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CHARACTERIZING CONTINUA BY DISCONNECTION PROPERTIES

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ABSTRACT. We study Hausdorff continua in which every set of certain cardinality contains a subset which disconnects the space. We show that such continua are rim-finite. We give characterizations of this class among metric continua. As an application of our methods, we show that continua in which each countably infinite set disconnects are generalized graphs. This extends a result of Nadler for metric continua.

1**. Introduction.** The idea of characterizing spaces by using disconnection properties goes back at least to Janiszewski [Ja] who in 1912 characterized simple arcs as continua with exactly two non-separating points. Later, A. J. Ward [Wa] in 1936 characterized the real line topologically as a connected, locally connected, separable metric space which is separated by each of its points into exactly two components. Bing [Bi] in 1946 characterized the 2-sphere as a locally connected metric continuum which is not separated by any pair of points, but which is separated by each of its simple closed curves.

Nadler [Na1] defined the *disconnection number* $D(X)$ of a connected space X to be the smallest cardinal number κ such that X becomes disconnected upon removal of any set A with $|A| = \kappa$ (*i.e.*, cardinality of A is κ) provided κ exists. Otherwise, $D(X)$ is not defined.

Shimrat [Sh, Theorem 2] extended Ward's result by characterizing locally connected, separable, metric spaces X with $D(X) = 1$ as connected, separable, metric spaces which have no endpoints, contain no simple closed curves and are locally arc connected. Stone [St] gave a characterization of the class of locally connected, connected, separable, metric spaces X with $D(X) \leq \aleph_0$. Examples of Gladdines [Gl], Pierce [Pi] and Martin [Ma] show that separability, local connectedness and metrizability, respectively, are all necessary in Stone's theorem. Nadler [Na1] proved that every metric continuum X with $D(X) \leq \aleph_0$ is a graph. Nadler's proof depends on second countability.

We write $X \in E_\kappa$ if each set of cardinality κ contains a subset which disconnects X. It is clear that if each non-empty open set in X is uncountable then $X \in E_{\aleph_0}$ if and only if $D(X) \leq \aleph_0$. Further, $E_\kappa \subset E_\gamma$ for $\kappa < \gamma$. We show that if X is a continuum in E_κ where κ is an infinite cardinal number then each connected subset of X is in E_κ. Compactness is necessary in the above as is shown by the wedge of countably many lines. We show

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that each continuum X in E_c is rim-finite and all but countably many of its points are local separating points. Among metric continua the latter property characterizes E_c. As an application we extend Nadler's theorem to the non-metric case by proving that a Hausdorff continuum in $E_{\aleph_0}$ is a generalized graph.

We recall that a compact and connected Hausdorff space is called a *continuum*. A *generalized arc* is a continuum with exactly two non-separating points. A continuum is called a *generalized graph* if it is a union of finitely many generalized arcs any two of which intersect only in a subset of their sets of endpoints. A generalized arc Y can be linearly ordered in such a way that the order topology and the original topology coincide. We will denote Y by $[a, b]$ where a and b are the two non-separating points of Y. A Hausdorff continuum is *indecomposable* if it is non-degenerate and if it is not the union of two of its proper subcontinua. If X is a continuum and $p \in X$, then the set of all $x \in X$ such that $\{p, x\}$ is contained in a proper subcontinuum of X is called a *composant* of X.

The reader may look up the definitions of continuum theory terms in Whyburn [Wh] or Kuratowski [Ku].

2. Main Results.

In this section, unless stated otherwise, X denotes a non-degenerate continuum.

We are going to use the following two theorems.

BELLAMY'S THEOREM ([BE], COROLLARY 5). *If X is a non-degenerate indecomposable continuum, then X contains an indecomposable subcontinuum Y with at least c composants.*

GORDH'S THEOREM ([GOR], THEOREM 2.8). *If X is a continuum which is irreducible between a pair of points and contains no indecomposable subcontinuum with interior, then there exists a monotone continuous map f of X onto a generalized arc such that each point inverse under f has empty interior.*

LEMMA 1. *If $X \in E_\kappa$ and Y is a non-degenerate connected subset of X, then the cardinality of the set of components of $X \setminus \mathrm{cl}(Y)$ is less than κ.*

PROOF. Let Y be a proper connected subset of X. If K is a component of $X \setminus Y$ then $K \cup Y$ is connected by the Boundary Bumping Theorem [Na1, Theorem 5.4, p. 73]. Also, if $x \in K$ such that $K \setminus \{x\}$ is connected then $(K \setminus \{x\}) \cup Y$ is connected. If the cardinality of the set of components of $X \setminus \mathrm{cl}(Y)$ is not less than κ then we could choose κ distinct components, $\{C_\alpha\}_{\alpha<\kappa}$, of $X \setminus \mathrm{cl}(Y)$. Since $C_\alpha \cup \mathrm{cl}(Y)$ is a continuum for each α, by the Non-Separating Point Existence Theorem [Wh1, (6.1), p. 54], no proper connected subset of $C_\alpha \cup \mathrm{cl}(Y)$ contains the set of all non-separating points of $C_\alpha \cup \mathrm{cl}(Y)$. For each α let p_α be a non-separating point of $C_\alpha \cup \mathrm{cl}(Y)$ such that $p_\alpha \in C_\alpha$. Then $X \setminus \bigcup\{p_\alpha\}_{\alpha<\kappa}$ is connected and dense and, hence, no subset of $\bigcup\{p_\alpha\}_{\alpha<\kappa}$ separates X. This contradicts that $X \in E_\kappa$ and the lemma is proved.

LEMMA 2. *If $X \in E_\kappa$ for κ an infinite cardinal number and Y is a non-degenerate connected subset of X, then $Y \in E_\kappa$.*

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PROOF. Let Y be a proper connected subset of X, and let $A \subset Y$ with $|A| = \kappa$. Suppose that no subset of A separates Y. In particular, A has no interior in Y.

For each component C of $X \setminus \mathrm{cl}(Y)$ let $x_C \in \mathrm{cl}(Y) \cap \mathrm{cl}(C)$ and $A' = A \setminus \{x_C : C \text{ is a component of } X \setminus \mathrm{cl}(Y)\}$. By Lemma 1, $|A'| = \kappa$. Since

$$\begin{aligned} Y \setminus A &\subset \mathrm{cl}(Y) \setminus A \subset \big(\mathrm{cl}(Y) \setminus A\big) \cup \{x_C : C \text{ is a component of } X \setminus \mathrm{cl}(Y)\} \\ &\subset \mathrm{cl}(Y) = \mathrm{cl}(Y \setminus A), \end{aligned}$$

we have

$$\big(\mathrm{cl}(Y) \setminus A\big) \cup \big\{x_C : C \text{ is a component of } X \setminus \mathrm{cl}(Y)\big\}$$

is connected. Hence,

$$X \setminus A' = \bigcup \big\{C \cup \{x_C\} : C \text{ is a component of } X \setminus \mathrm{cl}(Y)\big\} \bigcup \big(\mathrm{cl}(Y) \setminus A\big)$$

is connected and no subset of A' separates X. This contradicts that $X \in E_\kappa$ and Lemma 2 is proved.

LEMMA 3. *If $X \in E_c$, then X is hereditarily decomposable.*

PROOF. If there exists an indecomposable subcontinuum Y in X, by Bellamy's theorem, Y contains an indecomposable subcontinuum Z with at least c composants. By Lemma 2 $Z \in E_c$. Let L be a composant of Z. Then $|L| \geq c$ but no subset of L separates Z. This is contrary to $Z \in E_\kappa$ and the lemma is proved.

LEMMA 4. *If $X \in E_c$, then every non-degenerate subcontinuum of X is connected by generalized arcs.*

PROOF. It suffices to show that if Y is a subcontinuum of X which is irreducible between a pair of points, then Y is a generalized arc. By Lemma 2 and Lemma 3 we know that $Y \in E_c$ and Y is a hereditarily decomposable continuum. Using Gordh's theorem, let f be a monotone continuous map from Y onto a generalized arc $[a, b]$ with a and b two non-separating points of $[a, b]$ such that $\mathrm{Int}\big(f^{-1}(t)\big) = \emptyset$ for each $t \in [a, b]$. We only need to show that for each $t \in [a, b]$ $f^{-1}(t)$ is a singleton. If not, there exists a $t_0 \in [a, b]$ such that $f^{-1}(t_0)$ is non-degenerate and connected and, hence, uncountable. If $t_0 = a$ (or $t_0 = b$) then $f^{-1}(a, b]$ (or $f^{-1}[a, b)$) is a connected dense subset in Y since f is monotone and $\mathrm{Int}\big(f^{-1}(t)\big) = \emptyset$ for each $t \in [a, b]$. Hence, if A is a subset of $f^{-1}(t_0)$ with $|A| = c$, the subset $Y \setminus A$ is still connected. This is contrary to $Y \in E_c$. If $a < t_0 < b$ then $\Big(\mathrm{cl}\big(f^{-1}[a, t_0)\big) \cap f^{-1}(t_0)\Big) \bigcup \Big(\mathrm{cl}\big(f^{-1}(t_0, b]\big) \cap f^{-1}(t_0)\Big) = f^{-1}(t_0)$ since $\mathrm{Int}\big(f^{-1}(t_0)\big) = \emptyset$. Without loss of generality we assume $\mathrm{cl}\big(f^{-1}[a, t_0)\big) \cap f^{-1}(t_0)$ is uncountable. Since $f^{-1}[a, t_0)$ is connected and dense in $\mathrm{cl}\big(f^{-1}[a, t_0)\big)$, $\mathrm{cl}\big(f^{-1}[a, t_0)\big) \cap f^{-1}(t_0)$ is a subset of cardinality $\geq c$ which does not separates $\mathrm{cl}\big(f^{-1}[a, t_0)\big)$. This is contrary to Lemma 2 and the proof of Lemma 4 is completed.

A connected space is *hereditarily locally connected* if each of its connected subsets is locally connected.

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LEMMA 5. *If $X \in E_c$, then X is hereditarily locally connected.*

PROOF. If X is not hereditarily locally connected then, by [Si, Theorem 3], there exists a convergence continuum K in X with a net of continua $\{K_\lambda\}_{\lambda \in \Lambda}$ such that $\operatorname{Lim} K_\lambda = K$, $K_{\lambda'} \cap K_\lambda = K_\lambda$ or $K_{\lambda'} \cap K_\lambda = \emptyset$ for $\lambda', \lambda \in \Lambda$ and $K_\lambda \cap K = \emptyset$ for each λ. Since K is non-degenerate, by Lemma 3, $K = A \cup B$ where A and B are two proper subcontinua of K. By Lemma 4, for each $\lambda \in \Lambda$, let L_λ be an irreducible generalized arc from K_λ to a point a_λ of K such that $L_\lambda \cap K = \{a_\lambda\}$. Since $\bigcup\{a_\lambda\}_{\lambda \in \Lambda} \subset A \cup B$, either A or B contains a subnet of $\{a_\lambda\}_{\lambda \in \Lambda}$. We assume by passing to a subnet if necessary that $\bigcup\{a_\lambda\}_{\lambda \in \Lambda} \subset A$. Then $Y = \operatorname{cl}(K \cup \bigcup_{\lambda \in \Lambda} K_\lambda \cup \bigcup_{\lambda \in \Lambda} L_\lambda)$ is a subcontinuum of X with $A \cup \bigcup_{\lambda \in \Lambda} K_\lambda \cup \bigcup_{\lambda \in \Lambda} L_\lambda$ connected and dense in Y. Let $C \subset B \setminus A$ with $|C| = c$. Then $Y \setminus C$ is connected. This is contrary to $Y \in E_c$ and Lemma 5 is proved.

THEOREM 6. *If κ is an infinite cardinal, $\kappa \leq c$, and X is a continuum in E_κ then the set of non-local separating points of X has cardinality less than κ.*

PROOF. Let

$$A_0 = \{x \in X : x \text{ is not a local separating point of } X\}.$$

If $|A_0| \geq \kappa$, then there is $A_1 \subset A_0$ which separates X. Since X is locally connected, by Mazurkiewicz's Theorem [Ku, Section 49, Theorem 3, p. 244], we may assume A_1 is an irreducible separator of X between some two points a and b in X and A_1 is closed. We shall consider two cases.

If A_1 contains an isolated point, let $d \in A_1$ be an isolated point of A_1 and let U be a connected open neighborhood of d such that $U \cap A_1 = \{d\}$. Then $\{d\}$ separates U which is a contradiction. If A_1 contains no isolated point, then A_1 is perfect, so $|A_1| \geq c$. Let U be the component of $X \setminus A_1$ containing a. Then $\operatorname{Bd}(U) = A_1$. By Lemma 1 the cardinality of the set of components of $X \setminus \operatorname{cl}(U)$ is less than κ. For each component C of $X \setminus \operatorname{cl}(U)$ let $x_C \in \operatorname{Bd}(C)$ and $A_1' = A_1 \setminus \{x_C : C \text{ is a component of } X \setminus \operatorname{cl}(U)\}$. Then $|A'| \geq \kappa$ and no subset of A_1' separates X which is again a contradiction. The theorem is proved.

The proof of Theorem 6 serves to prove the following.

THEOREM 7. *Let κ be an infinite cardinal, $\kappa \leq c$, and X a continuum in E_κ. If A is an irreducible separator between two points of X, then $|A| < \kappa$.*

Let X be a continuum. A subset Y of X is said to be a *cyclic element* of X if Y is connected and maximal with respect to the property of containing no separating point of itself. We shall say that X is *cyclic* if X has no separating point. A subset A of X is said to be a *T-set* in X if A is closed and $|\operatorname{Bd}(J)| = 2$ for each component J of $X \setminus A$. For the space X a property is *cyclicly extensible* provided that if each cyclic element of X has this property then X itself has this property.

A space X is said to be *rim-finite* if it has a basis $\mathcal{B}$ such that $|\operatorname{Bd}(U)| < \aleph_0$ for each $U \in \mathcal{B}$. A point p of a space X is said to have *order less than or equal to n in X* provided that for each open neighborhood U of p there exists an open neighborhood V of p such that $V \subset U$ and $|\operatorname{Bd}(V)| \leq n$. If p is of order less than or equal to n but not of order less than or equal to $n - 1$ in X, p is said to be of *order n in X*.

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THEOREM 8. *If X is a continuum in E_c, then X is rim-finite.*

PROOF. Since rim-finiteness is a cyclicly extensible property (the proof in [Wh, Theorem 11.5, p. 83] works also in the non-metric setting) we may suppose X is cyclic. Let a and b be two points of X. It suffices to show since X is compact that there is a finite set which separates a and b in X. Let C be a closed set which separates a and b in X. We may suppose by Mazurkiewicz's Theorem [Ku, Section 49, Theorem 3, p. 244] that C is an irreducible separator of X between a and b. By Lemma 5 and [Ni2, Theorem 3.4] X is a continuous image of an arc and, hence, by [GNST, Theorem 1], C is metrizable. By [Ni1, Theorem 4.9] there is a metrizable T-set A such that $\{a, b\} \cup C \subset A$. Then each component of $X \setminus A$ has two point boundary. Note that no component of $X \setminus A$ contains both a and b in its closure since $C \subset A$.

If a and b lie in different components of A, let $A = B \cup D$, where B and D are separated sets with $b \in B$ and $a \in D$. Since X is locally connected there exist at most finitely many components $C_1, \ldots, C_n$ of $X \setminus A$ which meet both B and D. For each i let $a_i \in \mathrm{cl}(C_i) \setminus (C_i \cup \{a, b\})$. Then $\{a_1, \ldots, a_n\}$ separates a and b in X.

Now suppose a and b lie in the same component E of A. Since E is metrizable and $E \in E_c$ by Lemma 2, by Theorem 6, all but countably many points of E are local separating points of E. By [Wh, (9.2), p. 61] all but countably many of these points are of order 2 in E. Let F be an irreducible separator of E between a and b such that all points of F are local separating points of E and of order 2 in E. We claim that F is finite. Just suppose $x_0 \in F$ is a limit point of F. Let $\{x_i\}$ be a sequence in $F \setminus \{x_0\}$ converging to x_0. Since $F \setminus \{x_0\}$ does not separate a and b, by [GNST, Theorem 4], there is an arc P from a to b in $A \setminus (F \setminus \{x_0\})$. Since the order of x_0 in E is 2 and x_0 is a local separating point of the locally connected continuum E there is a connected neighborhood U of x_0 in E such that $U \cap P$ is connected and $U \cap P \setminus \{x_0\}$ meets two components of $U \setminus \{x_0\}$. Since the order of E at x_0 is 2 there does not exist an arc A_0 in E with $A_0 \cap P = \{x_0\}$. Let G (respectively, H) be the component of $E \setminus F$ which contains a (respectively, b). Since $\mathrm{cl}(G)$ and $\mathrm{cl}(H)$ are locally connected continua, let A_i and B_i be arcs in $\mathrm{cl}(G)$ (respectively, $\mathrm{cl}(H)$) which are irreducible from x_i to P and such that $\mathrm{Lim}_i A_i = \mathrm{Lim}_i B_i = \{x_0\}$. Then $x_0 \notin A_i \cup B_i$ for $i > 0$. Thus, for each sufficiently large i $A_i \cup B_i$ is an arc in $U \setminus \{x_0\}$ which meets both components of $P \cap U$. This is a contradiction and the proof of the claim is completed.

Since A is closed and metric and X is compact and locally connected, it follows that $X \setminus A$ has at most countably many components. Let $C_1, C_2, \ldots$ be the components of $X \setminus A$. For each i, $\mathrm{cl}(C_i) \cap A = \{a_i, b_i\}$. For each i, let $f_i : \mathrm{cl}(C_i) \longrightarrow [0, 1] \times \{i\}$ be a continuous function such that $f_i(a_i) = 0$ and $f_i(b_i) = 1$. Let $\tilde{X} = \left(A \cup \bigcup_{i=1}^{\infty}([0, 1] \times \{i\})\right) / \sim$, where $\sim$ is the smallest equivalence relation on $A \cup \bigcup_{i=1}^{\infty}([0, 1] \times \{i\})$ which identifies a_i with $(0, i)$ and b_i with $(1, i)$ for each i. Define $f : X \longrightarrow \tilde{X}$ by setting

$$f(x) = \begin{cases} x & \text{if } x \in A \\ f_i(x) & \text{if } x \in C_i \text{ for some } i. \end{cases}$$

Let $\tilde{X}$ have the topology induced by f. Then $\tilde{X}$ is a metric continuum and $\tilde{X} \in E_c$. By the argument of the previous paragraph applied to $\tilde{X}$ in place of E, a finite set F separates

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a and b in $\tilde{X}$. Then $\tilde{X} \setminus F = W \cup V$ where W and V are separated sets with $a \in W$ and $b \in V$. Now it follows that only finitely many components, say $C_{i_1}, \dots, C_{i_n}$, of $X \setminus A$ meet both W and V. For each $i = i_1, \dots, i_n$, let $c_i \in \{a_i, b_i\} \setminus \{a, b\}$. Then as in the second paragraph of the proof we have $(F \cap A) \cup \{c_{i_1}, \dots, c_{i_n}\}$ is a finite set which separates a and b in X. The theorem is proved.

THEOREM 9. *Let X be a Peano continuum. Then $X \in E_c$ if and only if the set of non-local separating points of X is countable.*

PROOF. Suppose first that X is a Peano continuum in E_c. Let A_0 denote the set of non-local separating points of X. It is well-known (see [Wh, p. 63]) that since X is a Peano continuum A_0 is a G_δ-set in X. If A_0 were uncountable it would contain a Cantor set contrary to Theorem 6.

To prove the sufficiency we assume X is a Peano continuum and the set of non-local separating points of X is countable but $X \notin E_c$. Then there is a set A of X such that $|A| = c$ and no subset of A separates X. Since X is a metric continuum, by [Wh (9.21), p. 62] and by the hypothesis, we may assume each point of A is a local separating point and is of order 2 relative to A. Let $a \in A$. Then there is a neighborhood U of a such that $\mathrm{Bd}(U) \subset A$ and $|\mathrm{Bd}(U)| = 2$. But $\mathrm{Bd}(U)$ separates X. This contradicts the assumption that no subset of A separates X and Theorem 9 is proved.

A *dendrite* is a locally connected metric continuum which contains no simple closed curve. A dendrite minus its endpoints is connected. The *Gehman dendrite* is the topologically unique dendrite whose endpoints form a Cantor set and whose branch points are all of order 3. For a space X let $\mathcal{C}(X)$ denote the set of all nonempty subcontinua of X.

THEOREM 10. *Let X is a metric continuum. Then $X \in E_c$ if and only if*

(1) X is locally connected and

(2) X contains no Gehman dendrite.

PROOF. The necessity follows by Lemma 2 and Lemma 5. To prove the sufficiency we suppose X satisfies (1) and (2) but $X \notin E_c$. As in the proof of Theorem 9, the set A_0 of non-local separating points of X contains a Cantor set C. We shall consider the following two cases.

Case I. X is hereditarily locally connected. Let $X = U_0$ and let $p_0 \in X \setminus C$ and let $U_{0,0}$ and $U_{0,1}$ be connected open sets with disjoint closures such that $p_0 \notin \mathrm{cl}(U_{0,0} \cup U_{0,1})$, $U_{0,1} \cap C \neq \emptyset$ and $\mathrm{Bd}(U_{0,i}) \cap C = \emptyset$ for $i = 0, 1$. Since the points of C are not local separating points of X we may suppose $U_0 \setminus \mathrm{cl}(U_{0,0} \cup U_{0,1})$ is connected. For $i = 0, 1$, let L_i be an irreducible arc joining p_0 to $\mathrm{Bd}(U_{0,i})$ and $L_i \subset U_0 \setminus \mathrm{cl}(U_{0,j})$ for $j \neq i$. We may suppose that $L_0 \cap L_1$ is connected. Let $p_{0,i} \in L_i \cap \mathrm{Bd}(U_{0,i})$ for $i = 0, 1$. Note $\mathrm{cl}(U_{0,i})$ is locally connected by our assumption and $p_{0,i}$ is an accessible point of $U_{0,i}$.

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Suppose n is a positive integer and for $1 \leq j \leq n$; $\{U_{0,i_1,\ldots,i_j} : i_k = 0, 1 \text{ for } k = 1, \ldots, j\}$ are open connected sets with pairwise disjoint closures,

$$\mathrm{cl}(U_{0,i_1,\ldots,i_j}) \subset U_{0,i_1,\ldots,i_{j-1}},$$
$$\mathrm{Bd}(U_{0,i_1,\ldots,i_j}) \cap C = \emptyset,$$
$$U_{0,i_1,\ldots,i_j} \cap C \neq \emptyset$$

and

$$U_{0,i_1,\ldots,i_{j-1}} \setminus \big(\mathrm{cl}(U_{0,i_1,\ldots,i_{j-1},0}) \cup \mathrm{cl}(U_{0,i_1,\ldots,i_{j-1},1})\big) \text{ is connected.}$$

Suppose $L_{i_1,\ldots,i_j}$ is an arc with

$$L_{i_1,\ldots,i_j} \subset U_{0,i_1,\ldots,i_{j-1}} \cup \{p_{0,i_1,\ldots,i_{j-1}}\}$$

irreducible from $p_{0,i_1,\ldots,i_{j-1}} \in \mathrm{Bd}(U_{0,i_1,\ldots,i_{j-1}})$ to $\mathrm{Bd}(U_{0,i_1,\ldots,i_j})$ with

$$L_{i_1,\ldots,i_{j-1},0} \cap L_{i_1,\ldots,i_{j-1},1} \text{ a connected set}$$

and

$$L_{i_1,\ldots,i_{j-1},k} \cap \mathrm{cl}(U_{0,i_1,\ldots,i_{j-1},m}) = \emptyset \quad \text{for } k \neq m.$$

Let $p_{0,i_1,\ldots,i_j} \in L_{i_1,\ldots,i_j} \cap \mathrm{Bd}(U_{0,i_1,\ldots,i_j})$. As in Step 1 we construct connected open sets $U_{0,i_1,\ldots,i_n,j}$, $j = 0, 1$ with disjoint closures and with

$$\mathrm{cl}(U_{0,i_1,\ldots,i_n,j}) \subset U_{0,i_1,\ldots,i_n},$$
$$\mathrm{Bd}(U_{0,i_1,\ldots,i_n,j}) \cap C = \emptyset,$$
$$U_{0,i_1,\ldots,i_n,j} \cap C \neq \emptyset,$$
$$U_{0,i_1,\ldots,i_n} \setminus \big(\mathrm{cl}(U_{0,i_1,\ldots,i_n,0}) \cup \mathrm{cl}(U_{0,i_1,\ldots,i_n,1})\big) \text{ connected,}$$

and construct arcs $L_{i_1,\ldots,i_n,j}$, $j = 0, 1$ in $U_{0,i_1,\ldots,i_n} \cup \{p_{0,i_1,\ldots,i_n}\} \setminus \mathrm{cl}(U_{0,i_1,\ldots,i_n,k})$ for $k \neq j$ irreducible from $p_{0,i_1,\ldots,i_n}$ to $\mathrm{Bd}(U_{0,i_1,\ldots,i_n,j})$ with

$$L_{i_1,\ldots,i_n,0} \cap L_{i_1,\ldots,i_n,1} \text{ a connected set.}$$

Let $M = \mathrm{cl}(\bigcup_{n=1}^{\infty} \bigcup\{L_{i_1,\ldots,i_n} : i_1, \ldots, i_n = 0, 1\})$. Then M contains a Gehman dendrite.

Case II. X is not hereditarily locally connected. Then there exists a convergence continuum in X, *i.e.*, there is a sequence $\{K_i\}_{i=0}^{\infty}$ of pairwise disjoint continua such that $\mathrm{Lim}\, K_i = K_0$. Since X is locally connected we may suppose K_i is locally connected for each $i \geq 1$.

Let U be a connected open set in X of diameter < 1 such that $K_0 \cap U \neq \emptyset$. Let $H_0 \in \mathcal{C}(K_0) \cap \mathrm{Limsup}\, \mathcal{C}(K_i)$ with $H_0 \subset U$ and $\mathrm{diam}\,(H_0) > 0$. By passing to a subsequence if necessary we may suppose $H_0 \in \mathrm{Lim}\, \mathcal{C}(K_i)$. For each i sufficiently large let $H_i \in \mathcal{C}(K_i \cap U)$ and H_i locally connected such that $\mathrm{Lim}\, H_i = H_0$. Let $i_1 \geq 1$ be an integer so that $U \cap K_{i_1} \neq \emptyset$. Let $x_1 \in H_{i_1}$. Let L_1' be an arc in U irreducible from x_1 to H_0. Let U_0

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and U_1 be open connected sets of diameter $< \frac{1}{2}$ with closures disjoint from L_1' and from each other, $\mathrm{cl}(U_i) \subset U$ and U_0 and U_1 each meet H_0. Let i_2 be an integer so large that $K_{i_2} \cap U_i \neq \emptyset$ for $i = 0, 1$ and there is an arc L_1 in $U \setminus \mathrm{cl}(U_0 \cup U_1)$ irreducible from x_1 to H_{i_2}. Let M_1 be an arc or simple triod with $M_1 \subset L_1 \cup H_{i_2}$ such that $x_1 \in M_1$, $M_1 \cap U_0 \neq \emptyset$ and $M_1 \cap U_1 \neq \emptyset$. Let $H_{0,i} \in \mathcal{C}(H_0 \cap U_i) \cap \mathrm{Limsup}\, \mathcal{C}(H_j)$. We may suppose $H_{0,i} \in \mathrm{Lim}\, \mathcal{C}(H_j)$. Let $H_{2,i,j} \in \mathcal{C}(H_j \cap U_i)$ for j sufficiently large such that $\mathrm{Lim}\, H_{2,i,j} = H_{0,i}$. We may suppose $H_{2,i,j}$ is a locally connected continuum for each j and that $M_1 \cap H_{2,i,i_2} \neq \emptyset$ for $i = 0, 1$.

Repeat the above argument in $H_{0,i}$ with a point of H_{2,i,i_2} in place of x_1 to get continua $M_{1,i} \subset U_i$ which are arcs or simple triods and meet M_1 and K_{i_3} and so on inductively. Then $M = \mathrm{cl}(\bigcup_{n=1}^{\infty} \bigcup \{M_{i_1,\ldots,i_n} : i_j = 0, 1\})$ contains a dendrite D with an uncountable set of endpoints. Every dendrite with an uncountable set of endpoints, it is easy to see, contains a Gehman dendrite.

THEOREM 11. *If X is a metric continuum in E_c then X is the union of countably many arcs.*

PROOF. Let A_0 be the set of non-local separating points of X. By Lemma 5 and Theorem 9 A_0 is countable. Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of X and let $\{U_i\}_{i=1}^{\infty}$ be a countable basis for X with each U_i connected. For each $x \in X \setminus A_0$, by [Wh1, (9.1), p. 61] there exists an integer k such that $x \in U_k$ and $\{x\}$ disconnects U_k. Since $\bigcup \{a_i\}_{i=1}^{\infty}$ is dense there exist $a_i, a_j \in U_k$ which are separated by x in U_k. Put

$$L_{ij}^k = \{x \in U_k : x \text{ separates } a_i \text{ and } a_j \text{ in } U_k\} \cup \{a_i, a_j\}.$$

Since U_k is connected and locally connected, L_{ij}^k is contained in each arc A_{ij}^k in U_k from a_i to a_j. Since A_0 is countable, this completes the proof of Theorem 11.

An arc A is said to be *free* in a continuum X if $A \subset X$ and $\mathrm{Bd}(A)$ is exactly the set of endpoints of A. A continuum X is said to be a *free arc continuum* if every subcontinuum of X has a free arc in X. A free arc continuum is rim-finite. Example 1 is a continuum in E_c which is not a free arc continuum. By Theorem 11 and the Baire Category Theorem every metric continuum in E_c contains a free arc. We can prove from the following theorem that every continuum in E_c contains a free arc.

THEOREM 12. *If X is a cyclic continuum in E_c then X is a free arc continuum.*

PROOF. The proof is by contradiction. Suppose A is an arc in X with no interior. Then the set of branch points of X in A is dense in A. Since X is rim-finite and cyclic, for each $x \in A$ and each neighborhood U of x, there is an arc $B \subset U$ which meets A exactly in the set of endpoints of B. Give A a natural order. There is an arc C_0 in X such that $C_0 \cap A = \{a_0, b_0\}$ with $a_0 < b_0$ since X is cyclic.

Suppose n is an integer and we have constructed pairwise disjoint arcs

$$C_{0,i_1,\ldots,i_j},\ i_k = 0, 1 \quad \text{and} \quad j = 0, \ldots, n$$

with endpoints

$$C_{0,i_1,\ldots,i_j} \cap A = \{a_{0,i_1,\ldots,i_j}, b_{0,i_1,\ldots,i_j}\}$$

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where for $1 \le j \le n$

$$a_{0,i_1,\dots,i_{j-1}} < a_{0,i_1,\dots,i_{j-1},0} < b_{0,i_1,\dots,i_{j-1},0} < a_{0,i_1,\dots,i_{j-1},1} < b_{0,i_1,\dots,i_{j-1},1} < b_{0,i_1,\dots,i_{j-1}}.$$

Now for $i_1,\dots,i_n$ there exist points

$$a_{0,i_1,\dots,i_n} < a_{0,i_1,\dots,i_n,0} < b_{0,i_1,\dots,i_n,0} < a_{0,i_1,\dots,i_n,1} < b_{0,i_1,\dots,i_n,1} < b_{0,i_1,\dots,i_n}$$

and arcs $C_{0,i_1,\dots,i_n,0}$ and $C_{0,i_1,\dots,i_n,1}$ with

$$C_{0,i_1,\dots,i_n,j} \cap A = \{a_{0,i_1,\dots,i_n,j}, b_{0,i_1,\dots,i_n,j}\} \quad \text{for } j = 0, 1$$

and

$$C_{0,i_1,\dots,i_{n+1}} \cap C_{0,m_1,\dots,m_k} = \emptyset \text{ for each } k \le n+1 \text{ if } (0, i_1, \dots, i_{n+1}) \ne (0, m_1, \dots, m_k).$$

Let $C = \bigcap_{k=0}^{\infty} \bigcup \{[a_{0,i_1,\dots,i_k}, b_{0,i_1,\dots,i_k}] : i_j = 0, 1 \text{ and } j = 1, \dots, k\}$. Then C is a second countable, perfect and 0-dimensional subset of the arc A since X is hereditarily locally connected. Let

$$B = A \cup \bigcup_{k=0}^{\infty} \{C_{0,i_1,\dots,i_k} : i_j = 0, 1 \text{ and } j = 1, \dots, k\}.$$

Then B is a subcontinuum of X. By Lemma 2, $B \in E_c$, but B contains C as a set of non-local separating points of B contrary to Theorem 6. Therefore, X is a free arc continuum.

In the following we give an application of the above theorems to extend Nadler's Theorem [Na1, Theorem 9.24, p. 153] in metric continua to the class of Hausdorff continua.

LEMMA 13. *If X is a Hausdorff continuum and $X \in E_{\aleph_0}$, then* $\operatorname{ord}(x, X) \le 2$ *for all but finitely many $x \in X$.*

PROOF. Suppose there exists an infinite subset C of X such that for each $x \in C$ $\operatorname{ord}(x, X) \ge 3$. Without loss of generality, we assume the set C is countable and contains no cluster point of itself. We shall define a subcontinuum L of X such that the set of endpoints of L is infinite which is contrary to $L \in E_{\aleph_0}$, and, hence, completes the proof.

Suppose first that there exists a generalized arc A such that A contains an infinite subset $\{x_1, \dots, x_n, \dots\}$ of C. Since for each i, $\operatorname{ord}(x_i, X) \ge 3$, $\operatorname{ord}(x_i, A) \le 2$ and X is rim-finite, let U_i be an open connected neighborhood of x_i and $p_i \in U_i \setminus A$ such that $U_i \cap U_j = \emptyset$ for $i \ne j$ and let L_i be an irreducible generalized arc in U_i from p_i to A. Then $L = \operatorname{cl}(A \cup \bigcup_{i=1}^{\infty} L_i)$ is a subcontinuum with $\bigcup_{i=1}^{\infty} \{p_i\}$ in its set of endpoints.

We assume now that no generalized arc contains infinitely many points of C. Let x_0 be a limit point of C. Let U_1 be a connected open neighborhood of x_0 and take $x_1 \in U_1 \cap C$. Let L_1 be a generalized arc in U_1 from x_1 to x_0. By induction, suppose we have defined

$x_1, \ldots, x_n$, $U_1, \ldots, U_n$ and $L_1, \ldots, L_n$ such that each U_i is a connected open neighborhood of x, $\mathrm{cl}(U_{i+1}) \subset U_i$, L_i is a generalized arc in U_i from x_i to x_0 and $x_j \notin \mathrm{cl}(U_i)$ for $j < i$. Let U_{n+1} be a connected open neighborhood of x_0 such that $\mathrm{cl}(U_{n+1}) \subset U_n$ and $x_i \notin \mathrm{cl}(U_{n+1})$ for each $i \leq n$. Take $x_{n+1} \in U_{n+1} \cap C \setminus \bigcup_{i=1}^{n} L_i$ and let L_{n+1} be a generalized arc in U_{n+1} from x_{n+1} to x_0. With this construction we have that for each i, $x_i \notin \mathrm{cl}(\bigcup_{j \neq i} L_j)$. Then the subcontinuum $L = \mathrm{cl}(\bigcup_{i=1}^{\infty} L_i)$ has $\{x_i\}_{i=1}^{\infty}$ contained in its set of endpoints as required.

THEOREM 14. *A Hausdorff continuum X is a generalized graph if and only if $X \in E_{\aleph_0}$.*

PROOF. The necessity is clear. To prove sufficiency let X be a Hausdorff continuum and $X \in E_{\aleph_0}$. By Lemma 13, let $\{p_1, \ldots, p_n\}$ be the points of order ≥ 3. Then each component of $X \setminus \{p_1, \ldots, p_n\}$ is a generalized ray or generalized half-line, *i.e.*, an open connected set in which each subcontinuum is a generalized arc. Let T be a finite tree which contains $\{p_1, \ldots, p_n\}$. Then $X \setminus T$ has finitely many components by Lemma 1 and by Lemma 5 each of these is a ray or a half-line whose closure meets T in either one or two points. Therefore, X is a finite graph. This completes the proof of Theorem 14.

EXAMPLE 1. A metric continuum in E_c which is not a free arc continuum.

In the plane $\Re^2$, for q and n integers with $0 \leq q \leq 2^n$, let $L_{q,n} = \{\frac{q}{2^n}\} \times [0, \frac{1}{2^n}]$. Let

$$X = [0,1] \times \{0\} \cup \bigcup_{n=0}^{\infty} \bigcup_{q=0}^{2^n} L_{q,n}.$$

Then X is a metric continuum in E_c which is not a free arc continuum.

EXAMPLE 2. A metric continuum in E_c which contains an infinite irreducible cutting.

In the plane $\Re^2$ we denote $O = (0,0)$, $A_i = (\frac{1}{2^i}, 0)$ and $B_i = (\frac{1}{2^i}, \frac{1}{2^i})$ for $i \geq 0$. For two points P and Q we denote $\overline{PQ}$ the segment from P to Q. Let $X = \overline{OA_0} \cup \overline{OB_0} \cup \bigcup_{i=1}^{\infty} \overline{A_iB_i}$. Then X is a metric continuum in E_c which contains an infinite irreducible cutting.

The following question seems to be of some interest.

QUESTION. *If X is a cyclic continuum in E_κ where κ is an uncountable cardinal number $\leq c$, does there exist $A \subset X$ such that $|X \setminus A| < \kappa$ and each point of A is of order 2 in X?*

(Comment: If X is a metric continuum in E_c then there is a set $A \subset X$ with $|X \setminus A| \leq \aleph_0$ and each point of A is of order 2 in X.)

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