 7$,

$$n = 7^2 - 1 - 2 = 46,$$

which is impossible by the Bruck–Ryser Theorem (1.2). We are left with $m = 8$, $n = 8^2 - 1 - 2 = 61$ $(= 6^2 + 5^2)$.

If π contains no pencil, then equations (8) and (10) of 4.6 apply. Putting $g = 1$, $m = 8$ we obtain

$$C_1 + 2C_2 = 169, \qquad 2C_1 + 6C_2 = 504,$$

giving $C_1 = 3$, $C_2 = 83$.

If π contains a single pencil, this pencil contains 425 points of $\pi - \pi_0$ (see proof of 2.1). The remaining eight parallel-classes each contain (as in 4.6)

$$m(m^2 - m - k) - gB_g - 2gB_{2g} = 432 - B_1 - 2B_2$$

points of $\pi - \pi_0$. Thus π contains

$$8 \times 432 - C_1 - 2C_2 + 425 + 8^2$$

points. But π contains $61^2 + 61 + 1$ points. Hence

$$3945 - C_1 - 2C_2 = 3783, \qquad \text{so} \quad C_1 + 2C_2 = 162.$$

Summing (9) over the m parallel-classes that are not pencils, we have

$$g(g + 1)C_g + 2g(2g + 1)C_{2g} = m^2(m - 1) \quad \text{or} \quad 2C_1 + 6C_2 = 448.$$

Solving the two equations, we find that $C_1 = 38$, $C_2 = 62$.

Finally we show that both values for C_1 are impossible. Let S be a 1-point (see Figure 8). There are $6 \times 8 = 48$ points of π_0 such as Q, of valency 8,

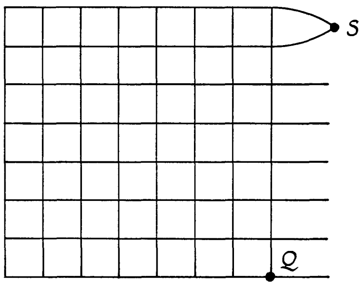

FIGURE 8

not lying on a line of π_0 through S. Each line such as SQ has valency $m+k = 10$ (by 3.1) and S and Q contribute 9 to this valency. Hence each line such as SQ must contain a single 1-point apart from S, to bring the valency up to 10. The 48 lines such as SQ are all distinct, so π must contain at least 49 1-points (counting S as well). Thus $C_1 \geqslant 49$, so we cannot have $C_1 = 3$ or $C_1 = 38$. Hence we cannot have $g = 1$, $k = 2$.

We can combine 4.2, 4.6, 4.7 into a single result, namely

LEMMA 4.8. $g < \frac{1}{2}k$.

LEMMA 4.9. *If* $g > \frac{1}{3}k$, *there cannot be a pencil, except possibly when* $g = \frac{1}{3}(k + 1)$.

Proof. If $g = 1$, then $g > \frac{1}{3}k$ implies that $k = 2$. This is impossible (by 4.7). Hence we may assume that $g > 1$.

Suppose P is the vertex of a pencil. Then P is an $(m - 1)$-point. Let G be a g-point (see Figure 7, with K replaced by G). PG is not a line of π_0, so its valency is $m + k$ (by 3.1). P and G contribute $m - 1 + g$ to this valency, so the remaining points of PG must have total valency

$$(m + k) - (m - 1 + g) = k - g + 1.$$

Now $k > k - g + 1 > k - g$ (since $g > 1$), so there can be no single point of valency $k - g + 1$ (by 4.3). Thus the valency $k - g + 1$ must come from at least two points of positive valency, each of which must have valency g or more. Thus $k - g + 1 \geqslant 2g$, so $g \leqslant \frac{1}{3}(k + 1)$. Since $g > \frac{1}{3}k$ and g and k are integers, we must therefore have $g = \frac{1}{3}(k + 1)$.

LEMMA 4.10. *If* $g = \frac{1}{3}(k + 1)$ *there cannot be a pencil, except possibly when* $g = 2$.

Proof. We may assume that $g > 1$; for if $g = 1$, then $k = 2$, which is impossible (by 4.7). Suppose there is a pencil, with vertex P. We show first that a line of $\pi - \pi_0$ through a point of π_0 cannot contain more than two other points of positive valency, nor can a line of $\pi - \pi_0$ through P, except in the case of three points of valency g. For if a line of $\pi - \pi_0$ through a point π_0 contained three or more other points of positive valency (i.e., of valency g at least) the valency of this line would be at least $m + 3g = m + k + 1$, which is impossible by 3.1. A similar argument holds for lines through P, except when we have three g-points on the line, when the valency is

$$(m - 1) + 3g = m + k.$$

Next we show that there must be at least one point of valency s for every s such that $g \leqslant s \leqslant k - g = 2g - 1$. Let A_0 be a g-point (such a point certainly exists). Let Q_0 be a point of π_0 not lying on a line of π_0 through A_0 (see Figure 9, with $i = 0$). On $A_0 Q_0$ there must be just one more point B_0

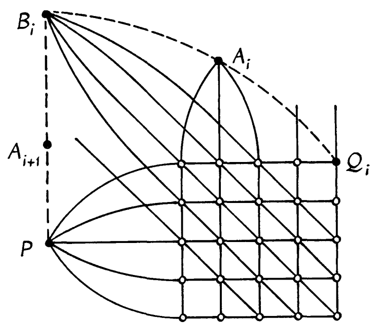

FIGURE 9

of positive valency. The valency of B_0 is $(m + k) - g - m = 2g - 1$. On $B_0 P$ there must be just one more point A_1 of positive valency $(2g - 1 > g$, since $g > 1)$. The valency of A_1 is $(m + k) - (2g - 1) - (m - 1) = g + 1$. Let Q_1 be a point of π_0 not lying on a line of π_0 through A_1. On $A_1 Q_1$ there is a point B_1 of valency $2g - 2$. On $B_1 P$ there is a point A_2 of valency $g + 2$; and so on. Continuing in this way (by induction) we obtain the required result.

Now let S be a $(2g - 1 - r)$-point, where $0 \leqslant r \leqslant g - 1$. There are $2g - r$ lines of π_0 through S, so there are $m(m - 2g + r)$ points of π_0, such as Q in Figure 6, not lying on these lines. Joining these points to S, we obtain $m(m - 2g + r)$ distinct lines through S that are not lines of π_0. Each of these lines must contain another point of positive valency

$$(m + k) - (2g - 1 - r) - m = g + r$$

distinct from S. (We must say "distinct from S" here, since $g + r = 2g-1-r$ if $r = \frac{1}{2}(g - 1)$.) Hence π contains at least $m(m - 2g + r)$ points of valency $g + r$, so

(11) $$C_{g+r} \geqslant m(m - 2g + r).$$

We now show that there are too many points in the plane. There are no k-points (by 4.1*). Through P there are $n + 1 - m = m^2 - m - k$ lines of $\pi - \pi_0$. Every point of valency $g, g + 1, \ldots, 2g - 1$ must lie on exactly one such line, and as we saw earlier, each such line contains (in addition to P) just three points of valency g or just two points of valency greater than g. Thus

$$\frac{1}{3}C_g + \frac{1}{2}\sum_{r=1}^{g-1} C_{g+r} = m^2 - m - k.$$

Hence from (11)

$$\frac{1}{3}m(m - 2g) + \frac{1}{2}\sum_{r=1}^{g-1} m(m - 2g + r) \leqslant m^2 - m - k = m^2 - m - (3g - 1).$$

This reduces to

(12) $$(3g - 7)m^2 - \tfrac{1}{2}(9g^2 - g - 12)m + 6(3g - 1) \leqslant 0.$$

Now $k = 3g - 1$ and $k + 1 \leqslant \frac{1}{2}(m + 1)$, so $m \geqslant 6g - 1$. Denote the left-hand side of (12) by $f(m)$. Then

$$\begin{aligned} f'(6g - 1) &= 2(3g - 7)(6g - 1) - \tfrac{1}{2}(9g^2 - g - 12) \\ &= \tfrac{1}{2}[63g(g - 3) + 10g + 40] > 0 \qquad \text{if } g \geqslant 3. \end{aligned}$$

Also $f''(m) = 2(3g - 7) > 0$ if $g \geqslant 3$, so that $f'(m)$ is an increasing function. Hence $f'(m) > 0$ if $m \geqslant 6g - 1$ and $g \geqslant 3$. Hence $f(m)$ is an increasing function if $m \geqslant 6g - 1$ and $g \geqslant 3$.

Finally

$$\begin{aligned} f(6g-1) &= \tfrac{1}{2}(162g^3 - 561g^2 + 281g - 38) \\ &= \tfrac{1}{2}[162g^2(g-4) + 87g^2 + 281g - 38] \\ &> 0 \qquad \text{if } g \geqslant 4 \\ &> 0 \qquad \text{if } g = 3 \text{ by direct calculation.} \end{aligned}$$

Hence $f(m) > 0$ if $m \geqslant 6g - 1$ and $g \geqslant 3$. This contradicts (12). Since $g > 1$, we are left with the case $g = 2$.

LEMMA 4.11. *If $g = 2$ and $k = 3g - 1 = 5$, then there cannot be a pencil.*

Proof. Suppose there is a pencil, with vertex P. As in 4.10 there exist 2-points and 3-points but no 5-points. Each of the $m^2 - m - 5$ lines of $\pi - \pi_0$ through P contains either three 2-points or two 3-points since each such line has valency $(m-1) + 6$. Moreover, each 2-point and 3-point lies on one such line. Thus

$$\tfrac{1}{3}C_2 + \tfrac{1}{2}C_3 = m^2 - m - 5. \tag{13}$$

Through every 2-point there pass $m(m-3)$ lines of $\pi - \pi_0$ containing a point of π_0, such as SQ in Figure 10. Each such line contains a single 3-point

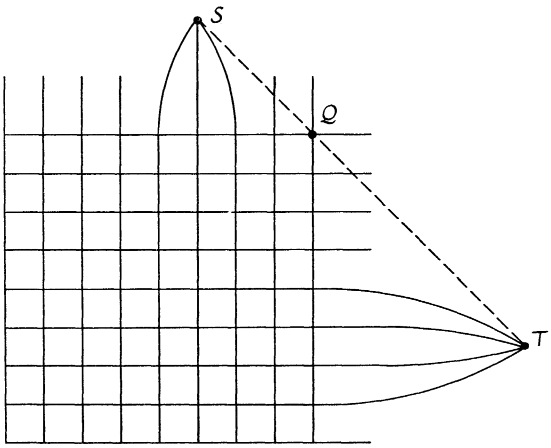

FIGURE 10

such as T (by consideration of valencies). Moreover (considering such lines through every 2-point), every 3-point occurs in this manner, on just $m(m-4)$ of the lines (since through every 3-point there pass $m(m-4)$ lines of $\pi - \pi_0$ containing a point of π_0, and each such line contains a 2-point). Thus

$$m(m-3)C_2 = m(m-4)C_3.$$

Hence from (13)

$$\tfrac{1}{3}C_2 + \tfrac{1}{2}(m-3)C_2/(m-4) = m^2 - m - 5.$$

Thus

(14) $$C_2 = 6(m-4)(m^2-m-5)/(5m-17).$$

Furthermore, every line of $\pi - \pi_0$ through a point of π_0 must contain a 2-point (and a 3-point). There are $m^2(n-m) = m^2(m^2-m-6)$ such lines, and $m(m-3)$ of them pass through each 2-point. Hence

$$m(m-3)C_2 = m^2(m^2-m-6)$$

so

(15) $$C_2 = m(m+2).$$

From (14) and (15) we obtain

$$\phi(m)(\text{say}) \equiv m^3 - 23m^2 + 28m + 120 = 0.$$

$\phi(0) > 0$, so the equation has a negative root. $\phi(3) > 0$, $\phi(4) < 0$; $\phi(21) < 0$, $\phi(22) > 0$. Hence the equation has no positive integral roots. This proves the result.

We can combine 4.7, 4.9, 4.10, 4.11 into a single result, namely

THEOREM 4.12. *If $g > \frac{1}{3}k$, there cannot be a pencil.*

We end this section with a lemma which gives another upper bound for g in terms of k. The method of proof is essentially the same as that used in 2.3 and 3.5, but it is convenient to use the formulae given in 5.1 at the beginning of the next section.

LEMMA 4.13.

$$g + 1 \leqslant \frac{m(m^2-1)}{(m^2-1) + k(m^2-m-1) - k^2}.$$

Proof. Condensing the formulae of 5.1, we have

(16) $$\sum_{s=g}^{m-1} sC_s = R,$$

(17) $$\sum_{s=g}^{m-1} s(s+1)C_s = m(m^2-1).$$

From (17)

$$(g+1)\sum sC_s = \sum s(g+1)C_s \leqslant \sum s(s+1)C_s = m(m^2-1)$$

so $(g+1)R \leqslant m(m^2-1)$. Dividing both sides by R (which is certainly positive, we have the result.

If we put $k = \frac{1}{2}m - \frac{1}{2}$ (the largest possible value), the right-hand side of 4.13 is less than $2 + 1/m$. Thus $g = 1$. This is only to be expected since in the proof of 2.3 we had to allow points of valency 1 in order to obtain the upper bound for k.

If we put $k = (m+1)^{\frac{1}{2}} - 1$ (the smallest possible value), we obtain an inequality from which we can deduce only $g < k$, which again is what we should expect.

Thus for "large" values of k, 4.13 gives a new upper bound for g, while for "small" values of k, 4.8 gives a better upper bound.

5. Further restrictions on k. The technique of this section is to decrease our present upper bound for the number of points of $\pi + \pi_0$ in a parallel-class (for given values of m, k, g, and z) by finding as many points of low valency as we can. (See the remarks before 3.5.) This is a refinement of the method used in 3.5. We then use this upper bound to obtain inequalities as in 3.6. We shall in fact consider all the parallel-classes together, and the technique is somewhat obscured by the quick methods used to obtain the inequalities.

We shall write $R = (m^2 - 1) + k(m^2 - m - 1) - k^2$. It is useful in some of the calculations to note that $R - m + 1 = (k + 1)(m^2 - k - m)$.

LEMMA 5.1.

(a) $\sum_{s=g}^{k-z} sC_s + kC_k + (m - 1)C_{m-1} = R,$

(b) $\sum_{s=g}^{k-z} s(s + 1)C_s + k(k + 1)C_k + m(m - 1)C_{m-1} = m(m^2 - 1).$

Proof. (a) Summing equation (5) of 3.5 over the $m + 1$ parallel-classes we see that the total number of points in $\pi - \pi_0$ is

$$\sum C_s = (m + 1)m(m^2 - m - k) - \sum sC_s,$$

the summation being taken from 0 to $m - 1$, since a pencil may occur. Thus the number of points of π is

$$m^2 + \sum_{s=0}^{m-1} C_s = m^2 + (m + 1)m(n + 1 - m) - \sum_{s=0}^{m-1} sC_s.$$

Equating this to $n^2 + n + 1$, putting $n = m^2 - 1 - k$, and remembering the restrictions on the values of s for which s-points can occur, we obtain the result.

(b) We obtain this result by summing 3.4 over the $m + 1$ parallel-classes.

The next two lemmas give information about the number of points of valency less than k.

LEMMA 5.2.

$$\sum_{s=g}^{k-z} sC_s \geqslant m(m - g - 1)(k - g) + g.$$

Proof. Let S be a g-point, and let Q be a point of π_0 not lying on any of the $g + 1$ lines of π_0 through S. (See Figure 6.) There are $m(m - g - 1)$ choices for Q, giving $m(m-g-1)$ distinct lines such as SQ. (If SQ_1 and SQ_2 were to coincide, then the line SQ_1Q_2, joining Q_1 and Q_2, would be a line of π_0, contradicting the fact that SQ_1 is not a line of π_0.) On a typical line SQ the points other than S and Q must have total valency $k-g$ (to bring up the total valency of the line to $m+k$). No s-point, where $s > k-g$, can contribute to this total, so each line contributes a term $k-g$ to the sum on the left-hand side. All the lines together contribute $m(m-g-1)(k-g)$ and S itself contributes a term g. This gives the result.

LEMMA 5.3. *If* $z < \frac{1}{2}k$, *then*

$$\sum_{s=g}^{z} sC_s \geqslant m(m-k+z-1)z.$$

Proof. The inequality for z implies that $z < k-z$. We use the method of 5.2, but we take S to be a $(k-z)$-point. There are now $m(m-k+z-1)$ choices for Q, and on each line SQ the points other than S and Q must have total valency $(m+k)-m-(k-z)=z$. No s-point, where $s > z$, can contribute to this total. Using an argument similar to that of 5.2, we deduce the result.

Note. If $z \geqslant k-z$ we can only take the summation as far as $k-z$ and the result is simply an inequality which is weaker than 5.2.

We now consider separately four cases. Using the results of § 4 we see that they exhaust all possibilities.

A1. $g \leqslant \frac{1}{3}k$, $C_{m-1}=0$.

A2. $g \leqslant \frac{1}{3}k$, $C_{m-1}=1$, $C_k > 0$, $g=1$.

A3. $g \leqslant \frac{1}{3}k$, $C_{m-1}=1$, $C_k=0$.

B. $\frac{1}{3}k < g < \frac{1}{2}k$, $C_{m-1}=0$.

*Case A*1. It is convenient not to put $C_{m-1}=0$ at this stage. Multiplying 5.1(a) by $k+1$ and subtracting 5.1(b) we eliminate C_k and obtain

$$\sum s(k-s)C_s + (k+1-m)(m-1)C_{m-1} = (k+1)R - m(m^2-1)$$

or

$$(k+1)R + (m-k-1)(m-1)C_{m-1} - m(m^2-1) = \sum_{s=g}^{k-z} s(k-s)C_s. \tag{18}$$

If we simply use the fact that the right-hand side of (18) is greater than or equal to zero, we obtain 3.6; but we can now say more than this. It is possible to obtain the ensuing inequalities by a method that appears to use the fact that $C_k \geqslant 0$, but we have now eliminated C_k so we never really use this information either in 3.6 or anywhere else. We shall, however, use the fact that $C_k=0$ in the discussion of the case A3.

Now

$$\sum_{s=g}^{k-z} s(k-s)C_s \geqslant z\sum_{s=g}^{k-z} sC_s \geqslant z[m(m-g-1)(k-g)+g] \qquad \text{(by 5.2)}. \tag{19}$$

Furthermore, if $z < \frac{1}{2}k$ we have

$$(20)\quad \sum_{s=g}^{k-z} s(k-s)C_s = \sum_{s=g}^{z} s(k-s)C_s + \sum_{z+1}^{k-z} s(k-s)C_s$$
$$\geqslant (k-z)\sum_{s=g}^{z} sC_s + z\sum_{z+1}^{k-z} sC_s$$
$$= (k-z)\sum_{s=g}^{z} sC_s + z\left[\sum_{g}^{k-z} sC_s - \sum_{g}^{z} sC_s\right]$$
$$= (k-2z)\sum_{s=g}^{z} sC_s + z\sum_{s=g}^{k-z} sC_s$$
$$\geqslant (k-2z)m(m-k+z-1)z$$
$$+ z[m(m-g-1)(k-g)+g] \qquad \text{(by 5.2 and 5.3)}$$
$$= m[-2z^3 - (2m-3k-2)z^2 + (mk-k^2-k)z]$$
$$+ z[mg^2 - (m^2+mk-m-1)g + (m^2k-mk)]$$
$$= m\theta(z) + z\phi(g), \qquad \text{say.}$$

Denote by $\psi(z)$ the function that is equal to $m\theta(z) + z\phi(g)$ when $z < \frac{1}{2}k$ and equal to $z\phi(g)$ when $z \geqslant \frac{1}{2}k$. These two expressions are equal when $z = \frac{1}{2}k$ so, regarding z as a continuous variable, $\psi(z)$ is a continuous function. By (19) and (20) we have

$$(21)\qquad \sum_{s=g}^{k-z} s(k-s)C_s \geqslant \psi(z).$$

We wish to find the minimum value of $\psi(z)$ in the interval $[g, k-g]$. In the interval $[g, \frac{1}{2}k]$, $\psi(z) = m\theta(z) + z\phi(g)$. $\psi'(z)$ need not have constant sign, but $\psi''(z) = m[-12z - 2(2m-3k-2)]$, which is negative. Hence the minimum value of $\psi(z)$ occurs when $z = g$ or $z = \frac{1}{2}k$. Now

$$\begin{aligned}\psi(g) - \psi(\tfrac{1}{2}k) &= m\theta(g) + g\phi(g) - \tfrac{1}{2}k\phi(g), \qquad \text{since } \theta(\tfrac{1}{2}k) = 0,\\ &= \tfrac{1}{2}[2m\theta(g) - (k-2g)\phi(g)]\\ &= \tfrac{1}{2}(k-2g)[2mg(m-k+g-1) - m(m-g-1)(k-g) - g]\\ &= \tfrac{1}{2}(k-2g)[mg^2 + (3m^2-mk-3m-1)g - m^2k - mk].\end{aligned}$$

The derivative with respect to g of the expression in square brackets is $2mg + (3m^2 - mk - 3m - 1)$, which is positive. Hence the expression itself is an increasing function of g. Its value when $g = \frac{1}{3}k$ is $-\frac{2}{9}mk^2 - \frac{1}{3}k$, so the expression is always negative (in the possible range of values for g). Hence $\psi(g) < \psi(\frac{1}{2}k)$. Hence the minimum value of $\psi(z)$ in the interval $[g, \frac{1}{2}k]$ is $\psi(g)$. Moreover, when $z \geqslant \frac{1}{2}k$, $\psi(z)$ is an increasing function. Hence, in the interval $[g, k-g]$,

$$(22)\qquad \psi(z) \geqslant \psi(g) = m\theta(g) + g\phi(g).$$

Combining (18), (21), and (22) we see that

$$(23)\qquad (k+1)R + (m-k-1)(m-1)C_{m-1} - m(m^2-1) \geqslant m\theta(g) + g\phi(g).$$

Now

$$m\theta(g) + g\phi(g) = -mg^3 - (3m^2 - 2mk - 3m - 1)g^2 + (2m^2k - mk^2 - 2mk)g,$$

whose derivative is

$$-3mg^2 - 2(3m^2 - 2mk - 3m - 1)g + (2m^2k - mk^2 - 2mk).$$

This is decreasing, since the second derivative is clearly negative. The value of the derivative when $g = \frac{1}{3}k$ is $\frac{2}{3}k$. Hence the derivative is always positive, so the right-hand side of (23) is an increasing function. Hence

$$m\theta(g) + g\phi(g) \geqslant m\theta(1) + 1\phi(1) = m(k-2)(m-k) + m(m-2)(k-1) + 1,$$

and so from (23)

$$(24)\qquad (k+1)R + (m-k-1)(m-1)C_{m-1} - m(m^2-1) \geqslant m(k-2)(m-k) + m(m-2)(k-1) + 1.$$

Substituting the value for R in (24) and putting $C_{m-1} = 0$ we obtain

$$(25)\qquad m^3 - m^2(k^2+4) + m(k+1) + (k^3 + 2k^2 + 2k + 2) \leqslant 0.$$

Hence $m^3 - m^2(k^2+4) < 0$, so $m < k^2 + 4$, Hence $m \leqslant k^2 + 3$. Putting $m = k^2 + 3$ in the left-hand side of (25) we obtain

$$-(k^4 - 2k^3 + 3k^2 - 5k + 4).$$

This is negative since $k \geqslant 3$ (by 4.7), so $m = k^2 + 3$ satisfies the inequality. Hence we cannot improve the result $m \leqslant k^2 + 3$. Thus in the case A1 we have

$$(26)\qquad k \geqslant (m-3)^{\frac{1}{2}}.$$

Case A2. We still have (18), with $C_{m-1} = 1$, and the calculations of case A1 up to the inequality (23) are still valid, except that now $g = 1$. Thus we have (24), but now we must put $C_{m-1} = 1$ and (24) becomes

$$(27)\qquad m^3 - m^2(k^2+5) + m(2k+3) + (k^3 + 2k + k + 1) \leqslant 0.$$

Hence $m^3 - m^2(k^2+5) < 0$, so $m < k^2 + 5$. Hence $m \leqslant k^2 + 4$. Putting $m = k^2 + 4$ in the left-hand side of (27), we obtain

$$-(k^4 - 3k^3 + 3k^2 - 9k + 3).$$

This is negative since $k \geqslant 3$ (by 4.7) so $m = k^2 + 4$ satisfies the inequality. Hence we cannot improve the result $m \leqslant k^2 + 4$. Thus in the case A2 we have

$$(28)\qquad k \geqslant (m-4)^{\frac{1}{2}}.$$

Case A3. Putting $C_k = 0$, $C_{m-1} = 1$ in 5.1, we obtain

$$\sum_{s=g}^{k-z} sC_s = R - m + 1, \tag{29}$$

$$\sum_{s=g}^{k-z} s(s+1)C_s = m^2(m-1). \tag{30}$$

By (30),

$$(k - z + 1)\sum_{s=g}^{k-z} sC_s \geqslant m^2(m-1),$$

so, by (29), $(k - z + 1)(R - m + 1) \geqslant m^2(m - 1)$, which gives

$$(k+1)(R-m+1) - m^2(m-1) \geqslant z(R-m+1). \tag{31}$$

Furthermore, if $z < \frac{1}{2}k$ we have, by (29),

$$R - m + 1 = \sum_{s=g}^{z} sC_s + \sum_{z+1}^{k-z} sC_s.$$

Hence

$$(k-z+1)(R-m+1) \geqslant (k-z+1)\sum_{s=g}^{z} sC_s + \sum_{z+1}^{k-z} s(s+1)C_s.$$

Also, by (30),

$$m^2(m-1) = \sum_{s=g}^{z} s(s+1)C_s + \sum_{z+1}^{k-z} s(s+1)C_s.$$

Hence, by subtraction,

$$\begin{aligned}(k-z+1)(R-m+1) - m^2(m-1) &\geqslant \sum_{s=g}^{z} s(k-z-s)C_s \\ &\geqslant (k-2z)\sum_{s=g}^{z} sC_s \\ &\geqslant (k-2z)m(m-k-z-1)z \\ &\qquad\qquad \text{(by 5.3).}\end{aligned}$$

Thus

$$\begin{aligned}(k+1)(R-m+1) - m^2(m-1) &\geqslant (k-2z)m(m-k-z-1)z \\ &\qquad + z(R-m+1) \\ &= m\theta(z) + z(R-m+1),\end{aligned} \tag{32}$$

using the notation of (20).

Denote by $\chi(z)$ the function that is equal to $m\theta(z) + z(R - m + 1)$ when $z < \frac{1}{2}k$ and equal to $z(R - m + 1)$ when $z \geqslant \frac{1}{2}k$. These two expressions are equal when $z = \frac{1}{2}k$ so, regarding z as a continuous variable, $\chi(z)$ is a continuous function. By (31) and (32)

$$(k+1)(R-m+1) - m^2(m-1) \geqslant \chi(z). \tag{33}$$

Now $\chi(z) - \psi(z) = z[R - m + 1 - \phi(g)]$, using the notation of (20) and (21). Hence

$$\chi'(z) - \psi'(z) = R - m + 1 - \phi(g)$$
$$= [m^2 - k(k+1)] + mg[m - g - 1] + [m(kg - 1) - g],$$

which is positive. Thus $\chi(z) - \psi(z)$ is an increasing function. But the minimum value of $\psi(z)$ in the interval $[g, k - g]$ occurs when $z = g$. Hence the minimum value of $\chi(z)$ occurs when $z = g$ also. Hence $\chi(z) \geqslant \chi(g)$ so, by (33),

(34) $$(k+1)(R - m + 1) - m^2(m-1) \geqslant m\theta(g) + g(R - m + 1).$$

The derivative of the right-hand side with respect to g is

$$m[-6g^2 - 2(2m - 3k - 2)g + (mk - k^2 - k)] + (R - m + 1).$$

This is a decreasing function of g. When $g = \frac{1}{3}k$, its value is

$$\tfrac{2}{3}m^2k + \tfrac{1}{3}mk^2 - \tfrac{2}{3}mk + m^2 - k^2 - k - m,$$

which is positive. Hence the derivative is always positive, so the right-hand side of (34) is an increasing function of g. Its least value therefore occurs when $g = 1$, so

$$m\theta(g) + g(R - m + 1) \geqslant m\theta(1) + 1(R - m + 1).$$

Combining this with (34), we have

$$(k+1)(R - m + 1) - m^2(m-1) \geqslant m(k-2)(m-k) + (R - m + 1).$$

This simplifies to

(35) $$m^3 - (k^2 + 3)m^2 + 3km + (k^3 + k^2) \leqslant 0.$$

Hence $m^3 - (k^2 + 3)m^2 < 0$, $m < k^2 + 3$, so $m \leqslant k^2 + 2$.

Putting $m = k^2 + 2$ on the left-hand side of (35) we obtain

$$-(k^4 - 4k^3 + 3k^2 - 6k + 4).$$

This is positive when $k = 3$ and negative when $k \geqslant 4$. Thus, if

$$m \geqslant 4^2 + 2 = 18,$$

$m = k^2 + 2$ satisfies the inequality (33), so we cannot improve the result $m \leqslant k^2 + 2$. But if $m < 18$, we cannot have $m = k^2 + 2$ (since this implies $k = 3$, for we are considering only $k \geqslant 3$), so $m < k^2 + 2$ and hence $m \leqslant k^2 + 1$. Putting $m = k^2 + 1$, on the left-hand side of (33), we have

$$-(2k^4 - 4k^3 + 3k^2 - 3k + 2),$$

which is negative since $k \geqslant 3$. Hence we cannot improve upon the result

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$m \leqslant k^2 + 1$. Thus in the case A3 we have

(36) $$\begin{cases} k \geqslant (m-2)^{\frac{1}{2}} & \text{if } m \geqslant 18, \\ k \geqslant (m-1)^{\frac{1}{2}} & \text{if } m \leqslant 17. \end{cases}$$

Note. We have not used 5.2 in these calculations. The reason for this is that (29) gives us more information than 5.2, since

$$R - m + 1 > m(m-g-1)(k-g) + g$$

as may easily be shown.

Case B. Let S be a g-point. There are $m(m-g-1)$ points of π_0, such as Q in Figure 6, not lying on any of the $g+1$ lines of π_0 through S. The $m(m-g-1)$ lines such as SQ are all distinct, and apart from S and Q each such line must contain points of total valency $(m+k) - g - m = k - g$. Since $k - g < 2g$, this extra valency must come from a single $(k-g)$-point. Hence there are at least $m(m-g-1)$ points of valency $k-g$. Thus

$$C_{k-g} \geqslant m(m-g-1).$$

Similarly, starting with a $(k-g)$-point, we can show that

$$C_g \geqslant m(m-k+g-1).$$

Since $k - g \neq g$ (by 4.8), $(k-g)$-points are not the same as g-points. Thus, substituting these inequalities in (18) and putting $C_{m-1} = 0$, we have

(37) $$\begin{aligned}(k+1)R - m(m^2-1) &\geqslant g(k-g)m[m-g-1+m-k+g-1] \\ &= g(k-g)m(2m-k-2).\end{aligned}$$

Now $g > \frac{1}{3}k$ and g and k are integers, so $g \geqslant \frac{1}{3}k + \frac{1}{3}$. Also $g(k-g)$ is an increasing function of g if $g < \frac{1}{2}k$, so the minimum value of $g(k-g)$ in the range $\frac{1}{3}k + \frac{1}{3} \leqslant g < \frac{1}{2}k$ is

$$(\tfrac{1}{3}k + \tfrac{1}{3})(k - \tfrac{1}{3}k - \tfrac{1}{3}) = \tfrac{1}{9}(2k^2 + k - 1).$$

Thus from (37) we deduce that

$$(k+1)R - m(m^2-1) \geqslant \tfrac{1}{9}(2k^2+k-1)m(2m-k-2).$$

This simplifies to

(38) $$9m^3 - m^2(5k^2 + 16k + 11) - m(2k^3 - 4k^2 - 8k + 7) + (9k^3 + 18k^2 + 18k + 9) \leqslant 0.$$

Suppose $k^2 < 2m$. Then (38) can be written

$$9m^3 - m^2(5k^2 + 20k + 11) + 2mk(2m - k^2) + m(4k^2 + 8k - 7) + (9k^3 + 18k^2 + 18k + 9) \leqslant 0.$$

Hence $9m^3 - m^2(5k^2 + 20k + 11) < 0$, so $9(m+1) < 5(k+2)^2$.

If $k^2 \geqslant 2m$, then certainly $9(m+1) < 5(k+2)^2$. Hence in any case

$$k + 2 > [9(m+1)/5]^{\frac{1}{2}}. \tag{39}$$

The results for the cases A1, A2, A3 are, by (26), (28), and (36),

A1. $k \geqslant (m-3)^{\frac{1}{2}}$,

A2. $k \geqslant (m-4)^{\frac{1}{2}}$,

A3. $\begin{cases} k \geqslant (m-2)^{\frac{1}{2}} & \text{if } m \geqslant 18, \\ k \geqslant (m-1)^{\frac{1}{2}} & \text{if } m \leqslant 17. \end{cases}$

Combining these results, we can say that if $g \leqslant \frac{1}{3}k$, then $k \geqslant (m-4)^{\frac{1}{2}}$. Since we cannot have $k = 2$, we cannot have $k = (m-4)^{\frac{1}{2}}$ if $m \leqslant 12$, so $k > (m-4)^{\frac{1}{2}}$ and thus

$$k \geqslant (m-3)^{\frac{1}{2}} \quad \text{if } m < 13. \tag{40}$$

In case B, we have $\frac{1}{3}k < g < \frac{1}{2}k$, which is impossible unless $k \geqslant 5$. Then $\frac{1}{2}(m+1) \geqslant k+1 \geqslant 6$ (by 2.3) so $m \geqslant 11$. If $m = 11$ or 12, both (39) and (40) give $k \geqslant 3$. If $m \geqslant 13$ we may easily verify that

$$[9(m+1)/5]^{\frac{1}{2}} - 2 > (m-4)^{\frac{1}{2}}.$$

Hence the inequalities obtained in case A are also valid in case B.

Since we are now considering $m \geqslant 7$, our new inequalities are better than

$$k + 1 \geqslant (m+1)^{\frac{1}{2}}$$

obtained in 3.6. We have now proved

THEOREM 5.4. *Either* $m = 2$, $k = 0$, $n = 3$; *or* $m = 3$, $k = 4$, $n = 4$; *or* $m = 3$, $k = 1$, $n = 7$; *or* $7 \leqslant m \leqslant 12$ *and* $\frac{1}{2}m - \frac{1}{2} \geqslant k \geqslant (m-3)^{\frac{1}{2}}$ *which implies that* $m^2 - 1 - (m-3)^{\frac{1}{2}} \geqslant n \geqslant m^2 - \frac{1}{2}m - \frac{1}{2}$; *or* $m \geqslant 13$ *and*

$$\tfrac{1}{2}m - \tfrac{1}{2} \geqslant k \geqslant (m-4)^{\frac{1}{2}}$$

which implies that $m^2 - 1 - (m-4)^{\frac{1}{2}} \geqslant n \geqslant m^2 - \frac{1}{2}m - \frac{1}{2}$. *Moreover,* $k \neq 2$.

6. The Desarguesian case. A projective plane is *Desarguesian* if it satisfies the axiom of Desargues, i.e., if any two triangles in central perspective are also in axial perspective.

An affine plane is *Desarguesian* if its projective extension (as defined in § 1)is Desarguesian.

It is well known that a Desarguesian plane (projective or affine) may be co-ordinatized using elements of a unique skew field (e.g., **1**, p. 374). The *characteristic* of a Desarguesian plane is the characteristic of the skew field of co-ordinates.

An affine plane and its projective extension have the same skew field of co-ordinates, so they have the same characteristic.

Ostrom and Sherk (**5**, p. 556) have investigated the conditions under which an affine plane of order 3 can be embedded in a Desarguesian projective plane of finite order. The proof of their result can easily be adapted to the infinite case, using a skew field, and with a little extra calculation we can extend their result to

THEOREM 6.1. *The Desarguesian projective plane* π, *co-ordinatized by the skew field* k, *contains an affine subplane* π_0 *of order* 3 *if and only if* k *contains an element* t *such that* $t^2 + t + 1 = 0$. *This means that either* (a) k *has characteristic* 3 *and* $t = 1$ *or* (b) k *contains a primitive cube root of unity, which, if* k *is finite, occurs if and only if the order of* k *is congruent to* 1 (mod 3).

The bundles of parallels in π_0 *form pencils of concurrent lines in* π *if and only if* k *has characteristic* 3, *in which case the four vertices of these pencils are collinear in* π, *so that* π *contains the projective extension of* π_0.

It is natural to ask what happens if the order of π_0 is greater than 3. We shall prove

THEOREM 6.2. *Let* π *be a Desarguesian projective plane containing an affine subplane* π_0 *of order greater than* 3. *Then*

(a) π_0 *is Desarguesian,*
(b) *each bundle of parallels in* π_0 *forms a pencil of concurrent lines in* π,
(c) *the vertices of all these pencils are collinear in* π, *so that* π *contains the projective extension of* π_0,
(d) π_0 *has the same characteristics as* π.
(e) *If* π *is finite, the order of* π *is a power of the order of* π_0.

Before giving the proof, we need a lemma.

LEMMA 6.3. *Let* L, M *be two points of an affine plane* π_0, *and let* f *be a line through* L *and* g *a line through* M, *both distinct from the line* $h = LM$. *Let* $l \to l^*$ *be a one–one mapping of the pencil of lines through* L *onto the pencil of lines through* M *such that* $h \to h$ *and* $f \to g$. *If this mapping has the property that* l *and* l^* *are parallel whenever* $l \neq f$ *and* $l \neq h$, *then* f *and* g *are parallel.*

Proof. Suppose f and g are not parallel. Let l_0 be the line through L parallel to g. Then $l_0 \neq h$ and $l_0 \neq f$. Hence l_0^* is parallel to l_0, and $l_0^* \neq g$ (since the mapping is one–one). Thus we have two distinct lines through M parallel to l_0, namely l_0^* and g, a contradiction. Hence f and g are parallel.

Proof of 6.2. Let a, b, c be any three parallel lines of π_0. Let A, B, C be any three non-collinear points on a, b, c (Figure 11). Since π_0 has order greater than 3, there exists a point L on BC, $L \neq B$, C, and L not lying on a. Let the line h through L parallel to a, b, c meet AB, AC at N, M. (The line h cannot be parallel to AB or AC, so M, N exist in π_0.)

Let X be a general point of a, and let NX meet b at Y. Write $LY = l$, $MX = l^*$. Then the mapping $h \to h$, $l \to l^*$ is a one–one mapping of the pencil of lines through L onto the pencil of lines through M, with the property $f \to g$, where $f = LB$, $y = MA$ (taking $X = A$). By 6.3, if l were

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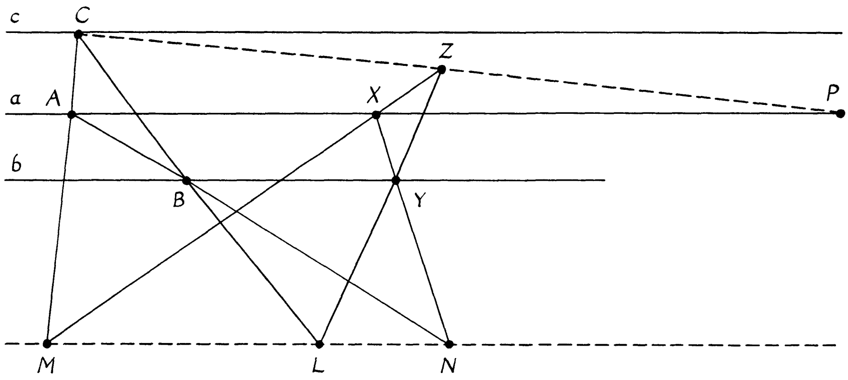

FIGURE 11

parallel to l^* whenever $X \neq A$, then f and g would be parallel, a contradiction since f and g meet at C. Hence there exists a point $X \neq A$ on a, and a corresponding point $Y \neq B$ on b (N, X, Y being collinear) such that LY and MX are not parallel. (The symbol X will now denote this particular point rather than a general point of a; similarly for Y.) Let $LY \cap MX = Z$, where $Z \in \pi_0$.

Suppose Z does not lie on c. Then CZ is not parallel to a, so CZ meets a at P, say, where $P \in \pi_0$. Then the triangles LCZ, NAX are in central perspective from M. Also $CZ \cap AX = P$, $ZL \cap XN = Y$, and $LC \cap NA = B$. Hence by the axiom of Desargues in π, P, Y, B are collinear. Thus $BY = b$ meets a at $P \in \pi_0$, a contradiction since b and a are parallel. Hence $Z \in c$.

Now let P denote the point of π where the parallel lines CZ, AX meet. By the above argument, P, Y, B are collinear in π. Hence a, b, c are concurrent in π.

It follows that all the lines of any bundle of parallels are concurrent in π. Thus we have proved (b).

Now let P, Q, R be the vertices, in π, of three distinct bundles of parallels in π_0, and let ABC be any triangle in π_0 such that BC, CA, AB pass through P, Q, R respectively (Figure 12). Let O be a point of π_0 not on a side of the triangle, and let A^*, B^* be points of π_0 on OA, OB such that A^*B^* and AB are parallel. Then A^*B^* passes through R.

Let C' be a point of π_0 on OC such that A^*C' is *not* parallel to AC. If C' is distinct from O and C, let $A^*C' \cap AC = Q'$. Let BC meet the line $Q'R$ parallel to AB in P'. Then $P' \in \pi_0$ since BC and AB are not parallel. Applying the axiom of Desargues in π to triangles BCO, $RQ'A^*$, in central perspective from A, we see that C', B^*, P' are collinear. Thus B^*C' and BC are not parallel. Also if $C' = O$ or $C' = C$, then B^*C' and BC are not parallel.

Hence if C^* is the point on OC such that A^*C^* *is* parallel to AC, then B^*C^* must be parallel to BC. Thus A^*C^* passes through Q and B^*C^* passes through P. Applying the axiom of Desargues in π to the triangles ABC,

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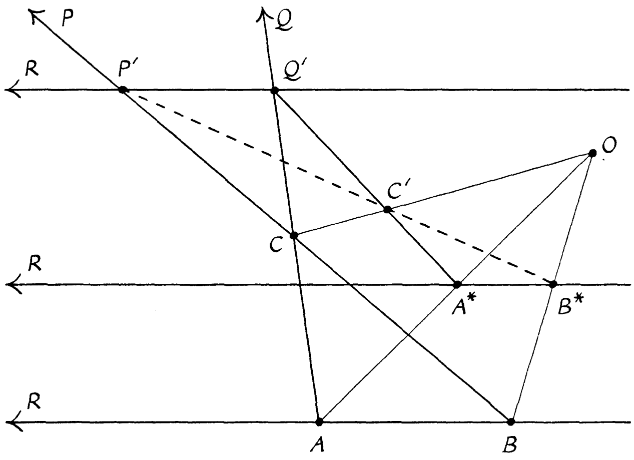

FIGURE 12

$A^*B^*C^*$, in central perspective from O, we see that P, Q, R are collinear in π. It follows that the vertices of all bundles are collinear. Thus we have proved (c).

Now the projective extension π^* of π_0, being a projective subplane of the Desarguesian projective plane π, must be Desarguesian. Hence π_0 is Desarguesian by definition. Thus we have proved (a).

A Desarguesian plane of given characteristic is characterized by an incidence theorem giving rise to a configuration which occurs only in such planes. If such a configuration occurs in π^*, then it will occur in π. Thus π, π^* have the same characteristic. Hence π, π_0 have the same characteristic. Thus we have proved (d).

It is easily shown that the skew field of co-ordinates of π_0 is a sub-skew-field of the skew field of co-ordinates of π. Hence if π is finite, the order of π is a power of the order of π_0. Thus we have proved (e).

From Theorems 6.1 and 6.2 we see that the planes π of the type considered in §§ 2–5 cannot be Desarguesian, except in the examples already considered ($m = 2$, $n = 3$; $m = 3$, $n = 4$; and $m = 3$, $n = 7$). Also the number $n = m^2 - 1 - k$, where $k \leqslant \frac{1}{2}m - \frac{1}{2}$, cannot be a square, since it lies between m^2 and $(m-1)^2$. Thus π cannot be a Hughes plane (**1**, p. 416), nor can it be a plane co-ordinatized by a Hall system (**1**, p. 364).

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