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SHIFTED CONVOLUTION SUM OF d_3 AND THE FOURIER COEFFICIENT OF HECKE–MAASS FORMS

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Abstract

Let $\{\phi_j(z) : j \geq 1\}$ be an orthonormal basis of Hecke–Maass cusp forms with Laplace eigenvalue $1/4 + t_j^2$. Let $\lambda_j(n)$ be the nth Fourier coefficient of ϕ_j and $d_3(n)$ the divisor function of order three. In this paper, by the circle method and the Voronoi summation formula, the average value of the shifted convolution sum for $d_3(n)$ and $\lambda_j(n)$ is considered, leading to the estimate

$$\sum_{n \leq X} d_3(n)\lambda_j(n-1) \ll X^{29/30+\varepsilon},$$

where the implied constant depends only on t_j and ε.

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1. Introduction

Let $\Gamma = SL_2(\mathbb{Z})$ be the modular group and let $\mathbb{H}$ denote the upper half-plane. Recall that the non-Euclidean Laplace operator

$$\Delta = -y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)$$

acts on $L^2(\Gamma\backslash\mathbb{H})$ and has a spectral decomposition

$$L^2(\Gamma\backslash\mathbb{H}) = C \oplus C(\Gamma\backslash\mathbb{H}) \oplus \mathcal{E}(\Gamma\backslash\mathbb{H}).$$

Here, C is the space of constant functions, $C(\Gamma\backslash\mathbb{H})$ the space spanned by Maass cusp forms and $\mathcal{E}(\Gamma\backslash\mathbb{H})$ the space spanned by the incomplete Eisenstein series.

Let $\mathcal{U} = \{\phi_j\}_{j\geq 1}$ be an orthonormal basis of Hecke–Maass forms with Laplace eigenvalues $1/4 + t_j^2$ in the space $C(\Gamma\backslash\mathbb{H})$. Here, $t_1, t_2, \ldots$ are real parameters which satisfy

$$\frac{1}{4} + t_j^2 \geq \frac{3\pi^2}{2}.$$

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Every ϕ_j has a Fourier expansion

$$\phi_j(z) = \sqrt{y}\sum_{n\neq 0}\rho_j(1)\lambda_j(n)K_{it_j}(2\pi|y|)e(x),$$

where $\rho_j(1) \neq 0$, $\lambda_j(n)$ is the eigenvalue of the nth Hecke operator T_n, $e(x) = e^{2\pi ix}$ and $K_s(y)$ is the K-Bessel function. Recall that $\lambda_j(n)$ satisfies the multiplicative property:

$$\lambda_j(m)\lambda_j(n) = \sum_{d|(m,n)} \lambda_j\Big(\frac{mn}{d^2}\Big).$$

Furthermore, towards the Ramanujan conjecture, Kim and Sarnak [4] proved that

$$\lambda_j(n) \ll n^{7/64+\varepsilon}.$$

By the Rankin–Selberg theory, it is well known that

$$\sum_{n\leq x}|\lambda_j(n)|^2 \ll_{t_j} x. \tag{1.1}$$

Let $d_3(n)$ be the divisor function of order three, that is, the coefficient of n^{-s} in the Dirichlet series for $\zeta^3(s)$. In this paper, we mainly focus on the shifted convolution sum of $d_3(n)$ and $\lambda_j(n)$. We define

$$\mathcal{S}(\phi_j, x) = \sum_{x\leq n\leq 2x} d_3(n)\lambda_j(n-1).$$

By the Voronoi summation formula for $d_3(n)$ and $\lambda_j(n)$ and the circle method, we get the following result, which generalises and improves the result of Munshi [6], who considered the same problem associated with the holomorphic Hecke eigenform.

Theorem 1.1. *We have*

$$\mathcal{S}(\phi_j, X) \ll X^{29/30+\varepsilon},$$

where the implied constant depends only on t_j and ε.

For the holomorphic Hecke eigenform $f(z)$ corresponding to the nth Fourier coefficient $\lambda_f(n)$, Pitt [8] considered the summation

$$\Psi(f, x) = \sum_{n\leq x} d_3(n)\lambda_f(n-1).$$

By analytical continuation of the Dirichlet series

$$\Phi(f, s) = \sum_{n=1}^{\infty}\frac{d_3(n)\lambda_f(n-1)}{n^s},$$

he proved that

$$\Psi(f, x) \ll x^{71/72+\varepsilon}.$$

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Recently, with the help of an idea based on shifted convolution sums for $GL(3) \times GL(2)$ [7], Munshi [6] improved the upper bound and obtained

$$\Psi(f, X) \ll X^{34/35+\varepsilon}.$$

Note that our improved bound is also valid for the holomorphic Hecke eigenform. A new difficulty we meet in proving Theorem 1.1 is that the Ramanujan conjecture for $\lambda_j(n)$ has not yet been proved. This problem is circumvented by using the estimate (1.1).

2. Outline of the proof

To prove the main theorem, we first give three lemmas. The first one is the Voronoi summation formula for $\lambda_j(n)$ given by Kowalski *et al.* [5], the second is the Voronoi summation formula for $d_3(n)$ proved by Ivić [2] and the third is a variant Jutila's version of the circle method.

Lemma 2.1. *Let q be a positive integer and a an integer with $(a, q) = 1$. Let g be a compactly supported smooth function on $\mathbb{R}^+$. Then*

$$\sum_{m=1}^{\infty} \lambda_j(m) e\Big(\frac{am}{q}\Big) g(m) = \frac{1}{q} \sum_{m=1}^{\infty} \lambda_j(m) e\Big(-\frac{\bar{a}m}{q}\Big) G_1\Big(\frac{m}{q^2}\Big) + \frac{1}{q} \sum_{m=1}^{\infty} \lambda_j(m) e\Big(\frac{\bar{a}m}{q}\Big) G_2\Big(\frac{m}{q^2}\Big), \tag{2.1}$$

where

$$G_1(y) = \int_0^{\infty} g(x) J_{\phi_j}(4\pi\sqrt{xy})\, dx, \quad G_2(y) = \int_0^{\infty} g(x) K_{\phi_j}(4\pi\sqrt{xy})\, dx$$

with

$$J_{\phi_j}(x) = \frac{-\pi}{\sin \pi i t_j}(J_{2it_j}(x) - J_{-2it_j}(x)), \quad K_{\phi_j}(x) = 4\varepsilon_{\phi_j} \cosh(\pi t_j) K_{2it_j}(x)$$

and $a\bar{a} \equiv 1 \pmod q$ and $\varepsilon_{\phi_j} = 1$ or -1 according as ϕ_j is even or odd.

If g is supported in $[AY, BY]$ (with $0 < A < B$), satisfying $y^k g^{(k)}(y) \ll_k 1$, then, by the asymptotic expansions of $J_\nu(z)$ and $K_\nu(z)$, the sums over m on the right-hand side of (2.1) can be restricted to $m \ll q^2(qY)^\varepsilon/Y$. By partial integration, the contribution from the tails $m \gg q^2(qY)^\varepsilon/Y$ is negligibly small. Trivially, we have the bound $G_1(m/q^2), G_2(m/q^2) \ll Y$.

A similar Voronoi-type summation formula for the divisor function $d_3(n)$ is as follows.

Lemma 2.2. *Let f be a compactly supported smooth function on $\mathbb{R}_+$ and $\tilde{f}(s) = \int_0^\infty f(x) x^s\, dx$. Define*

$$F_\pm(y) = \frac{1}{2\pi i} \int_{(\frac{1}{8})} (\pi^3 y)^{-s} \frac{\Gamma^3(\frac{1\pm1+2s}{4})}{\Gamma^3(\frac{3\pm1-2s}{4})} \tilde{f}(-s)\, ds.$$

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Then

$$\sum_{n=1}^{\infty} d_3(n) e\Big(\frac{an}{q}\Big) f(n) = \frac{1}{q}\int_0^{\infty} P(\log y, q) f(y)\,dy + \frac{\pi^{3/2}}{2q^3}\sum_{\pm}\sum_{n=1}^{\infty} D_{3,\pm}(a,q;n) F_{\pm}\Big(\frac{n}{q^3}\Big), \tag{2.2}$$

where $P(y,q) = A_0(q)y^2 + A_1(q)y + A_2(q)$ *is a quadratic polynomial whose coefficients depend only on* q *and satisfy the bound* $|A_i(q)| \ll q^{\varepsilon}$, *and the* $D_{3,\pm}(a,q;n)$ *are given by*

$$\sum_{n_1n_2n_3=n}\ \sum\sum_{b,c,d=1}^{q}\sum\Big\{e\Big(\frac{bn_1+cn_2+dn_3+abcd}{q}\Big) \mp e\Big(\frac{bn_1+cn_2+dn_3-abcd}{q}\Big)\Big\}.$$

Suppose that f is supported in $[AX, BX]$ and $x^k f^{(k)}(x) \ll_k H^k$. Shifting the line of integration for $F_{\pm}(y)$ to the right and integrating $\tilde{f}(s)$ by parts, we see that the sums over n on the right-hand side of (2.2) can be restricted to $n \ll q^3 H(qX)^{\varepsilon}/X$. The contribution from the tail $n \gg q^3 H(qX)^{\varepsilon}/X$ is negligibly small. For smaller n, we shift the contour left to $\sigma = \varepsilon$ and we obtain the bounds $F_{\pm}(y) \ll X$ and $y^k F_{\pm}^{(k)}(y) \ll XH$ $(k \geq 1)$.

For any set $S \subset \mathbb{R}$, we use $\mathbb{I}_S$ to denote the indicator function of S, defined by $\mathbb{I}_S(x) = 1$ for $x \in S$ and 0 otherwise. Let $\mathcal{Q}$ be a subset of $[1, Q]$ with integer elements (which we call the set of moduli) and let δ be a positive real number in the range $Q^{-2} \ll \delta \ll Q^{-1}$. Then we define the function

$$\tilde{I}_{\mathcal{Q},\delta}(x) = \frac{1}{2\delta L}\sum_{q\in\mathcal{Q}}\ \sideset{}{^*}\sum_{a \bmod q} \mathbb{I}_{[(a/q)-\delta,(a/q)+\delta]}(x),$$

which is an approximation for $\mathbb{I}_{[0,1]}$. Here, $L = \sum_{q\in\mathcal{Q}} \phi(q)$ and the star over the sum means that $(a, q) = 1$. For $\tilde{I}_{\mathcal{Q},\delta}(x)$, Jutila [3] proved the following result.

Lemma 2.3. *We have*

$$\int_0^1 |1 - \tilde{I}_{\mathcal{Q},\delta}(x)|^2\,dx \ll \frac{Q^{2+\varepsilon}}{\delta L^2}.$$

Proof of Theorem 1.1. Let $\Delta > 1$ and let $0 \leq W(x) \leq 1$ be a smooth function of compact support on $[1, 2]$, which is identically equal to 1 on $[1 + 1/\Delta, 2 - 1/\Delta]$ and satisfies $W^{(k)}(x) \ll_k \Delta^k$ for $k \geq 0$. Clearly,

$$\mathcal{S}(\phi_j, X) = \sum_{n=1}^{\infty} d_3(n)\lambda_j(n-1) W\Big(\frac{n}{X}\Big) + O\Big(\frac{X^{1+\varepsilon}}{\Delta} + \frac{X^{4/5+\varepsilon}}{\Delta^{1/2}}\Big).$$

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Let $V(x)$ be a smooth function supported in $[1/2, 3]$ satisfying $V(x) = 1$ for $x \in [3/4, 5/2]$, $V^{(j)}(x) \ll_j 1$, and put $Y = X$. Then

$$D := \sum_{n=1}^{\infty} d_3(n)\lambda_j(n-1)W\Big(\frac{n}{X}\Big) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_3(n)\lambda_j(m)W\Big(\frac{n}{X}\Big)V\Big(\frac{m}{Y}\Big)\delta(n-1, m)$$
$$= \int_0^1 e(-x)\sum_{n=1}^{\infty} d_3(n)e(xn)W\Big(\frac{n}{X}\Big)\sum_{m=1}^{\infty}\lambda_j(m)e(-xm)V\Big(\frac{m}{Y}\Big)\,dx,$$

where $\delta(m, n) = 1$ if $m = n$ and 0 otherwise. Suppose that $|Q| \gg Q^{1-\varepsilon}$, so that

$$L = \sum_{q\in Q}\phi(q) \gg \sum_{q\in Q}\frac{q}{\log\log q} \gg Q^{2-\varepsilon}.$$

Let $\delta = Y^{-1}$ and define

$$\tilde{D} := \int_0^1 \tilde{I}_{Q,\delta}(x)e(-x)\sum_{n=1}^{\infty} d_3(n)e(xn)W\Big(\frac{n}{X}\Big)\sum_{m=1}^{\infty}\lambda_j(m)e(-xm)V\Big(\frac{m}{Y}\Big)\,dx.$$

Thus,

$$\tilde{D} = \frac{1}{2\delta}\int_{-\delta}^{\delta}\tilde{D}(\alpha)e(-\alpha)\,d\alpha,$$

where

$$\tilde{D}(\alpha) = \frac{1}{L}\sum_{q\in Q}\ \sideset{}{^*}\sum_{a \bmod q} e\Big(-\frac{a}{q}\Big)\sum_{n=1}^{\infty} d_3(n)e\Big(\frac{an}{q}\Big)e(\alpha n)W\Big(\frac{n}{X}\Big)$$
$$\times\sum_{m=1}^{\infty}\lambda_j(m)e\Big(-\frac{am}{q}\Big)e(-\alpha m)V\Big(\frac{m}{Y}\Big). \tag{2.3}$$

Note that

$$D = \tilde{D} + O(|D - \tilde{D}|)$$

and that the error term satisfies

$$|D - \tilde{D}| \ll \int_0^1\Big|\sum_{n=1}^{\infty} d_3(n)e(xn)W\Big(\frac{n}{X}\Big)\Big|\Big|\sum_{m=1}^{\infty}\lambda_j(m)e(-xm)V\Big(\frac{m}{Y}\Big)\Big||1 - \tilde{I}_{Q,\delta}(x)|\,dx$$
$$\ll Y^{(1/2)+\varepsilon}\int_0^1\Big|\sum_{n=1}^{\infty} d_3(n)e(xn)W\Big(\frac{n}{X}\Big)\Big||1 - \tilde{I}_{Q,\delta}(x)|\,dx,$$

where we have used the bound (see Pitt [9])

$$\sum_{m=1}^{\infty}\lambda_j(m)e(-xm)V\Big(\frac{m}{Y}\Big) \ll Y^{1/2+\varepsilon}.$$

By Cauchy's inequality and Lemma 2.3,

$$\int_0^1\Big|\sum_{n=1}^{\infty} d_3(n)e(xn)W\Big(\frac{n}{X}\Big)\Big||1 - \tilde{I}_{Q,\delta}(x)|\,dx \ll X^{1/2+\varepsilon}\frac{Y^{1/2+\varepsilon}Q^{2\varepsilon}}{Q},$$

where we have used

$$\int_0^1 \left|\sum_{n=1}^{\infty} d_3(n)e(xn)W\left(\frac{n}{X}\right)\right|^2 dx = \sum_{n=1}^{\infty} d_3^2(n)W^2\left(\frac{n}{X}\right) \ll X^{1+\varepsilon}.$$

Taking $Q = YX^{-(1/2)+\gamma}$, $\Delta = X^{\gamma}$ for any $\gamma > 0$,

$$\mathcal{S}(\phi_j, x) = \tilde{D} + O(X^{1-\gamma+\varepsilon} + X^{4/5-\gamma/2+\varepsilon}).$$

For $\tilde{D}$, we have the following result, which will be proved in the next section.

Proposition 2.4. *For* $\gamma \le \frac{1}{30}$,

$$\tilde{D} \ll X^{9/10+2\gamma+\varepsilon}.$$

Hence, taking $\gamma = \frac{1}{30}$, we finally complete the proof. □

3. Proof of Proposition 2.4

Let

$$g(y) = V\left(\frac{y}{Y}\right)e(-\alpha y) \quad \text{and} \quad f(x) = W\left(\frac{x}{X}\right)e(\alpha x).$$

Inserting (2.1) and (2.2) into (2.3) gives exactly six terms. In fact, by the properties of the functions $G_1, G_2, D_{3,\pm}, F_{\pm}$ given by Lemmas 2.1 and 2.2, it suffices to investigate the following two summations:

$$\tilde{D}_1(\alpha) = \frac{1}{L}\sum_{q\in\mathcal{Q}} \frac{1}{q^2}\sum_{m=1}^{\infty} \lambda_j(m)S(1,m;q)G_1\left(\frac{m}{q^2}\right)\int_0^{\infty} P(\log x, q)f(x)\,dx$$

and

$$\tilde{D}_2(\alpha) = \frac{\pi^{3/2}}{2L}\sum_{q\in\mathcal{Q}} \frac{1}{q^4}\sum_{m=1}^{\infty} \lambda_j(m)\sum_{n=1}^{\infty} \mathcal{S}^{\star}(m,n;q)G_1\left(\frac{m}{q^2}\right)F_+\left(\frac{n}{q^3}\right),$$

where $S(1,m;q)$ is the Kloosterman sum and

$$\mathcal{S}^{\star}(m,n;q) := \sideset{}{^*}\sum_{a\,(\mathrm{mod}\,q)} e\left(\frac{-a+\bar{a}m}{q}\right)\sum_{n_1n_2n_3=n}\sum\sum_{b,c,d=1}^{q}\sum e\left(\frac{bn_1+cn_2+dn_3+abcd}{q}\right).$$

To estimate $\tilde{D}_1(\alpha), \tilde{D}_2(\alpha)$, we choose $\mathcal{Q}$ to be the product set $\mathcal{Q}_1\mathcal{Q}_2$, where

$$\mathcal{Q}_i = \{q_i \in [Q_i, 2Q_i] \mid q_i \text{ is a prime}\}, \quad i = 1, 2.$$

Here, $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$ and Q_1, Q_2 satisfy $Q_1Q_2 = Q$, which will be chosen later. In addition, the construction implies that $L \gg Q^{2-\varepsilon}$. For $\tilde{D}_1(\alpha)$, recall that the contribution of $m \gg q^2(qY)^{\varepsilon}/Y$ is negligible, so that

$$\tilde{D}_1(\alpha) \ll \frac{1}{L}\sum_{q\in\mathcal{Q}} \frac{1}{q^2}\sum_{m\ll(Q^2Y^{\varepsilon}/Y)} |\lambda_j(m)|q^{1/2}d(q)YX^{1+\varepsilon}q^{\varepsilon} + X^{-B}$$

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for any $B > 0$, where we have used the Weil bound for the Kloosterman sum, namely,

$$S(1, m; q) \ll q^{1/2}.$$

By Cauchy's inequality, (1.1) and the choice of Q,

$$\tilde{D}_1(\alpha) \ll \frac{X^{1+\varepsilon}}{\sqrt{Q}} \ll X^{3/4+\varepsilon}. \tag{3.1}$$

For $\tilde{D}_2(\alpha)$, we firstly estimate $S^\star(m, n; q)$. Assume that $q = q_1 q_2$ with $q_i \in Q_i$. Then

$$S^\star(m, n; q) = S^*(m, n, q_2; q_1) S^*(m, n, q_1; q_2)$$

with

$$\mathcal{S}^*(m, n, q_2; q_1) = \sum_{a=1}^{q_1-1} e\Big(\frac{-\overline{q_2}^3 a + q_2 \bar{a} m}{q_1}\Big) \sum_{n_1 n_2 n_3 = n} \sum\sum_{b,c,d=1}^{q_1}\sum e\Big(\frac{bn_1 + cn_2 + dn_3 + abcd}{q_1}\Big).$$

To compute $\mathcal{S}^*(m, n, q_2; q_1)$, we consider two cases: $q_1 \mid n$ and $q_1 \nmid n$. For the first case, suppose that $q_1 \mid n_1$; then

$$\begin{aligned}\sum\sum_{b,c,d=1}^{q_1}\sum e\Big(\frac{bn_1 + cn_2 + dn_3 + abcd}{q_1}\Big) &= q_1 \sum_{d=1}^{q_1} e\Big(\frac{dn_3}{q_1}\Big) + q_1 \sum_{c=1}^{q_1} e\Big(\frac{cn_2}{q_1}\Big) - q_1 \\ &\ll q_1 (q_1, n_2 n_3)\end{aligned}$$

by an elementary argument. Hence,

$$\mathcal{S}^*(m, q_1 n, q_2; q_1) \ll q_1^{3/2} (q_1, n) d_3(n).$$

For $q_1 \nmid n$, the sum over b, c, d is

$$q_1 \sum_{b=1}^{q_1-1} e\Big(\frac{b}{q_1}\Big) \sum_{\substack{c=1 \\ \bar{n}abc \equiv -1 (\bmod q_1)}}^{q_1-1} e\Big(\frac{c}{q_1}\Big) = q_1 \sum_{b=1}^{q_1-1} e\Big(\frac{b}{q_1}\Big) e\Big(\frac{-n\overline{ab}}{q_1}\Big) = S(1, -n\bar{a}; q_1).$$

Thus,

$$\mathcal{S}^*(m, n, q_2; q_1) = d_3(n) q_1 \sum_{a=1}^{q_1-1} e\Big(\frac{-\overline{q_2}^3 a + q_2 \bar{a} m}{q_1}\Big) S(1, -n\bar{a}; q_1) \ll d_3(n) q_1^2,$$

where we have used Corollary 4.3 of Adolphson and Sperber [1] to estimate the inner sum. Similar bounds can be obtained for $\mathcal{S}^*(m, n, q_2; q_2)$. Therefore,

$$\begin{aligned}
S^\star(m, n; q) &\ll q^{3/2} q_2^{1/2} \Big(q_1, \frac{n}{q_1}\Big) d_3^2(n) \quad \text{for } q_1 | n, q_2 \nmid n,\\
S^\star(m, n; q) &\ll q^{3/2} q_1^{1/2} \Big(q_2, \frac{n}{q_2}\Big) d_3^2(n) \quad \text{for } q_1 \nmid n, q_2 | n,\\
S^\star(m, n; q) &\ll q^{3/2} \Big(q_1, \frac{n}{q_1}\Big)\Big(q_2, \frac{n}{q_2}\Big) d_3^2(n) \quad \text{for } q_1 | n, q_2 | n.
\end{aligned}$$

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Recall that the contribution of $n \gg q^3H(qX)^\varepsilon/X$ is negligible, so it suffices to consider

$$\frac{1}{L}\sum_{q\in Q}\frac{1}{Q^4}\sum_{m\ll Q^2Y^\varepsilon/Y}|\lambda_j(m)|\sum_{n\ll Q^3HX^\varepsilon/\min\{Q_1,Q_2\}X} Q^{3/2}\sqrt{\max\{Q_1,Q_2\}}XY$$

for $(n,q)\neq 1$. Cauchy's inequality and (1.1) lead to the estimate

$$O\Big(\frac{Q^2HX^\varepsilon}{\min\{Q_1,Q_2\}^{3/2}}\Big).$$

So, we obtain

$$\tilde{D}_2(\alpha) = \frac{\pi^{3/2}}{2L}\sum_{q\in Q}\frac{1}{q^3}\sum_{m=1}^{M}\lambda_j(m)\sum_{\substack{n=1\\(n,q)=1}}^{N} d_3^2(n)\mathcal{S}^\sharp(m,n;q)G_1\Big(\frac{m}{q^2}\Big)F_+\Big(\frac{n}{q^3}\Big) + O\Big(\frac{X^{1+3\gamma+\varepsilon}}{\min\{Q_1,Q_2\}^{3/2}}\Big),$$

where $M = Q^{2+\varepsilon}Y^{-1} = X^{2\gamma+\varepsilon}$, $N = Q^{3+\varepsilon}HX^{-1} = X^{1/2+4\gamma+\varepsilon}$ and

$$\mathcal{S}^\sharp(m,n;q) = \sideset{}{^*}\sum_{a\,(\mathrm{mod}\,q)} e\Big(\frac{-a+\bar{a}m}{q}\Big)S(1,-n\bar{a};q).$$

Following the argument used above for $S^\star(m,n;q)$, we can get an exact bound for $\mathcal{S}^\sharp(m,n;q)$ for $(q,n)\neq 1$. So, the restriction $(n,q)=1$ can be removed with the error term unchanged. Define

$$\tilde{D}_3(\alpha) = \frac{\pi^{3/2}}{2L}\sum_{q\in Q}\frac{1}{q^3}\sum_{m=1}^{M}\lambda_j(m)\sum_{n=1}^{N} d_3^2(n)\mathcal{S}^\sharp(m,n;q)G_1\Big(\frac{m}{q^2}\Big)F_+\Big(\frac{n}{q^3}\Big).$$

By Cauchy's inequality,

$$\tilde{D}_3(\alpha) \ll \frac{M^{1/2}N^{1/2}}{Q^5}\sum_{q_2\in Q_2}\tilde{D}_4(\alpha)^{1/2},$$

where we have used the definition of L, Q and

$$\tilde{D}_4(\alpha) = \sum_{m=1}^{M}\sum_{n=1}^{N}\Big|\sum_{q_1\in Q_1}\mathcal{S}^\sharp(m,n;q_1q_2)G_1\Big(\frac{m}{q_1^2q_2^2}\Big)F_+\Big(\frac{n}{q_1^3q_2^3}\Big)\Big|^2.$$

Let h be a nonnegative smooth function on $(0,\infty)$, supported on $[1/2,2N]$, and satisfying $h(x)=1$ for $x\in[1,N]$ and $x^kh^{(k)}(x)\ll 1$. By expanding the square for the sum over q_1,

$$\tilde{D}_4(\alpha) \ll \sum_{m=1}^{M}\sum_{q_1\in Q_1}\sum_{\tilde{q}_1\in Q_1} G_1\Big(\frac{m}{q_1^2q_2^2}\Big)\bar{G}_1\Big(\frac{m}{\tilde{q}_1^2q_2^2}\Big) \times \sum_{n\in\mathbb{Z}} h(n)\mathcal{S}^\sharp(m,n;q_1q_2)\bar{\mathcal{S}}^\sharp(m,n;\tilde{q}_1q_2)F_+\Big(\frac{n}{q_1^3q_2^3}\Big)\bar{F}_+\Big(\frac{n}{\tilde{q}_1^3q_2^3}\Big).$$

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For the sum over n, we use the Poisson summation formula with modulus $q_1\tilde{q}_1q_2$ to get

$$\tilde{D}_4(\alpha) \ll \frac{1}{q_2}\sum_{m=1}^{M}\sum_{q_1\in\mathcal{Q}_1}\sum_{\tilde{q}_1\in\mathcal{Q}_1}\frac{1}{q_1\tilde{q}_1}G_1\Big(\frac{m}{q_1^2q_2^2}\Big)\bar{G}_1\Big(\frac{m}{\tilde{q}_1^2q_2^2}\Big) \times \sum_{n\in\mathbb{Z}}\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2)\mathcal{I}(n;q_1,\tilde{q}_1,q_2),$$

where

$$\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2) = \sum_{a\,(\mathrm{mod}\,q_1\tilde{q}_1q_2)}\mathcal{S}^{\sharp}(m,a;q_1q_2)\bar{\mathcal{S}}^{\sharp}(m,a;\tilde{q}_1q_2)e\Big(\frac{an}{q_1\tilde{q}_1q_2}\Big)$$

and

$$\mathcal{I}(n;q_1,\tilde{q}_1,q_2) = \int_{\mathbb{R}}h(x)F_+\Big(\frac{x}{q_1^3q_2^3}\Big)\bar{F}_+\Big(\frac{x}{\tilde{q}_1^3q_2^3}\Big)e\Big(-\frac{nx}{q_1\tilde{q}_1q_2}\Big)\,dx.$$

For $|n|\neq 0$,

$$\mathcal{I}(n;q_1,\tilde{q}_1,q_2) \ll \frac{X^2Hq_1\tilde{q}_1q_2}{|n|}$$

by using the bounds $F_+(y)\ll X, yF_+'(y)\ll XH$ and partial integration. Trivially,

$$\mathcal{I}(0;q_1,\tilde{q}_1,q_2) \ll X^2N.$$

For $\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2)$, following the argument of Lemmas 10 and 11 of Munshi [7], we arrive at the following result.

Lemma 3.1. *For $q_1\neq\tilde{q}_1$,*

$$\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2) = \begin{cases} O\Big(q_1^{3/2}\tilde{q}_1^{3/2}q_2^{5/2}(n,q_2)^{1/2}\Big) & \text{if } (n,q_1\tilde{q}_1)=1,\\ 0 & \text{otherwise.}\end{cases}$$

For $q_1=\tilde{q}_1$,

$$\mathcal{T}(m,n;q_1,q_1,q_2) = \begin{cases} O(q_1^{5/2}q_2^{5/2}(n/q_1,q_2)^{1/2}) & \text{if } q_1|n,\\ 0 & \text{otherwise.}\end{cases}$$

Using these bounds for $\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2), \mathcal{I}(m,n;q_1,\tilde{q}_1,q_2)$,

$$\begin{aligned}\tilde{D}_4(\alpha) &\ll X^2Y^2\sum_{q_1\in\mathcal{Q}_1}\sum_{\tilde{q}_1\in\mathcal{Q}_1}\Big\{\sum_{1\le|n|\le X^{2015}}\frac{H}{|n|}|\mathcal{T}(m,n;q_1,\tilde{q}_1,q_2)| + \frac{N}{QQ_1}|\mathcal{T}(m,0;q_1,\tilde{q}_1,q_2)|\Big\}\\ &\quad + X^{-B}\\ &\ll X^{2+\varepsilon}Y^2M(HQ_1^5Q_2^{5/2}+NQ^2),\end{aligned}$$

where $B>0$ is arbitrarily large. Finally,

$$\tilde{D}_2(\alpha) \ll \frac{X^{1+\varepsilon}YMN^{1/2}Q_2}{Q^5}\Big(H^{1/2}Q_1^{5/4}Q^{5/4}+N^{1/2}Q\Big) + \frac{X^{1+3\gamma+\varepsilon}}{\min\{Q_1,Q_2\}^{3/2}} \ll X^{9/10+2\gamma+\varepsilon} \tag{3.2}$$

provided that $Q_1 = X^{1/10+\gamma}, Q_2 = X^{2/5}, \gamma\le\frac{1}{30}$. Combining the estimates (3.1) and (3.2), we finally complete the proof. □

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