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ON THE HOMOTOPY PROPERTY OF NUSSBAUM'S FIXED POINT INDEX

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Introduction. In [**14**] R. D. Nussbaum generalized the fixed point index to a class of maps larger than the one in [**5**]. Unfortunately his homotopy property conditions are more restrictive than the often more readily verifiable ones of Eells-Fournier. In this paper we shall try to find an intermediate class of maps which will contain all the known examples of maps for which the index is defined and for which the condition of Eells-Fournier will imply the homotopy property.

In doing so, we shall give general conditions for which the sum of a compact map and a differentiable map will be a map having a fixed point index and for which the Lefschetz fixed point theorem is true.

1. Preliminaries.

1.1. *Admissible maps and compact attractor.* Consider the map $f\colon U \to X$ where U is an open subset of X. Denote by $\mathrm{Fix}(f)$ the set of fixed points of f (that is

$$\mathrm{Fix}(f) = \{x \in U\colon f(x) = x\}).$$

(1.1.1) *Definition.* A map $f\colon U \to X$ is called *admissible* provided (i) U is an open subset of X and (ii) $\mathrm{Fix}(f)$ is compact. A homotopy $h\colon U \times I \to X$ is said to be *admissible* provided (i) U is an open subset of X and (ii)

$$\mathrm{Fix}(h) = \cup \{\mathrm{Fix}(h_t)\colon t \in I\}$$

is compact.

In this paper, we shall make some use of the notion of compact attractor which is due to Nussbaum [**13**].

(1.1.2) *Definition.* Let X be a topological space and $f\colon X \to X$ a continuous map. A compact nonempty subset $M \subset X$ such that M is f-invariant (i.e., $f(M) \subset M$) will be called a *compact attractor* for f if, given any open neighbourhood U of M and any compact subset $K \subset X$, there exists an integer $n = n(K, U)$ such that $f^m(K) \subset U$ for $m \geqq n$.

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In the above situation, we say that M *attracts* the compact subsets of X.

1.2. *Leray endomorphisms and generalized Lefschetz number.* In this paper, we shall make an essential use of the notion of the generalized Lefschetz number in the sense given by Leray [**11**]. This notion has proved to be of great importance in fixed point theory (cf. [**7**]).

Let E be a graded vector space over the field of rational numbers, ϕ an endomorphism of degree zero of E and

$$N(\phi) = \cup \{\ker (\phi^n): n > 0\}.$$

Then ϕ is said to be a *Leray endomorphism* if and only if $\tilde{E} = E/N(\phi)$ is of finite type, that is (i) $\dim E_q < \infty$ for all q, and (ii) $E_q \neq 0$ only for a finite number of q. In that case, one defines

$$\mathrm{Tr}\ (\phi) = \mathrm{trace}\ (\tilde{\phi})$$

where $\tilde{\phi}: \tilde{E} \rightarrow \tilde{E}$ is the induced endomorphism.

Let H denote the singular homology functor with rational coefficients, and f_* denote $H(f)$, where $f: X \rightarrow X$ is a continuous map; f is said to be a *Lefschetz map* if and only if f_* is a Leray endomorphism and, in that case, the generalized Lefschetz number of f is defined to be

$$\Lambda(f) = \textstyle\sum_q (-1)^q \,\mathrm{Tr}\ (f_{*q}).$$

The reason for using singular homology is that it has compact support.

1.3. *Measure of non-compactness.* The notion of "measure of non-compactness" is due to Kuratowski [**9**, **10**].

Let (Y, d) be a metric space. We define the *measure of non-compactness* $\gamma(Y)$ of Y to be

$$\gamma(y) = \inf \{r > 0: \exists \text{ a finite covering of } Y \text{ by subsets of diameter at most } r\}.$$

Notice that $\gamma(Y) < \infty$ if and only if Y is bounded. Let $f: X \rightarrow Y$ be a continuous map where (X, d') and (Y, d) are metric spaces. We define the *measure of non-compactness* $\gamma(f)$ *of* f to be

$$\gamma(f) = \inf \{k: \gamma_Y(f(A)) \leqq k\, \gamma_X(A) \text{ for all } A \subset X\}.$$

This measure of non-compactness satisfies a number of properties (cf. [**9**, **13**]) among which are the following

(1.3.1) $A \subset B \subset Y$ implies that $\gamma(A) \leqq \gamma(B)$.
(1.3.2) $0 \leqq \gamma(Y) \leqq \delta(Y)$ where $\delta(Y)$ is the diameter of Y.
(1.3.3) $\gamma(A) = \gamma(\mathrm{cl}\, A)$ where $\mathrm{cl}\, A$ denotes the closure of A.
(1.3.4) $\gamma(A \cup B) \leqq \max \{\gamma(A), \gamma(B)\}$.
(1.3.5) If $g: Y \rightarrow Z$ is a continuous map, $\gamma(g \circ f) \leqq \gamma(g)\, \gamma(f)$.

Furthermore, if Y is a linear normed space, we have the following (cf. [**2**]):

(1.3.6) $\gamma(A + B) \leqq \gamma(A) + \gamma(B)$.
(1.3.7) $\gamma(\text{co } A) = \gamma(A)$ where co A denotes the closed convex hull of A.

2. Compacting maps.

2.1. *Compacting families.*

(2.1.1) *Definition.* A family $\{A_i\}_{i \in I}$ of closed subsets of X is *compacting* provided

(i) $\bigcap_{i \in I} A_i = A$ is compact and

(ii) for each open subset $U \subset X$ with $A \subset U$ there exists a finite subset J of I such that $\bigcap_{j \in J} A_j \subset U$.

(2.1.2) PROPOSITION. *If X is compact, any family of closed subsets of X is compacting.*

Proof. Since $\bigcap_{i \in I} A_i = A$ is closed, it is compact. Let U be an open neighbourhood of A in X. Then $\{A_i \cap \mathscr{C} U\}_{i \in I}$ (where $\mathscr{C} U$ denotes the complement of U in X) is a family of closed subsets whose intersection is empty. Hence there exists a finite subset $J \subset I$ such that

$$\bigcap_{j \in J} (A_j \cap \mathscr{C} U) = \emptyset.$$

Consequently, we get

$$\bigcap_{j \in J} A_j \subset U.$$

(2.1.3) PROPOSITION. *Let $\{A_i\}_{i \in I}$ be a compacting family and $\{B_i\}_{i \in I}$ be a family of closed subsets. Then $\{A_i \cap B_i\}_{i \in I}$ is compacting.*

Proof. Since

$$\bigcap_{i \in I} (A_i \cap B_i) = B \subset A$$

it is compact. Let U be an open subset such that $B \subset U$. Since the family $\{A \cap A_i \cap B_i\}_{i \in I}$ is compacting in A (by (2.2)), there exists $J \subset I$ such that

$$\left(\bigcap_{i \in J} (A_i \cap B_i)\right) \cap A \subset A \cap U \subset U.$$

Let $V = \mathscr{C}\left(\bigcap_{i \in J} (A_i \cap B_i)\right)$. Then $A \subset U \cup V$ and $U \cup V$ is open. Since $\{A_i\}_{i \in I}$ is compacting, there exists $J' \subset I$ such that

$$\bigcap_{i \in J'} A_i \subset U \cup V.$$

Consequently

$$\bigcap_{i \in J \cup J'} A_i \cap B_i \subset \bigcap_{i \in J'} A_i \cap \left(\bigcap_{i \in J} (A_i \cap B_i)\right) \subset (U \cup V) \cap \mathscr{C} V \subset U.$$

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(2.1.4) PROPOSITION ([**9**]). *Let* X *be a complete metric space and* $\{A_i\}_{i \in \mathbf{N}}$ *a family of closed subsets of* X *such that*

(i) $A_i \subset A_{i-1}$ *for all* i

(ii) $\lim_{i \to \infty} \gamma(A_i) = 0$.

Then $\{A_i\}_{i \in \mathbf{N}}$ *is compacting.*

(2.1.5) PROPOSITION. *If* $\{A_i\}_{i \in \mathbf{N}}$ *is compacting, then*

$$\lim_{n \to \infty} \gamma \bigcap_{i=1}^{n} A_i = 0.$$

2.2. *Compacting maps.*

(2.2.1) As in [**14**], we shall write $X \in \mathcal{F}$ if X is a closed subset of a Banach space from which it inherits its metric and if X has a locally finite covering $\{C_\alpha : \alpha \in A\}$ by closed, convex sets $C_\alpha \subset X$.

If furthermore A is finite, we shall write $X \in \mathcal{F}_0$.

Note that if $X \in \mathcal{F}$, it follows that X is an absolute neighbourhood retract ($X \in$ ANR).

(2.2.2) *Definition.* Let $X \in \mathcal{F}$ and $f: U \to X$ be an admissible map. Then f is a *weakly compacting map* if there exists a compacting decreasing sequence $K_n \in \mathcal{F}_0$ of subsets of X and an open subset W of U such that

(2.7.1) $\quad \text{Fix}(f) \subset W \subset K_1$

(2.7.2) $\quad f(W \cap K_n) \subset K_{n+1}$.

If, in addition, there exists $\epsilon_n > 0$ such that

$$f(W \cap N_{\epsilon_n}(K_n)) \subset K_{n+1} \text{ for all } n \in \mathbf{N},$$

we say that f is a compacting map. If in addition for any compact subset M of U, with $\text{Fix}(f) \subset f(M) \subset M$, we can choose W such that $M \subset W$ then we say that f is a *strongly compacting map.*

The following proposition gives an example of compacting maps. We shall assume that $f: U \to X$ is an admissible map.

(2.2.3) PROPOSITION. *Let* X *be a subset of a Banach space and* $f: U \to X$ *be a* k*-set contraction with* $k < 1$. *Then* f *is strongly compacting.*

Proof. Let α be a finite covering of $M \supseteq \text{Fix}(f)$ by open balls such that the closure, $\text{cl}(\bigcup \{B : B \in \alpha\})$ is contained in U. Define

$$K_1 = \bigcup \{\text{cl } B : B \in \alpha\}$$

and

$$W = \text{int}(K_1) \cap f^{-1}(\text{int}(K_1))$$

where $\text{int}(K_1)$ denotes the interior of K_1.

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Then Fix $(f) \subseteq M \subseteq W \subseteq K_1$. Take $k < 1$ such that $\gamma(f) < k$ and choose k', ϵ_1 such that $k < k' < 1$ and

$$\epsilon_1 < \frac{k' - k}{2k} \gamma(K_1).$$

Then

$$\begin{aligned}\gamma(f(W \cap N_{\epsilon_1}(k_1))) &\leqq k\gamma(W \cap N_{\epsilon_1}(K_1)) \\ &\leqq k\gamma(N_{\epsilon_1}(K_1)) \\ &= 2\epsilon_1 k + k\gamma(K_1) < k'\gamma(K_1).\end{aligned}$$

Hence $f(W \cap N_{\epsilon_1}(K_1))$ is a finite union of sets with diameter less than $k'\gamma(K_1)$. Define K_2 to be the union of the convex closure of these sets. Thus

$$f(W \cap N_{\epsilon_1}(K_1)) \subset K_2 \text{ and } \gamma(K_2) \leqq k'\gamma(K_1).$$

Repeating this procedure, take $\epsilon_n < \frac{1}{2}(k'k^{-1} - 1)\gamma(K_n)$ for each integer n, and we obtain

$$\gamma(K_{n+1}) \leqq k'\gamma(K_n).$$

Then

$$\lim_{n\to\infty} \gamma(K_n) \leqq \lim_{n\to\infty} k'^n \gamma(K_1) = 0.$$

Hence, by (2.1.4), $\{K_n\}$ is compacting. Thus f is compacting.

(2.2.4) *Definition*. Let $f\colon U \to X$ be a continuously Fréchet differentiable map and let $D \subset X$. Then f is *D-homogeneously eventually condensing* if there exists $k > 0$ and $m \in \mathbf{N}$ such that for any $\{x_0, x_1, \ldots, x_m\} \subset U$ with $x_{i-1} - f(x_i) \in D$ for all $i = 1, \ldots, m$, we have

$$\gamma(Df(x_0) \circ Df(x_1) \circ \ldots \circ Df(x_m)) < k < 1.$$

(2.2.5) LEMMA. *Define* $\psi\colon X^{m+1} \to Y^{m+1}$ *by*

$$\psi(y_m, \ldots, y_1, x) = (f(f(\ldots(f(x) + y_1)\ldots) + y_{m-1}) + y_m, \ldots, f(x) + y_1, x)$$

then

$$\psi(D_m \times U) \cap U^{m+1} = \{(x_0 \ldots x_m) \in U^{m+1}\colon x_{i-1} - f(x_i) \in D \textit{ for all } i = 1 \ldots m\}.$$

Proof. By the definition of ψ, since

$$f(f(\ldots(f(x) + y_1) + \ldots) + y_{i-1}) + y_i - f[f(\ldots(f(x) + y_1)\ldots) + y_{i-1}] = y_i \in D$$

we have the first inclusion. Now take $(x_0 \ldots x_m) \in U^{m+1}$ with $x_{i-1} -$

$f(x_i) \in D$ for all $i = 1, \ldots, m$. Define

$$y_i = x_{m-i} - f(x_{m-i-1}).$$

Then we can prove by induction on i that

$$x_{m-i} = f(\ldots (f(x) + y_1) + \ldots) + y_i.$$

Thus we have equality.

(2.2.6) LEMMA. *Let $f: U \to E$ be D-homogeneously eventually condensing. Then for any compact subset M of U and for any $\epsilon > 0$, there exists a neighbourhood V of M and $r > 0$ such that $A \subset V$ and $\delta A < r$ imply*

$$\|R_x(a) - R_x(b)\| < \epsilon \|a - b\| \text{ for all } x, a, b \in A$$

where $\delta(A)$ denotes the diameter of A and $R_x(a)$, the remainder term in Taylor's formula at the point x. Thus

$$f(A) \subset N_{\epsilon\delta A}(f(x) - Df(x)(x) + Df(x)(A)).$$

Proof. We know (cf [**3**] p. 164, 8.6.2) that co $\{a, b\} \subset U$ implies

$$(*) \quad \|R_x(a) - R_x(b)\| = \|f(a) - f(b) - Df(x)(a - b)\| \leqq \|a - b\| \sup \{\|Df(x) - Df(y)\|: y \in \text{co}\{a, b\}\}.$$

So, since f is continuously Fréchet-differentiable, choose $x_1, \ldots, x_n$ and $\delta_1, \ldots, \delta_n$ such that

(1) $N_{2\delta_i}(x_i) \subset U$

(2) $y \in N_{2\delta_i}(x_i)$ implies $\|Df(y) - Df(x_i)\| < \epsilon/2$

(3) $V = \cup \{N_{\delta_i}(x_i): i = 1, \ldots, n\} \supset M$.

Put $r = \min\{\delta_i: i = 1, \ldots, n\}$. If $A \subset V$ and $\delta A < r$, we have

$$A \cap N_{\delta_i}(x_i) \neq \emptyset \text{ for some } i;$$

hence $A \subset N_{\delta_i + \delta A}(x_i)$, thus

$$\text{co } A \subset \text{cl}(N_{\delta_i+\delta(A)}(x_i)) \subset N_{\delta_i+r}(x_i) \subset N_{2\delta_i}(x_i) \subset U.$$

However if $x, y \in N_{2\delta_i}(x_i)$ we get

$$\|Df(x) - Df(y)\| \leqq \|Df(x) - Df(x_i)\| + \|Df(x_i) - Df(y)\| \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$\sup\{\|Df(x) - Df(y)\|: y \in \text{co } A \subset N_{2\delta_i}(x_i)\} \leqq \epsilon.$$

That is, using (*), if $x, a, b \in A$, we get

$$\|R_x(a) - R_x(b)\| < \epsilon \|a - b\|.$$

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The second part of this lemma is evident from Taylor's formula.

(2.2.7) LEMMA. *Under the assumptions of* (2.2.6), *there exists a neighbourhood* V' *of* M *and* $s, k > 0$ *such that* $\delta A < s$ *implies that* $\delta f(A) \leqq k\delta A$ *for all* $A \subset V'$.

Proof. Denote

$$k = 1 + \sup \{ \|Df(x)\| : x \in M\}.$$

Then there exists an open neighbourhood W of M such that $x \in W$ implies that $\|Df(x)\| < k$. Since Fix (h) is compact, there exists $s > 0$ such that $N_{2s}(M) \subset W$. Put $V' = N_s(M)$. Then, if $A \subset V'$ and $\delta A < s$, it follows that $\text{co}A \subset N_{2s}(M) \subset W$. Hence by [**3**, p. 164, 8.5.4], we get

$$\begin{aligned}\| f(x) - f(y)\| &\leqq \|x - y\| \sup \{ \|Df(a)\| : a \in \text{co}\{x, y\} \subset \text{co } A\} \\ &\leqq k\|x - y\|\end{aligned}$$

for all $x, y \in A$; that is $\delta f(a) \leqq k\delta A$.

(2.2.8) LEMMA. *Assume that* $f: U \to X$ *is D-homogeneously eventually condensing where* D *is a compact subset of* X. *Then there exists a finite union* E *of closed convex subsets of* X *such that* $D \subset E$ *and* f is *E-homogeneously eventually condensing in a neighbourhood of any compact* $M \subset U$.

Proof. Choose V an open subset of U such that

$$M \subset V \subset \text{cl } V \subset U.$$

Consider the function $\phi: U^{m+1} \to \mathbf{R}^+$ defined by

$$\phi(x_1, \dots, x_m) = \gamma(Df(x_0) \circ \dots \circ Df(x_m));$$

since f is continuously (Fréchet) differentiable, ϕ is continuous. Hence $\phi^{-1}[0, k)$ is an open subset.

Now consider the map $\psi: X^{m+1} \to Y^{m+1}$ where Y is the whole Banach space, defined by

$$\begin{aligned}\psi(y_m, \dots, y_1, x) = (f(f(\dots (f(x) + y_1) + \dots) + y_{m-1}) \\ + y_m, \dots, f(x) + y_1, x);\end{aligned}$$

it is continuous. Furthermore by (2.2.5), we have that

$$\psi(D^m \times U) \cap U^{m+1} \subset \phi^{-1}[0, k),$$

thus

$$D^m \times M \subset D^m \times U \subset \psi^{-1}((U^{m+1} \cap \phi^{-1}[0, k)) \cup \mathscr{C}[\text{cl } V^{m+1}] = W$$

which is open. Since $D^m \times M$ is compact there exists $\epsilon > 0$ such that

$$N_\epsilon(D^m \times M) \subset W.$$

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The norm of the product being

$$\|(y_m, \dots, y_1, y_0)\| = \left(\sum_{i=0}^{m} y_i^2\right)^{1/2}$$

choose $0 < s < \epsilon\,(m+1)^{-1/2}$ such that $N_s(M) \subset V$. Then

$$(N_s(D)^m \times N_s(M) \subset N_\epsilon(D^m \times M)) \subset W.$$

Thus

$$\psi(N_s(D)^m \times N_s(M)) \cap (N_s(M))^{m+1} \subset U^{m+1} \cap \phi^{-1}[0, k) \subset \phi^{-1}[0, k),$$

that is $f: N_s(M) \to X$ is $N_s(D)$-homogeneously eventually condensing. Let α be a finite covering of D by balls of diameter less than $s/2$; then

$$E = \bigcup \{\text{cl}\, A \mid A \in \alpha\} \subset N_s(D)$$

and so E is the set we were looking for.

The following proposition for f, the zero constant map, and g, a self map, is due to Nussbaum [**3**, 367 Corollary 9]; if f is the zero constant map, it is due to Nussbaum [**14**] and Eells-Fournier [**5**]. If g is a linear map it is due to Nussbaum [**12**, p. 225, Corollary 3]; in fact, in this particular case, if f is compact and g is eventually condensing, the conclusion of (2.2.9) always follows.

(2.2.9) PROPOSITION. *Let f, $g: U \to X$ be two maps such that g is compact and f is* cl$g(U)$*-homogeneously eventually condensing. Then $h = f + g$ is a strongly compacting provided* Fix(h) *is compact.*

Proof. By (2.2.8), we may assume that U has the property that $f: U \to X$ is E-homogeneously eventually condensing, where cl $g(U) \subset E$ and E is a finite union of closed convex sets.

Since f is continuously differentiable and M is compact, with Fix $(h) \subset f(M) \subset M$, we define

$$d = 1 + \sup\{\|Df(x)\|: x \in M\}$$

and the open subset

$$V' = \{x: \|Df(x)\| < d\} \supset M.$$

Now let m and k be as in (2.2.4) and take t such that

$$1 < t < k^{-1};$$

then $1 - kt > 0$. Let V and r be as in (2.2.6) and choose $\epsilon > 0$ such that

(2.2.9.1) $\quad \epsilon < (2d^m(t+4)m + 2t + 2d + 6)^{-1}\,(1 - kt).$

Finally choose s such that $s < 1$, $s < \epsilon d^{-1}$ and $r' = d(M, \mathscr{C} V) - sr > 0$.

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Now define

$$U' = N_{r'}(M) \cap V'.$$

Notice that

(2.2.9.2) $\|Df(x)\| \leqq d < \epsilon/s$

for all $x \in U' \subset V'$ and

(2.2.9.3) $N_{sr}(U') \subset N_{sr+r'}(M) \subset V.$

We may now proceed with the inductive construction of the K_n's.

Choose $\mathfrak{A}_1$ a finite set of closed convex subsets of U', the union of which is a neighbourhood of M and such that $A \in \mathfrak{A}_1$ implies that

$$\delta(A) < r(t + \epsilon + 2s)^{-1}.$$

Put $K_1 = \cup \mathfrak{A}_1$ and $W = \text{int}(K_1) \cap h^{-1}(\text{int}(K_1))$. Then W is an open set and $\text{Fix}(h) \subset W \subset K_1$. We may assume that $A \in \mathfrak{A}_1$ implies that

$$\delta(A) < t\gamma(K_1).$$

Notice that

$$h(N_{s\gamma(K_1)}(K_1) \cap W) \subset h(W) \subset K_1.$$

Now assume that $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ and $K_i = \cup \mathfrak{A}_i$ $(i = 1, \ldots, n)$ are defined and satisfy the properties (i) $A \in \mathfrak{A}_i$ implies that $\delta(A) < t\gamma(K_i)$, (ii) there exists $A' \in \mathfrak{A}_{i-1}$ such that

$$A \subset f(N_{s\gamma(K_{i-1})}(A') \cap W)$$

(iii) $h(N_{s\gamma(K_n)}(K_n) \cap W) \subset h(W) \subset K_n$ and (iv) $K_i \subset K_{i-1}$ for all $i = 2, \ldots, n$. Now let us define $\mathfrak{A}_{n+1}$ and $K_{n+1} = \cup \mathfrak{A}_{n+1}$. Consider

(2.2.9.4) $h(N_{s\gamma(K_n)}(K_n) \cap W) \subset f(N_{s\gamma(K_n)}(K_n) \cap W) + \text{cl}\, g(W).$

Since g is compact, let $\mathfrak{B}_{n+1}$ be a finite covering of $\text{cl}\, g(W)$ by convex subsets of E, of diameter less than $\epsilon\gamma(K_n)$. Let $\tilde{\mathfrak{A}}_{n+1}$ be a set of subsets of $f(N_{s\gamma(K_n)}(K_n) \cap W)$ satisfying the following property:

for each $\tilde{A} \in \tilde{\mathfrak{A}}_{n+1}$, there exists $A \in \mathfrak{A}_n$ such that

$$\tilde{A} \subset f(N_{s\gamma(K_n)}(A) \cap W).$$

Define

(2.2.9.5) $\mathfrak{C}_{n+1} = \{A \cap (\text{co}\, \tilde{A} + B) : \tilde{A} \in \tilde{\mathfrak{A}}_{n+1}, A \in \mathfrak{A}_n \text{ and } B \in \mathfrak{B}_{n+1}\}$

and $K_{n+1} = \cup \mathfrak{C}_{n+1} \subset K_n$. Since $\gamma(K_{n+1}) \leqq \gamma(K_n)$ (cf (1.3.2)) which is finite, there exists a finite collection $\mathfrak{D}_{n+1}$ of subspaces of K_{n+1} such that $\cup \mathfrak{D}_{n+1} = K_{n+1}$ and $\delta(D) < t\gamma(K_{n+1})$ for all $D \in \mathfrak{D}_{n+1}$. Define

(2.2.9.6) $\mathfrak{A}_{n+1} = \{\text{co}\, D \cap C : C \in \mathfrak{C}_{n+1} \text{ and } D \in \mathfrak{D}_{n+1}\}.$

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Then if $A \in \mathfrak{A}_{n+1}$, it follows, from (1.3.7) and (2.2.9.6), that

(2.2.9.7) $\delta(A) \leqq \delta(\text{co } D) = \delta(D) < t\gamma(K_{n+1})$

for some $D \in \mathfrak{D}_{n+1}$. Clearly $\cup\, \mathfrak{A}_{n+1} = K_{n+1}$. Since

$$h(N_{s\gamma(K_n)}(K_n) \cap W) \subset K_n$$

we get (using (2.2.9.4) and (2.2.9.5))

$$h(N_{s\gamma(K_n)}(K_n) \cap W) \subset K_{n+1} \subset K_n$$

hence

$$h(N_{s\gamma(K_{n+1})}(K_{n+1}) \cap W) \subset h(N_{s\gamma(K_n)}(K_n) \cap W) \subset K_{n+1}.$$

It remains to show that

$$\lim_{n\to\infty} \gamma(K_n) = 0.$$

But

(2.2.9.8) $$\begin{aligned}\gamma(K_{n+1}) &\leqq \max\{\gamma(\text{co } \tilde{A} + B) : \tilde{A} \in \tilde{\mathfrak{A}}_{n+1} \text{ and } B \in \mathfrak{B}_{n+1}\}\\ &\leqq \max\{\gamma(\tilde{A}) + \gamma(B) : \tilde{A} \in \tilde{\mathfrak{A}}_{n+1} \text{ and } B \in \mathfrak{B}_{n+1}\}\\ &\leqq \max\{\gamma(f(N_{s\gamma(K_n)}(A) \cap W)) + \epsilon\gamma(K_n) : A \in \mathfrak{A}_n\}.\end{aligned}$$

Notice that $K_i \subset K_{i-1}$, hence $\gamma(K_n) \leqq \gamma(K_1)$ for all n and by construction, if $A \in \mathfrak{A}_n$ there exists $A' \in \mathfrak{A}_{n-1}$ such that $A \subset A'$, consequently we have that $\delta(A) < t\gamma(K_1)$ for all $A \in \mathfrak{A}_n$. But

$$\gamma(K_1) \leqq \max\{\delta A \mid A \in \mathfrak{A}_1\} < r(t + \epsilon + 2s)^{-1}$$

thus

$$\delta(N_{s\gamma(K_n)}(A)) \leqq (t + 2s)\,\gamma(K_1) < r.$$

Furthermore, since $K_n \subset K_1 \subset U'$, we obtain using (2.2.9.3), that

(2.2.9.9) $N_{s\gamma(K_n)}(A) \subset N_{sr}(A) \subset N_{sr}(U') \subset V.$

Hence if $A_0 \in \mathfrak{A}_n$, it follows from (2.2.9.9) and (2.2.6), that

(2.2.9.10) $f(N_{s\gamma(K_n)}(A_0) \cap W)$

$$\subset N_{\epsilon\delta(A_0)+2\epsilon s\gamma(K_n)}(f(x_0) + Df(x_0)(x_0) + Df(x_0)(N_{s\gamma(K_n)}(A_0)))$$

for any $x_0 \in A_0$. Then from (2.2.9.2), (2.2.9.8) and (2.2.9.10), we get, for all $x_0 \in A_0$

(2.2.9.11) $$\begin{aligned}\gamma(K_{n+1}) &\leqq \max_{A_0\in\mathfrak{A}_n} \gamma(Df(x_0)(N_{s\gamma(K_n)}(A_0)))\\ &\qquad + 2\epsilon\delta(A_0) + 4\epsilon s\gamma(K_n) + 2\epsilon\gamma(K_n)\\ &\leqq \max_{A_0\in\mathfrak{A}_n} \min_{x\in\mathfrak{A}_0} \gamma(Df(x)(A_0)) + (2ds + 2t + 4s + 2)\ \epsilon\gamma(K_n)\\ &\leqq \max_{A_0\in\mathfrak{A}_n} \min_{x\in\mathfrak{A}_0} \gamma(Df(x)(A_0)) + (2d + 2t + 6)\ \epsilon\gamma(K_n)\end{aligned}$$

for all $n > 0$. However, there exists $A_1 \in \mathfrak{A}_{n-1}$ and $B_1 \in \mathfrak{B}_{n-1}$ such that

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for any $x_1 \in A_1$ and any $b_1 \in B_1$, we have using (2.2.9.10), the construction of $\mathfrak{B}_{n-1}$ and (2.2.9.7) that

$$(2.2.9.12) \quad \begin{aligned} A_0 &\subset f(N_{s\gamma(K_{n-1})}(A_1) \cap W) + B_1 \\ &\subset B_1 + N_{r_1}(f(x_1) + Df(x_1)(x_1) + Df(x_1)(N_{s\gamma(K_{n-1})}(A_1))) \\ &\subset N_{r_2}(b_1 + f(x_1) + Df(x_1)(x_1) + Df(x_1)(N_{s\gamma(K_{n-1})}(A_1))) \end{aligned}$$

where

$$r_1 = \epsilon\delta(A_1) + 2\epsilon s\gamma(K_{n-1})$$

and

$$r_2 = \epsilon(t + 1 + 2s)\,\gamma(K_{n-1}).$$

Furthermore any $x_0 \in A_0$ can be written in the form $x_0 = f(x) + b$ for some $x \in N_{s\gamma(K_{n-1})}(A_1)$ and some $b \in B_1$. Hence we may assume that $x_0 = f(x_1) + b_1$. We can repeat this process and obtain the same inclusion ((2.2.9.11)) for the sets $A_0, \ldots, A_m, B_1, \ldots, B_m$

$$x_i \in N_{s\gamma(K_{n-i})}(A_i) \subset U \text{ and } b_i \in B_i$$

such that

$$f(x_i) + b_i = x_{i-1}$$

for $i = 1, \ldots, m < n$. Denote, for some $i \leqq m$,

$$\phi = Df(x_0) \circ \ldots \circ Df(x_{i-1});$$

then by (2.2.9.2), $\|\phi\| < d^{i-1} < d^m$. Hence, by (2.2.9.12) and the fact that $\gamma(K_\alpha) \leqq \gamma(K_\beta)$ for any $\beta \leqq \alpha$, we have

$$(2.2.9.13) \quad \begin{aligned} \gamma(\phi(A_{i-1})) &\leqq \gamma(\phi(N_{\epsilon(t+1+2s)\gamma(K_{n-i})}(b_i + f(x_i) + Df(x_i)(x_i) \\ &\qquad + Df(x_i)(N_{s\gamma(K_{n-i})}(A_i))))) \\ &\leqq \gamma(N_{r_3}(\phi(b_i) + \phi \circ f(x_i) + \phi \circ Df(x_i)(x_i) \\ &\qquad + \phi \circ Df(x_i)(N_{s\gamma(K_{n-i})}(A_i)))) \\ &\leqq 2d^m\,\epsilon(t + 3)\,\gamma(K_{n-i}) + \gamma(\phi \circ Df(x_i)(N_{s\gamma(K_{n-i})}(A_i))) \\ &\leqq 2d^m\,(\epsilon(t + 3) + \|Df(x_i)\|s)\,\gamma(K_{n-i}) + \gamma(\phi \circ Df(x_i)\,(A_i)) \\ &\leqq 2d^m\,\epsilon(t + 4)\,\gamma(K_{n-m}) + \gamma(\phi \circ Df(x_i)\,(A_i)) \end{aligned}$$

where

$$r_3 = \|\phi\|\,\epsilon(t + 3)\,\gamma(K_{n-i})$$

Applying the previous inequality successively, we obtain

$$\begin{aligned} \gamma(Df(x_0)(A_0)) &\leqq 2md^m\,\epsilon(t + 4)\,\gamma(K_{n-m}) \\ &\qquad + \gamma(Df(x_0) \circ \ldots \circ Df(x_i)\,(A_m)) \\ &\leqq 2md^m\,\epsilon(t + 4)\,\gamma(K_{n-m}) + k\delta(A_m) \\ &\leqq [2md^m\,\epsilon(t + 4) + kt]\,\gamma(K_{n-m}) \end{aligned}$$

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by (2.2.9.7) and (2.2.4). Hence, it follows from (2.2.9.11), that

$$\gamma(K_{n+1}) \leqq [\epsilon(2md^m (t + 4) + (2d + 2t + 6)) + kt]\, \gamma(K_{n-m}).$$

Put $L = \epsilon[2md^m (t + 4) + (2d + 2t + 6)] + kt$, then by (2.2.9.1), $L < 1$. Hence

$$\gamma(K_{n+1}) \leqq L\gamma(K_{n-m})$$

with $L < 1$. Since $n > m$ is arbitrary and $\gamma(K_n) < \gamma(K_1)$ for all n, it follows that

$$\lim_{n\to\infty} \gamma(K_n) = 0.$$

(2.2.10) *Examples*. Here we give some examples of D-homogeneously eventually condensing maps $f: U \to X$.

1) f is linear eventually condensing and D any subset of X (cf. [**12**]).
2) f is eventually condensing and $f(x + y) = f(x)$ for all $y \in D$.
3) There is a sequence $\{E_i\}_{i=1\ldots n}$ and $k, L > 0$ such that
a) E_1 is the whole Banach space
b) $Df(x)\ (E_i) \subset E_{i+1}$ which is a linear subspace of E_i for all $i = 1, \ldots, n - 1$ and all $x \in E_i$
c) $\gamma(Df(x)|_{E_n}) < k < 1$ for all $x \in E_i$
d) $\gamma(Df(x)) \leqq L$ for all x.
Then f is D-homogeneously eventually condensing for any $D \subset X$.
4) If conditions 3(a)–3(c) are satisfied and if D and M are two compact subsets of X then there exists $\epsilon > 0$ such that

$$f: N_\epsilon(M) \to X$$

is D-homogeneously eventually condensing.

3. Fixed point index. The main reference for this section is [**14**].

(3.1) Suppose that U and Y are open subsets of a space $X \in \mathscr{F}$ such that $U \subset Y$ and $f: U \to Y$ is a continuous map. Assume that Fix (f) is compact (possibly empty). Suppose there exists a bounded open neighbourhood W of Fix (f), $\bar{W} \subset U$, and a decreasing sequence of spaces $K_n \subset Y$, $K_n \in \mathscr{F}_0$, such that

(3.1.1) $K_1 \supset W$;

(3.1.2) $f(W \cap K_n) \subset K_{n+1}$;

(3.1.3) $\lim_{n\to\infty} \gamma(K_n) = 0$.

(3.2) *Definition* (Nussbaum). If the above conditions are satisfied for some W and some decreasing sequence $\{K_n\}$ we say that f *belongs to the*

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fixed point index class, and we define

$$\text{ind } (Y, f, U) = \lim_{n\to\infty} \text{ind } (K_n, f, W \cap K_n).$$

If K_n is empty for some n, we take ind (Y, f, U) to be zero.

Note that a map $f: U \to Y$ which has a compact set of fixed points and is weakly condensing (in the sense of [**5**]), is weakly compacting and belongs to the fixed point index class. The fixed point index defined in the above generality satisfies the familiar properties: e.g., the excision, additivity, fixed point and contraction properties. Since the properties of normalization, commutativity and homotopy have special hypotheses, we write them here. We need one more definition.

(3.3) *Definition* (see [**6**]). Suppose that $X \in \mathscr{F}$, Y is an open subset of X and $f: Y \to Y$ is a continuous map. Let $M \subset Y$ be a compact, f-invariant set. Assume that there exists an open neighbourhood W of M and a decreasing sequence of sets $K_n \in \mathscr{F}_0$, $K_n \subset Y$, such that $K_1 \supset W$, $f(W \cap K_n) \subset K_{n+1}$ and $\lim_{n\to\infty} \gamma(K_n) = 0$. Then we say that $f: Y \to Y$ has *property* $(\mathscr{L})$ in a neighbourhood of M.

Note that a strongly compacting map has property $(\mathscr{L})$ in a neighbourhood of any such M. We have the following properties.

(3.4) *Normalization*. Suppose that $X \in \mathscr{F}$, Y is open in X, and $f: Y \to Y$ is a continuous map which has a compact attractor M. Then, if f has property $(\mathscr{L})$ in a neighbourhood of M, f belongs to the fixed point index class, f is a Lefschetz map, and

$$\text{ind } (X, f, Y) = \Lambda(f: Y \to Y).$$

(3.5) *Commutativity*. Suppose that X, $Y \in \mathscr{F}$, that U and V are open subsets of X and Y respectively and that $f: U \to Y$ and $g: V \to X$ are continuous maps. Assume that

$$S = \{x \in f^{-1}(V): g \circ f(x) = x\}$$

is compact (possibly empty), so that

$$T = \{y \in g^{-1}(U): f \circ g(y) = y\}$$

is compact. Assume that there exists a decreasing sequence of sets $A_n \in \mathscr{F}_0$, $A_n \subset X$, indexed by nonnegative even integers and a decreasing sequence of sets $B_n \in \mathscr{F}_0$, $B_n \subset Y$, indexed by positive odd integers with the following properties:

(3.5.1) $f(U \cap A_n) \subset B_{n+1}$ and $g(V \cap B_{n+1}) \subset A_{n+2}$ for all even integers $n \geqq 0$.

(3.5.2) A_0 contains an open neighbourhood of S and B_1 contains an open neighbourhood of T.

(3.5.3) $\lim_{n\to\infty} \gamma(A_n) = 0$ and $\lim_{n\to\infty} \gamma(B_n) = 0$.

Then it follows that

$$\text{ind}\,(X, g \circ f, f^{-1}(V)) = \text{ind}\,(Y, f \circ g, g^{-1}(U)).$$

(3.6) *Homotopy*. Suppose that $X \in \mathscr{F}$, U, Y are open in X, and $h: U \times [0, 1] \rightarrow Y$ is a continuous map such that

$$S = \{x \in U: \text{there exists } t \text{ such that } h_t(x) = h(x, t) = x\}$$

is compact. Assume there exists a bounded open neighbourhood W of S with $\bar{W} \subset U$ and a decreasing sequence $K_n \in \mathscr{F}_0$, $K_n \subset Y$, such that

$$K_1 \supset W, h((W_n \cap K_n) \times [0, 1]) \subset K_{n+1} \text{ and } \lim_{n\to\infty} \gamma(K_n) = 0.$$

Then ind (Y, h_t, U) is defined and constant for $0 \leqq t \leqq 1$.

Furthermore this index has the following properties.

(3.7) PROPOSITION (Product). *Let $f: U \rightarrow X$ and $g: V \rightarrow Y$ belong to the fixed point index class. Then*

$$\text{ind}\,(X \times Y, f \times g, U \times V) = \text{ind}\,(X, f, U)\ \text{ind}\,(Y, g, V).$$

Proof. Let W and W' be open neighbourhoods of Fix (f) and Fix (g) respectively such that cl $W \subset U$ and cl $W' \subset U$. Let K_n, $K_n' \in \mathscr{F}_0$ $K_n \subset X$ and $K_n' \subset Y$ be two decreasing sequences of spaces satisfying conditions (3.1.1)–(3.1.3). Then $f \times g$ belongs to the fixed point index class since $W \times W'$ and the sequence $K_n \times K_n'$ satisfy all the required properties (notice that $\gamma(A \times B) \leqq \gamma(A) + \gamma(B)$ provided the norm of $X \times Y$ is defined by

$$\|(x, y)\| = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}).$$

Finally if $\pi \circ f$ and $\pi' \circ g$ are admissible approximations of f and g respectively, the map $(\pi \circ f) \times (\pi' \circ g)$ is an admissible approximation of $f \times g$ and

$$\begin{aligned}
&\text{ind}\,(X \times Y, f \times g, U \times V)\\
&= \lim_{n\to\infty} \text{ind}\,(K_n \times K_n', (\pi \circ f) \times (\pi' \circ g),\\
&\qquad\qquad (W \times W') \cap (K_n \times K_n'))\\
&= \lim_{n\to\infty} [(\text{ind}\,(K_n, \pi \circ f, W \cap K_n))\\
&\qquad\qquad \times (\text{ind}\,(K_n', \pi' \circ g, W' \times K_n'))]\\
&= \text{ind}\,(X, f, U) \times \text{ind}\,(Y, g, V).
\end{aligned}$$

The following proposition extends the previously stated homotopy property but only for compacting maps. The conditions are sometimes easier to verify.

(3.8) PROPOSITION (Homotopy). *Let U be an open subset of $X \times I$. Assume $h: U \rightarrow X$ is a homotopy. Let $H: U \rightarrow X \times I$ be the map defined by*

$$H(x, t) = (h(x, t), t).$$

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If H is a compacting map, it follows that

$$\text{ind}\,(X, h_0, U_0) = \text{ind}\,(X, h_t, U_t) \textit{ for all } t \in I$$
$$= \text{ind}\,(X \times I, H, U).$$

Proof. By (2.2.2), there exists an open subset W, $\epsilon_n > 0$ and $K_n \in \mathscr{F}_0$ for all n such that

(3.8.1) $\quad \text{Fix}\,(H) \subset W \subset K_1,$

(3.8.2) $\quad H(W \cap N_{\epsilon_n}(K_n)) \subset K_{n+1} \subset K_n,$

(3.8.3) $\quad \lim_{n\to\infty} \gamma(K_n) = 0.$

(i) Without loss of generality, it is sufficient to prove the first equality for $W = V \times I$. In fact, since Fix (H) is compact, it has a neighbourhood of the form

$$\bigcup_{i=1}^{n} V_i \times I_i \subset W$$

where V_i is an open subset of X and I_i is a closed interval for all $i = 1, \ldots, n$.

Let us well-order the extremities of the I_i, say $\{t_0, \ldots, t_m\}$, where $t_i < t_{i+1}$. We may assume that $t_0 = 0$: otherwise Fix $(h_0) = \emptyset$ and $U_{i=1}^{n} V_i \times I_i$ being a neighbourhood of Fix (H), it follows that

$$\text{Fix}\,(H) \cap (X \times \{t_0\}) = \emptyset,$$

that is Fix $(h_{t_0}) = \emptyset$ and consequently

$$\text{ind}\,(X, h_0, U_0) = \text{ind}\,(X, h_{t_0}, U_0) = 0.$$

For the same reason, we may assume that $t_m = 1$.

Now define $W_j = \cup \{V_i : t_j, t_{j-1} \in I_i\}$; it follows that

$$W_j \times [t_{j-1}, t_j] \subset \bigcup_{i=1}^{n} V_i \times I_i.$$

Furthermore

$$V_i \times I_i \subset \bigcup_{j=1}^{m} W_j \times [t_{j-1}, t_j]$$

since there exists j_0, j_1 such that $I_i = [t_{j_0}, t_{j_1}]$. Hence

$$\bigcup_{i=1}^{n} V_i \times I_i = \bigcup_{j=1}^{m} W_j \times [t_{j-1}, t_j].$$

Now $W_j \times [t_{j-1}, t_j]$ has the required form and satisfies the required hypothesis. Thus, if

$$\text{ind}\,(X, h_{t_{j-1}}, U_{t_{j-1}}) = \text{ind}\,(X, h_{t_j}, U_{t_j})$$

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for all $j = 1, \ldots, m$ it follows that

$$\text{ind}\,(X, h_0, U_0) = \text{ind}\,(X, h_1, U_1).$$

(ii) Now, we shall prove that ind (X, h_t, U_t) is a constant. In fact, since $K_\infty = \cap \{K_n : n \in \mathbf{N}\}$ is compact and

$$\text{Fix}\,(H) \cap (K_1 \backslash W) = \emptyset,$$

we know that

$$r = \sup \{\|y - H(y)\| : y \in (K_1 \backslash W) \cap K_\infty\} > 0.$$

Hence

$$W' = \{y : \|y - H(Y)\| > r/2\} \supset (K_1 \backslash W) \cap K_\infty.$$

Notice that $K_\infty \subset W \cup W'$: in fact, since $K_\infty \subset K_1$ it follows that

$$K_\infty \subset (K_1 \backslash W) \cup W$$

and consequently we have

$$K_\infty \subset ((K_1 \backslash W) \cap K_\infty) \cup W \subset W' \cup W.$$

Now take $\epsilon > 0$ such that $N_{2\epsilon}(K_\infty) \subset W' \cup W$ and take n_0 such that $K_n \subset N_\epsilon (K_\infty)$ for all $n \geqq n_0$. Then $N_\epsilon(K_n) \subset W' \cup W$ for all $n \geqq n_0$.

Choose $n_1 \geqq n_0$ such that $\gamma(K_n) < r/4$ for all $n \geqq n_1$; take $p \geqq n_1$, p even, and let $\mathscr{A}_p$ be a finite covering of K_p by closed convex subsets of K_p of diameter less than $r/4$. Choose s such that

$$s < \min \{r/4, \epsilon, \epsilon p/4, 1/p\}.$$

Take $t_0 \in I$ and consider the interval $J = [t_0, t_0 + s]$.

For all $A \in \mathscr{A}_p$, define

$$A_1 = \text{cl}\,\{x : \text{there exists } t \in J \text{ such that } (x, t) \in A\}$$

and let $A_j = A_1 \times J$. Then

$$A_J \subset \text{cl}\,N_s(A) \subset N_{2s}(A) \subset N_{\frac{1}{2}\epsilon_p}(K_p);$$

hence

$$H(W \cap \cup \{A_J : A \in \mathscr{A}_p\}) \subset K_p.$$

Denote: $K_p' = \cup \{A_1 : A \in \mathscr{A}_p\}$, then

$$K_{n+1}' \subset K_n',\ K_n \subset K_n' \times J \subset N_{\frac{1}{2}\epsilon_n}(K_n)$$

and

$$\gamma(K_n') \leqq \gamma(K_n) + \epsilon_n \leqq \gamma(K_n) + n^{-1}.$$

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Hence

$$H(W \cap (K_n' \times J)) = H(W \cap \bigcup \{A_J : A \in \mathscr{A}_p\}) \subset K_{p+1} \cap (W \times J) \subset K_{n+1}' \times J,$$

that is

$$h_t(V \cap K_n') \subset K_{n+1}' \subset K_n' \text{ for all } t \in J \text{ and all } n.$$

Furthermore $\text{Fix}\, h_t \subset V \subset K' \times J$. By definition, since K_n' is compacting, we have that

$$\text{ind}\,(X, h_t, U_t) = \text{ind}\,(K_p', h_t, V \cap K_p')$$

for all $t \in J$. But $h: (V \cap K_p') \times J \to K_p'$ is a homotopy without any fixed point on the boundary, thus

$$\text{ind}\,(K_p', h_{t_0}, V \cap K_p') = \text{ind}\,(K_p', h_t, V \cap K_p')$$

for all $t \in J$; that is $\text{ind}\,(X, h_t, U_t)$ is constant for all $t \in J$. Since J is arbitrary of length s, we get the conclusion.

(iii) Let us prove that $\text{ind}\,(X, h_0, U_0) = \text{ind}\,(X \times I, H, U)$. Define $H_s: B_s = (U \cap (X \times [0, s])) \cup U_s \times [s, 1] \to X \times I$ by

$$H_s(x, T) = \begin{cases} (H(x, t), t) & \text{if } t \leqq s \\ (H(x, s), s) & \text{if } t \geqq s. \end{cases}$$

It is sufficient to prove that there exists ϵ such that

$$\text{ind}\,(X, H_s, B_s) = \text{ind}\,(X, H_r, B_r) \text{ for all } s, r \text{ with } |s - r| < \epsilon.$$

Because, we would then have that $H = H_1$ and $H_0 = h_0 \times 0$

$$\begin{aligned} \text{ind}\,(X \times I, H, U) &= \text{ind}\,(X \times I, h_0 \times 0, U_0 \times I) \\ &= \text{ind}\,(X, h_0, U_0) \cdot \text{ind}\,(I, 0, I) \\ &= \text{ind}\,(X, h_0, U_0) \end{aligned}$$

by (3.7) and (3.4) since 0 is a constant map.

Take $\epsilon < \epsilon_1$. Without loss of generality, by (i), we may assume that $r - s < \epsilon$ and $s, r \in [t_{j-1}, t_j]$ for some $j = 1, \ldots, m$. Put

$$U^s = (W_1 \times [t_0, t_1] \cup \ldots \cup W_{j-1} \times [t_{j-2}, t_{j-1}] \cup W_j \times [t_{j-1}, 1]) \times I$$

and define $H^s: U^s \to X \times I \times I$ by

$$H^s(x, t, t') = (H_{(1-t')s+t'r}(x, t), t').$$

It is sufficient to prove that H^s is compacting, since by (ii) we would have the conclusion.

Furthermore since $\text{Fix}\, H$ is compact and thus $\bigcup_{t \in I} \text{Fix}\, h_t$ is compact, since $X \in \mathscr{F}$, by excision, we may assume that $X \in \mathscr{F}_0$.

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Now define

$$D_s = \{(t, t') : t \leqq (1 - t')\, s + t'r\}$$

and

$$E_1 = [K_1 \times I \cap X \times D_s] \text{ and } K_n = E_n \text{ for } n \geqq 2$$

and

$$K_1{}^s = E_1 \cup (\phi^{-1}(E_1)) \cap [X \times (I^2 \backslash D_s)]$$

where

$$\phi(x, t, t') = (x, (1 - t')\, s + t'r, t').$$

We have that $K_n{}^s \in \mathscr{F}_0$: since $K_n \in \mathscr{F}_0$ and since the finite product or the intersection of closed convex sets is closed and convex, we obtain that $E_n \in \mathscr{F}_0$; since $\phi: E \times I^2 \rightarrow E \times I^2$, where E is the whole Banach space, satisfies

$$\phi(a(x, t, t') + (1 - a)(x_1, t_1, t_1')) = a\phi(x, t, t') + (1 - a)\, \phi(x_1, t_1, t_1')$$

we obtain that the inverse image of a closed convex set is closed and convex, thus $K_1{}^s \in \mathscr{F}_0$.

Define

$$A = [W \times I \cap X \times D_s] \cup [\phi^{-1}(W \times I \cap X \times D_s) \cap X \times (I^2 \backslash D_s)]$$

and

$$W^s = \operatorname{int} A \cap U^s.$$

We have that $W^s \subset A \subset K_1{}^s$. Notice that $H^s(U^s) \subset X \times D_s$ and so $\operatorname{Fix} H^s \subset X \times D_s$, thus

$$\operatorname{Fix} H^s \subset (\operatorname{Fix} H \times I) \cap (X \times D_s)$$

and is a compact contained in the interior of $(W \times I) \cap (X \times D_s)$ in $X \times D_s$. Furthermore since $\phi(X \times I^2) \subset X \times D_s$ we get that

$$\phi^{-1}(W \times I \cap X \times D_s) \cap X \times (I^2 \backslash D_s)$$

is open in $X \times (I^2 \backslash D_s)$; moreover it is a neighbourhood of

$$\operatorname{Fix} H^s \cap [X \times (I^2 \backslash D_s)] = \operatorname{Fix} H^s \cap (X \times \partial D_s)$$

since ϕ is the identity on $X \times \partial D_s$. Thus $\operatorname{Fix} H^s \subset \operatorname{int} A$ and W^s is an open neighbourhood of $\operatorname{Fix} H^s$ in $X \times I \times I$. Since $K_{n+1}^s \subset K_n{}^s$ and $\gamma(K_n{}^s) \leqq \gamma(K_n \times I) = \gamma(K_n)$, it remains to prove that

$$H^s(K_n{}^s \cap W^s) \subset K_{n+1}^s.$$

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But if $n \geqq 2$,

$$H^s(K_n{}^s \cap W^s) \subset (H(W \cap K_n) \times I) \cap (X \times D_s) \subset K_{n+1} \times I \cap X \times D_s = K^s_{n+1}.$$

Similarly $H^s(E_1 \cap W^s) \subset K_2{}^s$. Furthermore

$$H^s(\phi^{-1}((W \times I) \cap (X \times D_s)) \cap X \times (I^2 \backslash D_s)) \subset H^s((W \times I) \times (X \times D_s)) \subset H^s(E_1 \times W^s) \subset K_2{}^s.$$

Thus we have $H^s(K_1{}^s \cap W^s) \subset K_2{}^s$.

Thus except for the commutativity property all the preceding results have nicer statements for the class of strongly compacting maps.

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