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THE STRUCTURE OF FINITE GROUPS IN WHICH PERMUTABILITY IS A TRANSITIVE RELATION

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Abstract

The structure of finite groups in which permutability is transitive (PT-groups) is studied in detail. In particular a finite PT-group has simple chief factors and the p-chief factors fall into at most two isomorphism classes. The structure of finite T-groups, that is, groups in which normality is transitive, is also discussed, as is that of groups generated by subnormal or normal PT-subgroups.

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1. Introduction

A subgroup H of a group G is called *permutable* (or *quasinormal*) if $HK = KH$ for all subgroups K of G. Here we are interested in groups G in which permutability is transitive, that is, H permutable in K and K permutable in G always imply that H is permutable in G; such groups are called *PT-groups*. In the sequel *all groups are understood to be finite*.

By a well-known theorem of Ore [11] a permutable subgroup is subnormal. Thus the (finite) PT-groups are exactly the groups in which all subnormal subgroups are permutable. A subclass of PT is the class of *T-groups*, or groups in which normality is transitive, that is, $H \lhd K \lhd G$ implies $H \lhd G$.

T-groups have as long history going back to Dedekind [5], while PT-groups were first studied by Zacher [16] in 1964. Soluble T-groups were classified by Gaschütz [6], while Zacher [16] classified soluble PT-groups. Zacher's main result asserts that a group G is a soluble PT-group if and only if it has a normal abelian Hall subgroup

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L of odd order such that G/L is a nilpotent modular group and elements of G induce power automorphisms in L. (A *modular group* is one whose subgroup lattice satisfies the modular law: for the structure of modular groups see Schmidt [15, page 55]). Gaschütz's theorem is obtained from Zacher's on replacing 'nilpotent modular group' by 'Dedekind group'.

Soluble T-groups and soluble PT-groups have been characterized in terms of their Sylow structure and also in terms of pronormality of subgroups by Peng [12], Robinson [13], and by Beidleman, Brewster and Robinson [2] respectively. For survey of results in the area see [14].

In the present work the emphasis is on insoluble PT-groups and T-groups. Since all simple groups are T-groups, there was little activity in this area prior to the classification of finite simple groups. Now that the CFSG is considered complete, and in particular the Schreier Conjecture has been verified, one can hope to make progress in determining the structures of PT-groups and T-groups.

We begin our study by showing in Section 2 that PT-groups have simple chief factors, (that is, they are *SC-groups*), and describing the structure of SC-groups. The more delicate task of identifying PT-groups and T-groups within the class of SC-groups is undertaken in Section 3 and Section 4, where two main structure theorems, Theorem 3.1 and Theorem 4.1, are established. These show that insoluble PT-groups and T-groups arise as extensions of covering groups of direct products of simple groups by soluble groups, subject to certain restrictions on the p-factors.

In Section 5 it is shown that the p-chief factors of a PT-group G fall into at most two G-isomorphism classes. This extends a previous result of Cossey [4]. In Section 6 the structure of groups generated by their normal or subnormal PT- or T-subgroups is investigated. In particular it is shown that a group generated by its normal PT-subgroups has simple chief factors, which generalizes another result of Cossey [3].

In Section 7 a number of counterexamples are described. These are designed to show that no inclusions exist between the various classes of groups under consideration other than the obvious ones, and also to shed light on differences between PT-groups and T-groups. In a final note we point out that the wider class of groups in which every subnormal subgroup permutes with all the Sylow subgroups, the so-called *PST-groups*, is amenable to a very similar treatment.

2. Groups with simple chief factors

DEFINITION. We will call a group an *SC-group* if all its chief factors are simple, and an *SNAC-group* if its non-abelian chief factors are simple.

The connection with PT-groups is shown by our first result:

PROPOSITION 2.1. *Every PT-group is an SC-group.*

In the proof we shall use the following result, which was proved in [2], (see Lemma 2).

LEMMA 2.2. *Let G be a PT-group. Then p'-elements of G induce power automorphisms in $O_p(G)$.*

PROOF OF PROPOSITION 2.1. Let N be a minimal normal subgroup of a PT-group G. We argue that N is simple. Suppose first that N is not abelian. Then $N = S_1 \times S_2 \times \cdots \times S_k$, where each S_i is a (non-abelian) simple group. Let $g \in G$. Now S_i is subnormal, and hence permutable, in G. Thus $\langle g\rangle S_i = S_i \langle g\rangle$ and $S_i^{\langle g\rangle} = S_i(S_i^{\langle g\rangle} \cap \langle g\rangle)$. Now $S_i^{\langle g\rangle}$ is a direct product of simple groups, yet $S_i^{\langle g\rangle}/S_i$ is cyclic. Hence $S_i^{\langle g\rangle} = S_i$ and $S_i \lhd G$. It follows that N is simple.

Next suppose that N is an elementary abelian p-group. By Lemma 2.2 each p'-element of G induces a power automorphism in N. Since power automorphisms belong to the centre of $\mathrm{Aut}(G)$, it follows that $\overline{G} = G/C_G(N)$ is (abelian p')-by-p. Let $\overline{P}$ be a Sylow p-subgroup of $\overline{G}$. Then $C_N(\overline{P}) \neq 1$ and $C_N(\overline{P})$ is left fixed by each p'-element of G. Therefore $C_N(\overline{P})$ is G-invariant and $N = C_N(\overline{P})$. Hence all elements of G induce power automorphisms in N, so that $|N| = p$. □

The next two results provide characterizations of $SNAC$-groups and SC-groups; in particular they give information about the structure of PT-groups.

PROPOSITION 2.3. *A group G is SNAC if and only if it has normal subgroups S and D such that S and G/D are soluble while D/S is a direct product of G-invariant simple groups.*

PROOF. Let G be an $SNAC$-group and put $D = G^{(\infty)}$, the limit of the derived series. Thus D is perfect and G/D is soluble. Factoring out by the soluble radical of D, we can assume that $D \neq 1$ is semisimple, that is, it has no non-trivial abelian normal subgroups. Choose a minimal normal subgroup N of G contained in D. Then N is simple, and the truth of the Schreier Conjecture shows that $D/N C_D(N)$ is soluble. Hence $D = N \times C_D(N)$. If $C_D(N) \neq 1$, choose a minimal normal subgroup of G contained in $C_D(N)$ and repeat the argument. After sufficiently many applications of this procedure we will find that D is the direct product of minimal normal subgroups of G each of which is simple.

Conversely, assume that G possesses normal subgroups S and D as described in the statement. If N is a non-abelian minimal normal subgroup of G, then $N \cap S = 1$ and $N \leq D$. Hence $N \overset{G}{\simeq} NS/S \leq D/S$ and it follows that N is simple. □

SC-groups can be characterized in a similar fashion.

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PROPOSITION 2.4. *A group G is an SC-group if and only if there is a perfect normal subgroup D such that G/D is supersoluble, $D/Z(D)$ is a direct product of G-invariant simple groups, and $Z(D)$ is supersolubly embedded in G, (that is, there is a G-admissible series in $Z(D)$ with cyclic factors).*

PROOF. Let G be an SC-group and put $D = G^{(\infty)}$; thus G/D is supersoluble. Denote by S the soluble radical of D and form a G-composition series in S, noting that its factors are cyclic and D is perfect. This implies that S lies in the hypercentre, and hence the centre, of D. Thus $S = Z(D)$. Evidently S is supersolubly embedded in G, and by the proof of Proposition 2.3 D/S is a direct product of G-invariant simple groups.

Conversely, let G have a normal subgroup D as indicated in the statement. If N is a minimal normal subgroup of G, then N is G-isomorphic with a chief factor in $Z(D)$, $D/Z(D)$ or G/D. In each case N is simple. □

COROLLARY 2.5. *An SC-group G is an extension of a perfect T-group by a supersoluble group.*

PROOF. Let $D = G^{(\infty)}$; we show that D is a T-group. If not, let H be a non-normal subnormal subgroup of D with least order. Then $H \not\leq Z := Z(D)$, so $HZ/Z \geq U/Z$, a simple normal subgroup. Thus $H' = (HZ)' \geq U'$. Since $U' \neq 1$, we get $H \lhd G$ by minimality. □

EXAMPLE. Let R be the Suzuki group $Sz(8)$. Its multiplicator $M(R)$ is a Klein 4-group, while $\mathrm{Out}(R)$ has order 3 and acts fixed-point-freely on $M(R)$. An automorphism of R with order 3 extends to an automorphism α of the covering group D of R. Let $G = \langle\alpha\rangle \ltimes D$, the semidirect product. Here $Z(G)$ is minimal normal in G and $Z(G) \simeq M(R)$. Thus G is not an SC-group, so that the condition of supersoluble embeddability in Proposition 2.4 is essential.

Further structural information about SC-groups is contained in:

LEMMA 2.6. *Let G be an SC-group, $D = G^{(\infty)}$ and $R \simeq D/Z(D)$. Then*

(i) *$C_G(D) = C_G(R)$ is the soluble radical S of G;*

(ii) *DS/D is the kernel of the coupling $\chi : G/D \to \mathrm{Out}(D)$ of the extension $D \rightarrowtail G \twoheadrightarrow G/D$.*

PROOF. (i) Let S be the soluble radical of G. Now $[D, S] \leq Z(D)$ and $D = D'$, so $[D, S] \leq [D, S, D] = 1$ and $S \leq C_G(D) \leq C_G(R)$. On the other hand, $C_G(R)$ is evidently soluble; hence $C_G(R) = S$.

(ii) This follows at once from (i). □

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Note that G/DS is isomorphic with a supersoluble subgroup of $\mathrm{Dr}_{i=1}^{k} \mathrm{Out}(R_i)$, where $R = R_1 \times R_2 \times \cdots \times R_k$ and R_i is simple. Furthermore, $\mathrm{Out}(R_i)$ is a soluble group of restricted type; for example, its derived length is ≤ 3. Also $Z(D)$ is a quotient of $M(R) \simeq M(R_1) \oplus M(R_2) \oplus \cdots \oplus M(R_k)$ and the $M(R_i)$ are known. Thus a rather clear picture of the structure of an SC-group emerges.

3. Characterizations of PT-groups

Consider the central product

$$G = SL_2(5) \,\mathrm{Y}\, \mathrm{Dih}(8),$$

where the centres of $SL_2(5)$ and $\mathrm{Dih}(8)$ are identified. Here $G'' = SL_2(5)$, $|Z(G'')| = 2$ and G/G'' is a Klein 4-group. Certainly, G is an SC-group, but it is not a PT-group since $\mathrm{Dih}(8)$ is not modular. This example shows that in a PT-group there must be additional restrictions on normal p-subgroups beyond what is implied by Proposition 2.4.

DEFINITION. Let p be a prime.

(i) A group G satisfies the *condition* $\mathbf{N_p}$ if, for all soluble normal subgroups N, the p'-elements of G induce power automorphisms in $O_p(G/N)$.

(ii) A group G satisfies the *condition* $\mathbf{P_p}$ if, for all soluble normal subgroups N, each subgroup of $O_p(G/N)$ is permutable in a Sylow p-subgroup of G/N.

Clearly every PT-group satisfies $\mathbf{P_p}$, and by Lemma 2.2 it also satisfies $\mathbf{N_p}$ for all primes p. Thus $\mathbf{N_p}$ and $\mathbf{P_p}$ are necessary conditions if a group is to be a PT-group. We shall show that these conditions on quotients are also sufficient for an SC-group to be PT. On combining $\mathbf{N_p}$ and $\mathbf{P_p}$ with other properties known to hold for SC-groups, a characterization of PT-groups is obtained.

THEOREM 3.1. *A group G is a PT-group if and only if it has a perfect normal subgroup D such that*:

(i) *G/D is a soluble PT-group*;

(ii) *$D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$ where $U_i/Z(D)$ is simple and $U_i \lhd G$*;

(iii) *if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$, where $1 \leq r < k$, then $G/U'_{i_1}U'_{i_2}\ldots U'_{i_r}$ satisfies $\mathbf{N_p}$ for all $p \in \pi(Z(D))$ and $\mathbf{P_p}$ for all $p \in \pi(D)$.*

PROOF. Only the sufficiency of the three conditions is in doubt. So assume that G satisfies the conditions but is not a PT-group, and that of all such groups G has smallest order. Let H be a subnormal subgroup of G which is not permutable.

Case (a): *H is insoluble.* Then $(H \cap D)Z/Z$ is non-trivial and subnormal in D/Z. By (ii) it must contain some U_i/Z, and therefore $H' \geq ((H \cap D)Z)' \geq U_i'$. Passing to G/U_i', which inherits the hypotheses on G, we conclude that H/U_i' is permutable in G/U_i', that is, H is permutable in G, a contradiction.

Case (b): *H is soluble.* Here H is contained in the soluble radical S of G. Put $K = \gamma_\infty(S)$, the limit of the lower central series of S. We claim that $H \cap K \lhd G$. Since G/D is a soluble PT-group, KD/D is abelian by Zacher's theorem. Also $K \cap D \leq Z(K)$ since $[D, S] = [D', S] \leq [D, S, D] = 1$. Hence K is nilpotent, and it is enough to show that $H \cap K_p \lhd G$ for all primes p.

If $K_p \leq Z := Z(D)$, then $[K_p, S] = 1$ and $[K, S] \neq K$. Hence $K_p \nleq Z$ and so $K_p \nleq D$. We can assume that $p \in \pi(Z)$. For otherwise $K_p \cap D = 1$ and $K_p \overset{G}{\simeq} K_pD/D \leq \gamma_\infty(G/D)$; therefore elements of G induce power automorphisms in K_p and $H \cap K_p \lhd G$.

Since $1 \neq K_pD/D \leq \gamma_\infty(G/D)$, which is a Hall subgroup of G/D, we see that p cannot divide $|G/D : \gamma_\infty(G/D)|$. Now consider $G/C_G(K_p)$; by $\mathbf{N_p}$ the p'-elements in this group form a normal subgroup $V/C_G(K_p)$ and G/V is a p-group. Therefore $\gamma_\infty(G/D) \leq V/D$ and consequently $V = G$, so that $H \cap K_p \lhd G$, as required.

Now pass to the group $G/H \cap K$ and use minimality of order to conclude that $H \cap K = 1$. Hence H is nilpotent, and obviously we can suppose it is a p-group. It is enough to show that $H\langle g \rangle = \langle g \rangle H$ where g is either a p-element or a p'-element. Let g be a p'-element. If $p \in \pi(Z)$, the condition $\mathbf{N_p}$ implies that $H^g = H$. If on the other hand $p \notin \pi(Z)$, then $H^G \cap Z = 1$, so that $H^G \cap D = 1$ and $H^G \overset{G}{\simeq} H^GD/D$. Since g induces power automorphisms in $O_p(G/D)$, we again obtain $H = H^g$.

Finally, suppose that g is a p-element. Let P be a Sylow p-subgroup containing g; then of course $H \leq P$. If $p \in \pi(D)$, we have $\langle g \rangle H = H\langle g \rangle$ by condition $\mathbf{P_p}$. If $p \notin \pi(D)$ on the other hand, $P \cap D = 1$ and $P \simeq PD/D$, showing that P is modular and $H\langle g \rangle = \langle g \rangle H$. □

There is a simpler, but weaker criterion for a group to be a PT-group.

THEOREM 3.2. *A group G is a PT-group if and only if its non-abelian chief factors are simple and each quotient of G satisfies* $\mathbf{N_p}$ *and* $\mathbf{P_p}$ *for all primes p.*

PROOF. Again only sufficiency is in question. Let G satisfy the condition and be a counterexample of smallest order. Suppose that N is an abelian minimal normal subgroup, and say it is an elementary abelian p-group. By $\mathbf{N_p}$ the group $G/C_G(N)$ has a p'-subgroup of power automorphisms with p-power index. Let P be a Sylow p-subgroup of G. Choose $a \neq 1$ from $N \cap Z(P)$. Then $\langle a \rangle \lhd G$, and hence $|N| = p$. It follows that G is an SC-group.

Next apply Proposition 2.4 to get a perfect normal subgroup D such that G/D is soluble and $D/Z(D)$ is a direct product of G-invariant simple groups. By Theorem 3.1

we reduce to the case $D = 1$. Hence G is supersoluble.

Let H be subnormal but not permutable in G. Put $K = \gamma_\infty(G)$, which is nilpotent since G is supersoluble. We will argue that $H \cap K \lhd G$. If this is false, $H \cap K_p$ is not normal in G for some p. Let L be a minimal normal subgroup of G contained in K_p. Thus $|L| = p$. Now G/L is a soluble PT-group. By Zacher's theorem p does not divide $|G/L : \gamma_\infty(G/L)|$ since $K_p \not\leq L$, (otherwise $H \cap K_p \lhd G$). Now argue as in the proof of Theorem 3.1 that each element of G induces power automorphisms in K_p, so that $H \cap K_p \lhd G$.

We have reached the stage where $H \cap K \lhd G$ and so $H \cap K = 1$. The rest of the argument is the same as for Theorem 3.1. □

EXAMPLE. An SC-group with $\mathbf{N_p}$ and $\mathbf{P_p}$ for all p need not be a PT-group.

Thus in Theorem 3.1 and Theorem 3.2 it is not sufficient to assume that the group satisfies $\mathbf{N_p}$ and $\mathbf{P_p}$.

Let $D_1 = SL_3(4)$ and $D_2 = PSL_2(8)$. Then D_1 and D_2 have field automorphisms α_1, α_2 of orders 2 and 3 respectively. Let $G_i = \langle \alpha_i \rangle \ltimes D_i$, the semidirect product, and put $G = G_1 \times G_2$. Clearly G is an SC-group. Note that $G'' = D = D_1 \times D_2$ and $Z(D) = Z(D_1)$ has order 3, while $G/D \simeq \mathbb{Z}_6$. Hence G satisfies $\mathbf{N_p}$ and $\mathbf{P_p}$ for all p. However even G/D_2 is not a PT-group since it does not satisfy $\mathbf{N_3}$.

4. Characterizations of T-groups

To obtain characterizations of T-groups corresponding to Theorem 3.1 and Theorem 3.2 the conditions $\mathbf{N_p}$ and $\mathbf{P_p}$ are replaced by a single stronger condition.

DEFINITION. Let p be a prime. A group G satisfies the *condition* $\mathbf{T_p}$ if, for all soluble normal subgroups N, elements of G induce power automorphisms in every G-invariant p-factor X/N of nilpotent class ≤ 2.

Clearly every T-group satisfies $\mathbf{T_p}$ for all p. Using this condition we formulate our characterization of T-groups.

THEOREM 4.1. *A group G is a T-group if and only if it has a perfect normal subgroup D such that*:

(i) *G/D is a soluble T-group*;

(ii) *$D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$, where $U_i/Z(D)$ is simple and $U_i \lhd G$*;

(iii) *if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$, where $1 \leq r < n$, the group $G/U'_{i_1} \ldots U'_{i_r}$ satisfies $\mathbf{T_p}$ for all $p \in \pi(Z(D))$.*

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PROOF. Only the sufficiency requires a proof. This is quite similar to the proof of Theorem 3.1, so we only indicate the differences. Let G be a counterexample of minimal order and let H be a non-normal subnormal subgroup of G. If H is insoluble, then argue as before that H contains some U_i'; a contradiction to minimality then ensues. Thus H is soluble and hence is contained in the soluble radical S.

Let $K = \gamma_3(S)$ and note that KD/D is abelian, so that K is nilpotent of class at most 2. Argue as before that $H \cap K \lhd G$; hence $H \cap K = 1$ and H is nilpotent of class ≤ 2. We can assume H is a p-group. If $p \notin \pi(Z)$, then $H^G \cap Z = 1$ and $H^G \overset{G}{\simeq} H^G D/D$; this implies that $H \lhd G$ since G/D is a T-group. Hence $p \in \pi(Z)$.

Next $H^G \leq HD$ since $HD \lhd G$, and therefore $H^G \leq HD \cap S = HZ$. Hence $\gamma_3(H^G) \leq \gamma_3(H) = 1$. By the condition $\mathbf{T_p}$ elements of G induce power automorphisms in H^G and thus $H \lhd G$. □

There is a simpler version of the criterion, just as in the case of PT-groups.

THEOREM 4.2. *A group G is a T-group if and only if its non-abelian chief factors are simple and each quotient of G satisfies $\mathbf{T_p}$ for all primes p.*

PROOF. To prove sufficiency, let G be a counterexample of smallest order. By the proof of Theorem 3.2 the group G is supersoluble. Let H be a non-normal subnormal subgroup of G and put $K = \gamma_3(G)$. Reduce to the case $H \cap K = 1$ as in Theorem 3.2. Hence H is nilpotent of class at most 2. Assume that H is a p-group. Now $K \neq 1$, so there is a minimal normal subgroup L of G contained in K. Hence $HL \lhd G$ and $H^G \leq HL$. But $[H, L] = 1$ since H lies in the Fitting subgroup. Hence $\gamma_3(H^G) = \gamma_3(H) = 1$; now apply $\mathbf{T_p}$ to H^G to get $H \lhd G$. □

An essentially equivalent result has been found by Cossey [4, Theorem 1]. Characterizations of some special types of T-group can be found in the same paper.

5. Chief factors

A distinguishing feature of PT-groups, as against SC-groups in general, is the behaviour of the abelian chief factors. In any PT-group the p-chief factors fall into at most two G-isomorphism classes.

THEOREM 5.1. *Let G be a PT-group and let p be any prime. Then all p-chief factors covered by G'' are G-isomorphic, as are all p-chief factors avoided by G''. Hence there are at most two G-isomorphism classes of p-chief factors.*

Recall here that G'' is said to *cover* a chief factor H/K if $HG'' = KG''$ and to *avoid* H/K if $H \cap G'' = K \cap G''$. On the other hand, it is easy to construct supersoluble groups G with $p - 1$ chief factors of order p no two of which are G-isomorphic.

COROLLARY 5.2. *In a soluble PT-group G all the p-chief factors are G-isomorphic.*

(For $G'' = 1$ if G is a soluble PT-group.)

PROOF OF THEOREM 5.1. Recall that p-chief factors of G have order p. Suppose first that G is soluble and put $L = \gamma_\infty(G)$. Then L is abelian, $\pi(L) \cap \pi(G/L)$ is empty, and elements of G induce power automorphisms in L. If $p \in \pi(L)$, then all p-chief factors are G-isomorphic since an element of G induces the same power automorphism in each factor. If $p \notin \pi(L)$, all p-chief factors are central.

Now for the general case. Let $D = G''$. A p-chief factor that is avoided by D is G-isomorphic to one of G/D. All such are G-isomorphic by the last paragraph. Consider a p-chief factor H/K which is covered by D, and hence by $Z(D)$. By Lemma 2.2 each p'-element induces a power automorphism in $Z(D)_p$, and hence in H/K. Of course p-elements centralize H/K. Consequently all such H/K are G-isomorphic. □

EXAMPLE (J. G. Thompson; see Cossey [4]). There is an insoluble T-group with two isomorphism classes of p-chief factors.

Let p be an odd prime and let q be a prime such that $q \equiv 1 (\operatorname{mod} p)$. Put $D = SL_p(q^p)$, and define two automorphisms τ, u of D as follows: $A^\tau = (A^{-1})^T$, while φ arises from the field automorphism $a \mapsto a^q$. Then $X = \langle \tau, \varphi \rangle$ is cyclic of order $2p$, and the semidirect product $G = X \ltimes D$ is a T-group by Theorem 4.1. But $Z(D)$ and $\langle \varphi D \rangle$ are non-G-isomorphic p-chief factors.

On the other hand, for certain primes p all the p-chief factors of an insoluble PT-group are isomorphic. The following result was established by Cossey [4] for T-groups.

THEOREM 5.3. *Let G be a PT-group with soluble radical S, and let p be a prime dividing $|S : Z(G'')|$. Then all p-chief factors are G-isomorphic.*

PROOF. We can assume that $O_{p'}(G) = 1$. Put $D = G''$. Since p divides $|SD : D|$, there is a p-chief factor H/K of G such that $D \le K < H \le SD$. Let L/M be another p-chief factor. If D avoids L/M, then $L/M \overset{G}{\simeq} LD/MD$ and hence $L/M \overset{G}{\simeq} H/K$ by Corollary 5.2. Assume therefore that D covers L/M, so that $L/M \overset{G}{\simeq} L \cap D/M \cap D \overset{G}{\simeq} L \cap Z/M \cap Z$ where $Z = Z(D)$. Note that Z is a p-group.

Next $O_{p'}(S/Z) = 1$; thus the Fitting subgroup of S/Z is a non-trivial p-group and contains a minimal normal subgroup N/Z of G/Z. Then $N = \langle a, Z \rangle$ is an abelian p-group since $[Z, S] = 1$. Further $N/Z \overset{G}{\simeq} ND/D \overset{G}{\simeq} H/K$. Since p'-elements of G induce power automorphisms in N, while p-elements centralize N/Z and L/M, we deduce that $L/M \overset{G}{\simeq} N/Z$. Hence $L/M \overset{G}{\simeq} H/K$. □

6. Groups generated by subnormal PT-subgroups

We will now broaden the investigation to include groups which are generated by their subnormal PT-subgroups. Although such groups need not be SC-groups, it will be seen that they are quite close to SC-groups. This has already been observed by Cossey [3] in the case of soluble groups generated by subnormal T-subgroups.

A special role is played by the subnormal perfect T-subgroups of a group: notice here that perfect SC-groups are T-groups by Corollary 2.5. We recall a theorem of Kegel [9]— see also [10, page 152]: *a subnormal perfect T-subgroup of a group G normalizes every subnormal subgroup, and so is contained in the Wielandt subgroup $\omega(G)$*. Since $\omega(G)$ is a T-group, it follows that there is a unique largest subnormal perfect T-subgroup in G.

This subgroup admits other descriptions, as is perhaps known.

LEMMA 6.1. *In any group G the following subgroups coincide*:

(i) *the unique maximum subnormal perfect T-subgroup*;

(ii) *$\omega''(G)$ where $\omega(G)$ is the Wielandt subgroup*;

(iii) *the layer $E(G)$, that is, the limit of the lower central series of the generalized Fitting subgroup.*

PROOF. Let $\tau(G)$ denote the subgroup in (i); then $\tau(G) \leq \omega(G)$. Also $\omega''(G) \leq \tau(G)$ because soluble T-groups are metabelian. But $\tau(G)$ is perfect, so $\tau(G) = \omega''(G)$.

Next $E(G)$ is perfect and $E(G)/Z(E(G))$ is a direct product of simple groups; thus $E(G)$ is a T-group and $E(G) \leq \tau(G)$. (For these and other facts about the subgroup $E(G)$ see [7, Section 13].) On the other hand, the structure of $\tau(G)$ shows that its elements induce inner automorphisms in chief factors of $\tau(G)$—see Proposition 2.4 and Theorem 4.1. Hence $\tau(G)$ is quasinilpotent and $\tau(G) \leq F^*(G)$, the generalized Fitting subgroup; thus $\tau(G) \leq \gamma_\infty(F^*(G)) = E(G)$. □

The main result on groups that are generated by subnormal PT-subgroups depends on work of Cossey [3].

THEOREM 6.2. *Let G be a group which is generated by subnormal PT-subgroups. Then*

(i) *non-abelian chief factors of G are simple, that is, G is a SNAC-group*;

(ii) *$\overline{G} := G/E(G)$ is metanilpotent, Sylow p-subgroups of $\overline{G}/\mathrm{Fit}(\overline{G})$ are abelian for odd p, and $O^2(\overline{G})$ is supersoluble of odd order.*

We precede the proof with two auxiliary results.

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LEMMA 6.3. *A soluble PT-group is generated by its subnormal T-subgroups.*

PROOF. Let G be a soluble PT-group and put $L = \gamma_\infty(G)$. Then L is an abelian Hall subgroup of G and elements of G induce power automorphisms in L. If $g \in G$, then $\langle g, L\rangle$ is a T-group, and it is subnormal in G. Thus G can be generated by subnormal T-subgroups. □

LEMMA 6.4. *If a group G is generated by subnormal SNAC-subgroups, then G is a SNAC-group.*

PROOF. By a standard induction on the subnormal defect it is enough to show that if $G = HK$ where H and K are normal *SNAC*-subgroups, then G is *SNAC*. To this end suppose that N is a non-abelian minimal normal subgroup of G which is not simple. If $N \not\leq H$, then $[N, H] = 1$ and so $[N, K] \neq 1$. Hence $N \leq K$ and N is minimal normal in K, which is impossible. This argument shows that $N \leq H \cap K$.

Now choose a minimal normal subgroup N_1 of H contained in N. Then N_1 is simple and $N = N_1^G = N_1^K$ is a direct product of simple groups. Choose a minimal normal subgroup N_2 of K contained in N. Then N_2 is also simple and $N_2 = N_1^k$ for some $k \in K$, that is, $N_2 = N_1$ and $N_1 = N$ is simple. □

PROOF OF THEOREM 6.2. Put $E = E(G)$ and let $H_1, \dots, H_r$ be subnormal PT-subgroups generating G. Now $E(H_i) = H_i''$ and $H_i/E(H_i)$ is a soluble PT-group by Corollary 2.5 and Lemma 6.1; thus $E(H_i) \leq E$. It follows that $\overline{G} := G/E$ is generated by the subnormal soluble PT-subgroups H_iE/E, so that $\overline{G}$ is soluble. By Lemma 6.3 $\overline{G}$ is even generated by subnomal T-subgroups. The rest of the statement now follows from Theorem 2 of Cossey [3]. □

We turn our attention next to groups that are generated by normal PT-subgroups. Here one should keep in mind that a product of normal supersoluble subgroups need not be supersoluble. Our aim is to show that, despite this negative result, a product of normal PT-subgroups is always an *SC*-group. We will in fact prove something more general.

THEOREM 6.5. *Let $G = HK$ where H is a normal PT-subgroup and K is a subnormal SC-subgroup of the group G. Then G is an SC-group.*

PROOF. First of all consider the case where $K \lhd G$. By Lemma 6.4 it is sufficient to prove that an abelian minimal normal subgroup N of G is cyclic. If $N \cap H = 1 = N \cap K$, then $N \leq Z(G)$ and all is clear. If $N \cap H = 1 \neq N \cap K$, then $[N, H] = 1$ and N is minimal normal in K and hence is cyclic. Therefore we reduce to the case where $N \leq H \cap K$.

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Put $D = H''$. Suppose first of all that $N \cap D = 1$, so that $N \overset{G}{\simeq} ND/D$ and we can assume that H is soluble. Put $L = \gamma_\infty(H)$. If $N \cap L = 1$, then $[N, H] = 1$, so that N is minimal normal in K and hence is cyclic. Otherwise $N \leq L$, and elements of H induce power automorphisms in L. Again N is minimal normal in K.

We are left with the case $N \leq D$, when of course $N \leq Z := Z(D)$. Suppose that N_1 is a minimal normal subgroup of H contained in N. Thus $|N_1| = p$, a prime, and $N = N_1^K$. If $k \in K$, then $N_1^k \lhd H$ and N_1^k is minimal normal in H. But H is a PT-group, so all the p-chief factors that are covered by D are isomorphic, by Theorem 5.1. Consequently an element of H induces the same power automorphism in each N_1^k, and thus induces a power automorphism in N. Hence N is minimal normal in K.

Finally, consider the general case. Here a familiar argument applies. Let $K = K_r \lhd K_{r-1} \lhd \cdots \lhd K_1 \lhd K_0 = G$ be the series of successive normal closures of K in G. Then $K_i = (H \cap K_i)K$ and $K_{i+1} = (H \cap K_{i+1})K_i$. Suppose that K_i is an SC-group. Since $H \cap K_{i+1}$ is a normal PT-subgroup of K_{i+1}, we deduce from the special case that K_{i+1} is an SC-group. Hence G is an SC-group. □

From this result we deduce at once:

THEOREM 6.6. *A group which is generated by normal PT-subgroups is an SC-group.*

7. Diagram of group classes

The eight classes of groups which feature in our investigation are displayed in the diagram (see Figure 1). Here, for example, $\langle sn\ PT\rangle$ is the class of groups generated by their subnormal PT-subgroups.

THEOREM 7.1. *There are no further inclusions between the eight classes of groups in the diagram.*

PROOF. It is sufficient to disprove four inclusions:

(i) $SC \not\subseteq \langle sn\ PT\rangle$.

Let $N = \langle a, b\rangle$ be a non-abelian group of order 5^3 and exponent 5. An automorphism τ of N with order 4 is defined by $a^\tau = a^2$, $b^\tau = b^3$. Then $G = \langle\tau\rangle \ltimes N$ is a supersoluble group of order 500. We show that G cannot be generated by subnormal PT-subgroups.

Let H be a subnormal PT-subgroup of G and suppose that $H \not\leq \langle\tau^2, N\rangle$. Then H contains an element τc with $c \in N$, and hence $H \geq [N, {}_m\tau c]$ for some $m > 0$. But

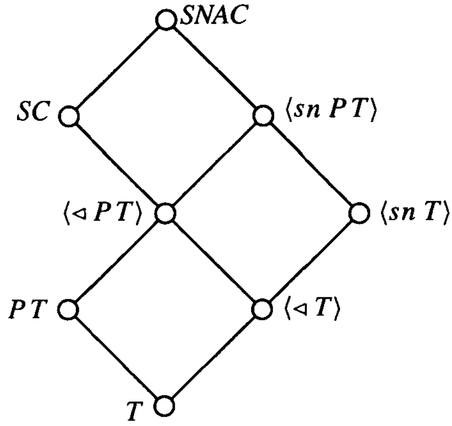

FIGURE 1. The eight classes of groups

$N = [N, \tau c]$, so $H \geq N$, and thus $H = G$. This is false since G is not PT-group. Therefore, all subnormal PT-subgroups are contained in $\langle \tau^2, N\rangle$.

(ii) $\langle sn\ T\rangle \not\subseteq SC$.

Let $N = \langle a, b\rangle$ be an elementary abelian group of order 9, and define automorphisms ξ and γ of N by

$$a^{\xi} = b, \quad b^{\xi} = a \quad \text{and} \quad a^{\eta} = a^{-1}, \quad b^{\eta} = b.$$

Then $X = \langle \xi, \eta\rangle$ is dihedral of order 8. Let $G = X \ltimes N$. Since N is minimal normal, G is not an SC-group. On the other hand, $H = \langle \xi, [N, \xi]\rangle$ and $K = \langle \eta, [N, \eta]\rangle$ are subnormal in G and $H \simeq S_3 \simeq K$. Thus G is generated by two subnormal T-subgroups.

(iii) $\langle \triangleleft\ T\rangle \not\subseteq PT$.

This is shown by the non-abelian group of order 27 and exponent 3.

(iv) $PT \not\subseteq \langle sn\ T\rangle$.

This is the most difficult non-inclusion to establish. Choose primes p and q such that $q \equiv 1 (\operatorname{mod} p)$ but $q \not\equiv 1 (\operatorname{mod} p^2)$, and let $R = PSL_{p^2}(q^p)$. Since p^2 divides $q^p - 1$, the subgroup $Z(SL_{p^2}(q^p))$ has an element of order p^2, and there is a stem extension $\langle a\rangle \rightarrowtail D \twoheadrightarrow R$, where $|a| = p^2$. □

The automorphism $f \mapsto f^q$ of $GF(q^p)$ gives rise to an automorphism α of $SL_{p^2}(q^p)$, and hence of D, with order p. Since $a^{\alpha} = a^q \neq a$ and $q \equiv 1 (\operatorname{mod} p)$, we can assume that $a^{\alpha} = a^{1+p}$, so that $(a^p)^{\alpha} = a^p$.

Let $\langle x\rangle$ be a cyclic group of order p^l where p^{l-1} exceeds the exponent of the Sylow p-subgroups of D. Then let x act on D via the automorphism α, and form the semidirect product $\langle x\rangle \ltimes D$. Now factor out by the central subgroup $\langle x^{p^{l-1}} a^p\rangle$ to obtain a group $G = \langle x\rangle D$ where $x^{p^{l-1}} = a^{-p}$. The soluble radical of G is

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$S = O_p(G) = \langle x^p, a\rangle$. Note that G/D is cyclic of order p^{l-1} and $Z(D) = \langle a\rangle$ has order p^2. Our aim is to establish:

LEMMA 7.2. *G is a PT-group, but it cannot be generated by subnormal T-subgroups.*

PROOF. There is a Sylow p-subgroup P of D such that $P^x = P$. Then $P_1 = \langle x\rangle P$ is a Sylow p-subgroup of G. We shall prove that every subgroup of S is permutable in P_1, thus verifying the condition $\mathbf{P_p}$.

Since $[a, x] = a^p$, we have $x^a = xa^{-p} = x^{1+p^{l-1}}$. Let $d \in P$. Then $(xd)^a = x^{1+p^{l-1}}d$. Now

$$
\begin{aligned}
(xd)^{p^{l-1}} &= x^{p^{l-1}} d^{x^{p^{l-1}-1}+x^{p^{l-1}-2}+\cdots+x+1} \\
&= x^{p^{l-1}} \left(d^{x^{p-1}+x^{p-2}+\cdots+x+1}\right)^{p^{l-2}} \\
&= x^{p^{l-1}},
\end{aligned}
$$

since $[D, x^p] = 1$ and $P^{p^{l-2}} = 1$. Hence

$$(xd)^a = x^{1+p^{l-1}}d = (xd)^{1+p^{l-1}}$$

and $x^{p^i}a$ normalizes $\langle xd\rangle$ for $i > 0$. Since $x^{p^i}a \in Z(\langle x^p, P\rangle)$, it follows that $\langle x^{p^i}a\rangle$ is permutable in P_1. Since all subgroups of $\langle x^p, a^p\rangle$ are normal in P_1, the condition $\mathbf{P_p}$ is verified.

Next all p'-elements of G lie in D and therefore centralize S. Thus the condition $\mathbf{N_p}$ holds in G. It follows from Theorem 3.1 that G is a PT-group.

Finally, let H be a subnormal T-subgroup of G. If H is insoluble, then $D \leq H$. Since x does not induce a power automorphism in S, the group G is not a T-group. Hence $H \leq \langle x^p, D\rangle$. This is also true if H is soluble. Consequently, G cannot be generated by subnormal T-subgroups. (The smallest example is for $p = 2$, $q = 3$). □

By Lemma 6.3 the classes $\langle sn\ PT\rangle$ and $\langle sn\ T\rangle$ coincide for soluble groups, so there are just seven classes of groups if we restrict attention to soluble groups. In fact no further inclusions hold amongst these seven classes. To see this just one further example is needed.

(v) There is a soluble PT-group which is not generated by normal T-subgroups.

For let G be the modular p-group $\langle x, a \mid a^x = a^{1+p}, a^{p^3} = 1 = x^{p^2}\rangle$, where p is an odd prime. Every normal T-subgroup is contained in $\langle x^p, a\rangle$.

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Further examples

Our main theorems in Section 3 and Section 4 show that PT-groups and T-groups have very similar structures. We conclude with two examples of PT-groups that are not T-groups to illustrate the differences between these types of group.

(I) *There is a PT-group which is generated by its normal T-subgroups but which is not a T-group.*

This is shown by the modular 2-group of order 16 with presentation $\langle x, a \mid a^x = a^5, x^2 = 1 = a^8 \rangle$. This is not a T-group but it is the product of two normal T-subgroups $\langle x, a^4 \rangle$ and $\langle a \rangle$.

It is a conspicuous feature of a T-group G that group elements induce power automorphisms in each normal nilpotent subgroup, and in particular in $Z(G'')$. We show by a more elaborate example that this last property may fail in a PT-group.

(II) *There is a PT-group G with an element which does not induce a power automorphism in $Z(G'')$.*

Let R be a simple group of Lie type whose multiplicator $M(R)$ has a subgroup of type $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then there is a subgroup V such that $M(R)/V \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and hence there is a stem extension $Z \rightarrowtail D \twoheadrightarrow R$ with $Z \simeq M(R)/V$. Assume that R has an automorphism α of order 2 which leaves V invariant and acts non-trivially on $M(R)/V$. Then α lifts to an automorphism of D.

Since α has a fixed point in Z, we can write $Z = \langle a \rangle \times \langle b \rangle$ where

$$a^\alpha = a \quad \text{and} \quad b^\alpha = ab.$$

Next choose l so that 2^{l-1} exceeds the exponent of a Sylow 2-subgroup of D. Let $\langle x \rangle$ be a cyclic group of order 2^l and form $\langle x \rangle \ltimes D$ where x acts on D according to α. Identify $x^{2^{l-1}}$ with a to obtain a group $G = \langle x \rangle D$ where $\langle x \rangle \cap D = \langle a \rangle$. The soluble radical of G is $O_2(G) = \langle x^2 \rangle Z$.

Let P be a Sylow 2-subgroup of D such that $P^x = P$. Then $P_1 = \langle x \rangle P$ is a Sylow 2-subgroup of G. By computations similar to those in the proof of Lemma 7.2, it can be shown that every subgroup of $O_2(G)$ is permutable in P_1. Hence G satisfies $\mathbf{P_2}$. Since $\mathbf{N_2}$ is obviously satisfied, G is a PT-group. Finally, conjugation by x in Z is not a power automorphism.

For example, one can take R to be $PSL_3(4)$ and $V \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Postscript: Sylow permutability

Let us say that a subgroup H of a group G is *Sylow permutable*, or *S-permutable*, if $HP = PH$ for all Sylow subgroups P of G. Kegel [8] proved that S-permutable subgroups are always subnormal. Therefore S-permutability is transitive in a group G if and only if every subnormal subgroup is S-permutable in G. We will call groups of this type *PST-groups*. Thus every PT-group is a *PST*-group.

It turns out that the theory of PT-groups extends readily to *PST*-groups, and in fact the proofs are very similar. In the first place the structure of soluble *PST*-groups has been determined by Agrawal [1], as follows.

(i) *A group G is a soluble PST-group if and only if it has a normal abelian Hall subgroup L of odd order such that G/L is nilpotent and elements of G induce power automorphisms in L.*

This, of course, is the analogue of Zacher's theorem. By arguments essentially identical to those given above, one can establish the following facts.

(ii) *There are characterizations of PST-groups corresponding to Theorems* 3.1 *and* 3.2 *which are obtained by omitting the conditions* $\mathbf{P}_p$ *in these results.*

Thus the difference between PT-groups and *PST*-groups is quite simply the property $\mathbf{P}_p$. Since $\mathbf{P}_\mathbf{p}$ is valid in any modular p-group, we deduce at once:

(iii) *A PST-group with modular Sylow subgroups is a PT-group.*

The theorems on p-chief factors of PT-groups and on groups generated by subnormal PT-subgroups also generalize to *PST*-groups.

(iv) *In a PST-group G all p-chief factors covered by G'' are G-isomorphic, as are all p-chief factors avoided by G''.*

(v) *A group which is generated by normal PST-subgroups is an SC-group.*

(I am grateful to J. C. Beidleman for bringing the class of *PST*-groups to my attention.)

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