PROOF OF A CONJECTURE OF RAMANUJAN

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1. Introduction. We write

$$f(x) = (1-x)(1-x^2)(1-x^3)\ldots$$

and
$$\sum_{n=0}^{\infty} p(n)x^n = 1/f(x),$$

so that $p(n)$ is the number of unrestricted partitions of n. Ramanujan [1] conjectured in 1919 that if $q = 5$, 7, or 11, and $24m \equiv 1 \pmod{q^n}$, then $p(m) \equiv 0 \pmod{q^n}$. He proved his conjecture for $n = 1$ and 2†, but it was not until 1938 that Watson [4] proved the conjecture for $q = 5$ and all n, and a suitably modified form for $q = 7$ and all n. (Chowla [5] had previously observed that the conjecture failed for $q = 7$ and $n = 3$.) Watson's method of modular equations, while theoretically available for the case $q = 11$, does not seem to be so in practice even with the help of present-day computers. Lehner [6, 7] has developed an essentially different method, which, while not as powerful as Watson's in the cases where $\Gamma_0(q)$ has genus zero, is applicable in principle to all primes q without prohibitive calculation. In particular he proved the conjecture for $q = 11$ and $n = 3$ in [7]. Here I shall prove the conjecture for $q = 11$ and all n, following Lehner's approach rather than Watson's. I also prove the analogous and essentially simpler result for $c(m)$, the Fourier coefficient‡ of Klein's modular invariant $j(\tau)$ as

THEOREM 1. *If* $m \equiv 0 \pmod{11^n}$, *then* $c(m) \equiv 0 \pmod{11^n}$.

The full truth with regard to Ramanujan's original conjecture is thus now known to be: *If* $24m \equiv 1 \pmod{5^a 7^b 11^c}$, *then* $p(m) \equiv 0 \pmod{5^a 7^\beta 11^c}$, *where* $\beta = [(b+2)/2]$.

In view of Watson's result we need only prove here

THEOREM 2. If $24m \equiv 1 \pmod{11^n}$, *then* $p(m) \equiv 0 \pmod{11^n}$.

The general plan of the paper is as follows. In § 2 we describe the notation and general theory required for the proof of Theorem 1. In § 3 we carry through sufficient detailed calculation to prove Theorem 1. In § 4 the additional theory required for the proof of Theorem 2 is given, and in § 5 Theorem 2 is proved. Necessary calculations which would unduly interrupt the main argument are given in Appendices.

2. Functions on $\Gamma_0(11)$.

2.1. We consider the subgroup $\Gamma_0(11)$ of the full modular group $\Gamma(1)$, defined by those transformations

$$\tau \to V\tau = \frac{a\tau + b}{c\tau + d} \quad (a, b, c, d, \text{ integral with } ad - bc = 1)$$

† Ramanujan [1, 2]. See also Rushforth [3].

‡ We take the Fourier series of $j(\tau)$ with leading coefficient unity and constant term zero. Thus $j(\tau) = x^{-1} + 196884x + \ldots$ with $x = e^{2\pi i\tau}$.

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of $\Gamma(1)$ that satisfy $c \equiv 0 \pmod{11}$. $\Gamma_0(11)$ is of genus 1, and its fundamental region has two cusps $\tau = i\infty$ and $\tau = 0$, with local variables $x = e^{2\pi i\tau}$, $x = e^{-2\pi i/11\tau}$, respectively. By " an entire modular function on $\Gamma_0(11)$ " we understand a function $F(\tau)$, regular in $\operatorname{Im}\tau > 0$, that satisfies $F(V\tau) = F(\tau)$ for $V \in \Gamma_0(11)$, and has at most polar singularities in the local variables at the two cusps of $\Gamma_0(11)$. For such $F(\tau)$ we shall write $F \in S$. If in addition $F(\tau)$ is zero at $\tau = i\infty$ we write $F \in S^\infty$. Finally, if $F(\tau)$ is zero at $\tau = 0$ we write $F \in S^0$.

We refer to the expansion of $F(\tau)$ in powers of $x = e^{2\pi i\tau}$ at $\tau = i\infty$ as its Fourier series (FS).

We have:

LEMMA 1. *If $F(\tau) \in S$, then $F^*(\tau) = F(-1/11\tau) \in S$.*

A simple proof is given by Newman [9, Lemma 1]. It is clear that the expansion of $F(\tau)$ at $\tau = 0$ is the FS of $F^*(\tau)$, and that $F \in S^\infty, S^0 \Leftrightarrow F^* \in S^0, S^\infty$.

We now introduce a linear operator U defined by

$$\left.\begin{aligned} 11UF(\tau) &= \sum_{r=0}^{10} F\left(\frac{\tau+r}{11}\right), \\ U^{n+1}F(\tau) &= U(U^nF(\tau)) \quad (n \geqq 1). \end{aligned}\right\} \tag{1}$$

Clearly $$U(a_1F_1 + a_2F_2) = a_1UF_1 + a_2UF_2,$$

if a_1, a_2 are constants. If the FS of $F(\tau)$ is

$$\sum_{r=r_0}^{\infty} \alpha_r x^r,$$

then the FS of $UF(\tau)$ is

$$\sum_{11r \geqq r_0} \alpha_{11r} x^r.$$

By $UF(-1/11\tau)$ we shall understand the effect of replacing τ by $-1/11\tau$ in $UF(\tau)$ and *not* " $UG(\tau)$ where $G(\tau) = F(-1/11\tau)$ ".

We also write

$$F_1(\tau) \equiv F_2(\tau) \pmod{m} \tag{2}$$

if all the respective coefficients in the FS of $F_1(\tau)$ and $F_2(\tau)$ are congruent modulo m. Thus nothing is asserted by (2) as to the expansions at $\tau = 0$.

It will be convenient in the sequel to assess divisibility by powers of 11 by using an exponential valuation. Accordingly, for integral a, we define $\pi(a)$ by

$$11^{\pi(a)} \mid a, \quad 11^{\pi(a)+1} \nmid a,$$

and for rational $a = b/c$ we define

$$\pi(a) = \pi(b) - \pi(c).$$

We write conventionally $\pi(0) = \infty$, and regard any inequality $\pi(0) \geqq k$ as valid.

We have

$$\pi(ab) = \pi(a)+\pi(b),$$
$$\pi(a+b) \geqq \min(\pi(a), \pi(b)), \tag{3}$$

with equality if $\pi(a) \neq \pi(b)$.

The crucial results on $UF(\tau)$ are given by Lehner (Theorem 8 and (8.81) of **[6]**), and are as follows.

LEMMA 2. *If $F(\tau) \in S$, then*

$$\text{(i) } UF(\tau) \in S, \quad \text{(ii) } 11UF(-1/11\tau) - 11UF(11\tau) = F(-1/121\tau) - F(\tau).$$

Note that in (ii) $UF(11\tau)$ and $F(-1/121\tau)$ are not themselves in S. It is also immediate that

$$F(\tau) \in S^{\infty} \Rightarrow UF(\tau) \in S^{\infty}. \tag{4}$$

((4) is *not* valid for S_0.)

Reverting now to the proof of Theorem 1, we see that, since $j(\tau) \in S$, then $U^n j(\tau) \in S^{\infty}$ for $n \geqq 1$. Theorem 1 is then equivalent to proving that the FS of $11^{-n}U^n j(\tau)$ has integral coefficients. To establish this, we obtain first a standard basis for the functions of S^{∞}, and then use Lemma 2 to obtain detailed information as to the effect of the operator U on these functions.

2.2. A linear basis for functions on $\Gamma_0(11)$. The following lemma is proved in Appendix A.

LEMMA 3. *For all integral $n \geqq 2$, there exist functions $G_n(\tau)$, $g_n(\tau)$, $h_n(\tau)$ with the following properties:*

(i) $$G_n(\tau) \in S^0, \; g_n(\tau) \in S^{\infty}, \; h_n(\tau) \in S^{\infty},$$

(ii) $$G_n(-1/11\tau) = h_n(\tau) = 11^{\theta(n)} g_n(\tau),$$

where $$\theta(n) = 6k+2, 3, 4, 6, 6$$

according as $$n = 5k+2, 3, 4, 5, 6 \qquad (k \geqq 0).$$

(iii) *The FS of $G_n(\tau)$ has integral coefficients with leading term x^{-n}.*

(iv) *The FS of $g_n(\tau)$ has integral coefficients with leading term $x^{\psi(n)}$,*

where $$\psi(n) = 5k+1, 2, 3, 5, 4$$

according as $$n = 5k+2, 3, 4, 5, 6 \quad (k \geqq 0).$$

Further, there exists a function $B(\tau) \in S$ with simple poles at $\tau = 0$ and $\tau = i\infty$, such that $B(-1/11\tau) = B(\tau)$. The FS of $B(\tau)$ has integral coefficients, with leading term x^{-1}.

Since the Riemann surface of $\Gamma_0(11)$ cannot support a univalent function, we have the immediate corollary:

LEMMA 4. *Suppose that $F(\tau)\in S$ has a pole of order M at $\tau=0$ and a pole of order N at $\tau=i\infty$. Then*

$$F(\tau)=\sum_{r=2}^{N}\lambda_{-r}G_r(\tau)+\lambda_{-1}B(\tau)+\lambda_0+\sum_{r=2}^{M}\lambda_r h_r(\tau),$$

where the λ_r $(-N\leqq r\leqq M)$ are constants.

Finally we restate Lemma 4 in the case of greatest interest to us.

LEMMA 5. *Suppose that $F(\tau)\in S^\infty$ has a pole of order M at $\tau=0$. Then*

$$F(\tau)=\sum_{r=2}^{M}\lambda_r h_r(\tau),\quad F(-1/11\tau)=\sum_{r=2}^{M}\lambda_r G_r(\tau).$$

For a given $F(\tau)\in S^\infty$, the constants λ_r in Lemma 5 can be determined from the FS of either $F(-1/11\tau)$ or $F(\tau)$. We are mainly concerned not with the exact value of λ_r, but with $\pi(\lambda_r)$. In §3 below we obtain suitable lower bounds for $\pi(\lambda_r)$ in the case when $F(\tau)=Ug_n(\tau)$. The calculations take a simpler form when we consider $F(\tau)=Uh_n(\tau)$; the transition to $Ug_n(\tau)$ is immediate from Lemma 3(ii).

3.1. Since $h_n(\tau)\in S^\infty$, we have

$$11Uh_n(\tau)=\sum c_{nr}h_r(\tau), \tag{5}$$

where the c_{nr} are constants, by Lemma 5. It is convenient to regard the sum in (5) as one from $r=2$ to ∞, although all but a finite number of the c_{nr} are zero. We have also

$$11Uh_n(-1/11\tau)=\sum c_{nr}G_r(\tau) \tag{6}$$

and, by Lemma 2(ii),

$$11Uh_n(-1/11\tau)-11Uh_n(11\tau)=-h_n(\tau)+h_n(-1/121\tau). \tag{7}$$

It follows from (7) that the principal part of the FS of $11Uh_n(-1/11\tau)$ is the same as that of $h_n(-1/121\tau)=G_n(11\tau)$, since $Uh_n(11\tau)$ and $h_n(\tau)$ are zero at $\tau=i\infty$. Hence the coefficients c_{nr} may be uniquely determined by the fact that the FS of

$$G_n(11\tau)-\sum c_{nr}G_r(\tau)$$

has no terms in $x^{-11n},\ldots,x^{-3},x^{-2}$. It will then necessarily have no term in x^{-1}, which provides a check in numerical work. It follows that the c_{nr} are integers (since each G_r has leading term x^{-r}, and the FS of $G_n(11\tau)$ has integral coefficients) and also that

$$c_{nr}=0\quad\text{if}\quad r>11n. \tag{8}$$

Considering next the determination of the c_{nr} from (5) we observe that for different r the FS of $h_r(\tau)$ commence with different powers† of x, by Lemma 3(iv).

† The linear basis used by Lehner [6, 7] does not have this property.

B

Thus, since every coefficient in the FS of $11Uh_n(\tau)$ is divisible by $11^{\theta(n)+1}$, and the leading term in the FS of $h_r(\tau)$ is $11^{\theta(r)}x^{\psi(r)}$, we have

$$\pi(c_{nr}) \geqq \theta(n)-\theta(r)+1 \tag{9}$$

and

$$c_{nr}=0 \quad \text{if} \quad 11\psi(r)<\psi(n). \tag{10}$$

We now establish certain conditions under which $\pi(c_{nr}) \geqq 3$.

3.2. The values of c_{nr} (mod 11^3). All congruences in this section are to the modulus 11^3. It follows from (6) and (7) that

$$G_2(11\tau)-11^2 g_2(\tau) \equiv \sum c_{2r}G_r(\tau), \quad G_n(11\tau) \equiv \sum c_{nr}G_r(\tau) \quad (n \geqq 3). \tag{11}$$

We shall use the symbol (k, l, m) to denote an expression of the form

$$11^2 \sum_{i=k}^{l-1} \lambda_i G_i(\tau) + 11 \sum_{i=l}^{m-1} \lambda_i G_i(\tau) + \sum_{i=m}^{N} \lambda_i G_i(\tau),$$

where $k < l < m \leqq N$ and the λ_i are integral constants. Then by direct calculation† we find

$$G_2(11\tau)-11^2 g_2(\tau) \equiv (2,9,19), \quad G_3(11\tau) \equiv (8,18,28). \tag{12}$$

Now Table 5 in Appendix B shows that

$$G_i(\tau)G_j(\tau) = \sum_{r=-3}^{0} \mu_r G_{i+j+r}(\tau),$$

where the μ_r are integral, and μ_{-3}, $\mu_{-2} \equiv 0$ (mod 11), for all i and j. It follows that

$$(k_1, l_1, m_1)(k_2, l_2, m_2) \equiv (k_3, l_3, m_3),$$

where

$$\begin{aligned} k_3 &= \min(k_1+m_2-1,\ k_2+m_1-1,\ l_1+l_2-1), \\ l_3 &= \min(l_1+m_2-1,\ l_2+m_1-1), \\ m_3 &= m_1+m_2-1. \end{aligned}$$

Further, from Table 6 we have, for $m \geqq 4$,

$$11^2 g_2(\tau).(k, l, m) \equiv (m-2, \infty, \infty).$$

Thus

$$\left.\begin{aligned} G_4(11\tau) &= G_2^2(11\tau)-11G_3(11\tau) \equiv (17,27,37), \\ G_5(11\tau) &= G_2(11\tau)G_3(11\tau)-11G_4(11\tau) \equiv (26,36,46), \\ G_6(11\tau) &= G_2(11\tau)G_4(11\tau) \equiv (35,45,55), \\ G_7(11\tau) &= G_2(11\tau)G_5(11\tau) \equiv (44,54,64). \end{aligned}\right\} \tag{13}$$

† This was done in three different ways on three different machines: firstly using Lemma 9 on a Diehl desk calculator at Durham University; next using (5) on an Elliott 803 computer at Durham University; and finally using (6) on the I.C.T. Atlas 1 computer at Chilton. The computing times were respectively one week, one hour, and ten seconds.

It is now easily seen by induction, since $G_{n+5}(\tau) = G_n(\tau)G_5(\tau)$, that

$$G_n(11\tau) \equiv (9n-19, 9n-9, 9n+1) \quad (n \geqq 3), \tag{14}$$

and thus, by (11),

$$\pi(c_{nr}) \geqq 3 \quad \text{if} \quad r \leqq 9n-20,\ n \geqq 3. \qquad (15)\dagger$$

3.3. The values of c_{nr} (mod 11^4). We have $11Uh_n(\tau) \equiv 0 \pmod{11^4}$ for $n \geqq 3$ (since $\theta(n)+1 \geqq 4$), and (5) with $\pi(c_{21}) \geqq 2$, $\pi(c_{22}) \geqq 1$ gives $11Uh_2(11\tau) \equiv 0 \pmod{11^4}$.

Then, by (6) and (7),

$$\left.\begin{aligned} G_2(11\tau) - 11^2 g_2(\tau) &\equiv \sum c_{2r} G_r(\tau) \pmod{11^4}, \\ G_3(11\tau) - 11^2 g_3(\tau) &\equiv \sum c_{3r} G_r(\tau) \pmod{11^4}, \\ G_n(11\tau) &\equiv \sum c_{nr} G_r(\tau) \pmod{11^4} \quad (n \geqq 4). \end{aligned}\right\} \tag{16}$$

Hence, by arguments similar to those of § 3.2, we obtain

$$c_{5r} \equiv 0 \pmod{11^4} \text{ for } r \leqq 15, \quad c_{7r} \equiv 0 \pmod{11^4} \text{ for } r \leqq 33.$$

A crude induction, using Table 5 and (15), now shows that

$$c_{5k+2,r} \equiv 0 \pmod{11^4} \text{ for } r \leqq 15k+18, \quad k \geqq 1. \tag{17}$$

We summarise our results on c_{nr} in the forms actually required later.

LEMMA 6.

$\pi(c_{nr}) \geqq 0$	*always,*	(from § 3.1)
$\pi(c_{nr}) \geqq \theta(n) - \theta(r) + 1$	*always,*	(from (9))
$\pi(c_{nr}) \geqq 1$	*for* $n = 2$ *or* 3, $9 \leqq r \leqq 11$,	(from (12))
$\pi(c_{nr}) \geqq 2$	*for* $n = 2$ *or* 3, $r \leqq 8$,	(from (12))
$\pi(c_{nr}) \geqq 3$	*for* $n = 4$, $r \leqq 16$,	(from (15))
$\pi(c_{nr}) \geqq 3$	*for* $n \geqq 5$, $r \leqq n+14$,	(from (15))
$\pi(c_{nr}) \geqq 4$	*for* $n \equiv 2 \pmod 5$, $n \geqq 7$, $r = n-1$ or $n-2$,	(from (17))
$\pi(c_{22}) = 2$, $\pi(c_{32}) = 3$, $\pi(c_{42}) = 4$.		(from Table 7)

3.4. We now use the results of § 3.3 to show that, in effect, functions of a suitable form remain of that form under the operation $11^{-1}U$. This is the basis of the proofs of Theorems

† This result is by no means best possible. We can, by consideration of cases (mod 5), establish results with $11n$ instead of $9n$ on the right-hand side of (14), but (15) suffices later.

1 and 2. Our Lemma 7 below is needed in §5, although a weaker form would suffice for Theorem 1.

We define

$$\left.\begin{array}{ll} & \xi(2)=0,\ \xi(3)=1 \\ \text{and} & \xi(n)=5k+1,3,3,4,5 \\ \text{according as} & n=5k+4,5,6,7,8 \quad (k \geqq 0). \end{array}\right\} \tag{18}$$

We also define $\eta(2)=0$, $\eta(3)=1$, $\eta(n)=\xi(n)+1$ $(n \geqq 4)$. We shall denote by X the class of functions $F(\tau)$ with

$$F(\tau)=\sum_{n=2}^{N} \lambda_n 11^{\xi(n)} g_n(\tau), \tag{19}$$

and by Y the class of functions $F(\tau)$ with

$$F(\tau)=\sum_{n=2}^{M} \mu_n 11^{\eta(n)} g_n(\tau), \tag{20}$$

where N, M, λ_n, and μ_n are any integral constants.

Lemma 7. *If* $F(\tau) \in X$, *then* $11^{-1}UF(\tau) \in Y$.

Proof. We have, by (5),

$$11^{-1}U \sum \lambda_n 11^{\xi(n)} g_n(\tau)=\sum\sum \lambda_n 11^{\xi(n)-2-\theta(n)+\theta(r)} c_{nr} g_r(\tau).$$

Thus we have to show that, for all n and r,

$$\xi(n)-2-\theta(n)+\theta(r)+\pi(c_{nr}) \geqq \eta(r). \tag{21}$$

The following table is given to clarify the details of the proof $(k \geqq 0)$.

n	2	3	$5k+4$	$5k+5$	$5k+6$	$5k+7$	$5k+8$	$5k+9$	$5k+10$
$\xi(n)$	0	1	$5k+1$	$5k+3$	$5k+3$	$5k+4$	$5k+5$	$5k+6$	$5k+8$
$\eta(n)$	0	1	$5k+2$	$5k+4$	$5k+4$	$5k+5$	$5k+6$	$5k+7$	$5k+9$
$\theta(n)$	2	3	$6k+4$	$6k+6$	$6k+6$	$6k+8$	$6k+9$	$6k+10$	$6k+12$
$\theta(n)-\eta(n)$	2	2	$k+2$	$k+2$	$k+2$	$k+3$	$k+3$	$k+3$	$k+3$
$\theta(n)-\xi(n)$	2	2	$k+3$	$k+3$	$k+3$	$k+4$	$k+4$	$k+4$	$k+4$

We quote the results of Lemma 6 without their formula numbers. Since $\pi(c_{nr}) \geqq 0$, (21) holds if $\theta(r)-\eta(r) \geqq \theta(n)-\xi(n)+2$. This is satisfied in the cases

$$n \geqq 5,\ r \geqq n+15; \quad n=4,\ r \geqq 17; \quad n=2 \text{ or } 3,\ r \geqq 12. \tag{22}$$

Also, since $\pi(c_{nr}) \geqq \theta(n)-\theta(r)+1$, (21) holds if $\xi(n) \geqq \eta(r)+1$. This is satisfied in the cases

$$n \geqq 5,\ n \not\equiv 2 \pmod 5,\quad r \leqq n-2;\quad n \geqq 5,\ n \equiv 2 \pmod 5,\ r \leqq n-3. \tag{23}$$

Now for $n \geqq 5$, $n-1 \leqq r \leqq n+14$, we have $\pi(c_{nr}) \geqq 3$, and so (21) holds if

$$\theta(n)-\xi(n) \leqq \theta(r)-\eta(r)+1.$$

This is valid unless $n \equiv 2 \pmod 5$, $r = n-1$. This gives the cases

$$n \geqq 5,\ n \not\equiv 2 \pmod 5,\ n-1 \leqq r \leqq n+14;\quad n \geqq 5,\ n \equiv 2 \pmod 5,\ r \geqq n. \tag{24}$$

Next, if $n \geqq 5$, $n \equiv 2 \pmod 5$, $r = n-1$ or $n-2$, we have $\pi(c_{nr}) \geqq 4$. Hence (21) holds if

$$n \geqq 5,\ n \equiv 2 \pmod 5,\quad r = n-1 \text{ or } n-2. \tag{25}$$

Now for $n=4$, (21) is $\pi(c_{4r}) \geqq \eta(r)-\theta(r)+5$. For $2 \leqq r \leqq 16$, we have $\pi(c_{4r}) \geqq 3$, $\eta(r)-\theta(r) \leqq -2$. This gives

$$n = 4,\quad 2 \leqq r \leqq 16. \tag{26}$$

Finally if $n = 2$ or 3, (21) is $\pi(c_{nr}) \geqq \eta(r)-\theta(r)+4$, and we have for $r \leqq 8$, $\pi(c_{nr}) \geqq 2$ and $\eta(r)-\theta(r) \leqq -2$; also for $9 \leqq r \leqq 11$ we have $\pi(c_{nr}) \geqq 1$ and $\eta(r)-\theta(r) \leqq -3$. This gives

$$n = 2 \text{ or } 3,\quad r \leqq 11. \tag{27}$$

Since (22) to (27) cover all integral n, r with $n \geqq 2$ and $r \geqq 2$, Lemma 7 is proved.

COROLLARY. *If* $F(\tau) \in X$, *then*

$$11^{-1}UF(\tau) \in X. \tag{28}$$

For $\xi(n) \leqq \eta(n)$.

It is desirable in some cases to prove that the congruences obtained by using Lemma 7 are best possible. To this end we define classes of functions X^0 and Y^0 as at the beginning of this section, but with the additional conditions $\pi(\lambda_2) = 0$, $\pi(\mu_2) = 0$. We now prove

LEMMA 8. *If* $F(\tau) \in X^0$, *then* $11^{-1}UF(\tau) \in Y^0$.

We have, as in the proof of Lemma 7,

$$\mu_2 = \sum \lambda_n 11^{\xi(n)-2-\theta(n)+2} c_{n2} = \sum \rho_n, \quad \text{say}.$$

Now $\pi(c_{n2}) \geqq \theta(n)-\theta(2)+1$, so that, for $n \geqq 5$, we have $\pi(\rho_n) \geqq 1$, since $\xi(n) \geqq 2$. For $n = 3$, $\pi(c_{32}) = 3$, $\xi(3)-\theta(3) = -2$ and so $\pi(\rho_3) \geqq 1$. For $n = 4$, $\pi(c_{42}) = 4$, $\xi(4)-\theta(4) = -3$, and so $\pi(\rho_4) \geqq 1$. Hence

$$\mu_2 \equiv \lambda_2 11^{-2} c_{22} \pmod{11}.$$

But $\pi(c_{22}) = 2$, and hence, if $\pi(\lambda_2) = 0$, then $\pi(\mu_2) = 0$. This proves the lemma.

3.5. Proof of Theorem 1.

We may express Lemma 2 (ii), in the form:

LEMMA 9. *If* $F(\tau)\in S$, *then* $F(-1/11\tau)+11UF(\tau)$ *is an entire function on the full modular group* $\Gamma(1)$.

Choosing $F(\tau)=B(\tau)$, we have, since the FS of $B(\tau)$ is $x^{-1}-5+\dots$,

$$60+B(\tau)+11UB(\tau)=j(\tau).$$

Now $U\{B(\tau)+5\}\in S^{\infty}$, and $11UB(-1/11\tau)$ has FS $x^{-11}+O(x^{-1})$. Hence

$$11\{UB(-1/11\tau)+5\}=\sum_{n=2}^{11}\alpha_n G_n(\tau),$$

where the α_n are integral constants, and so

$$11^{-1}\{UB(\tau)+5\}=\sum_{n=2}^{11}\alpha_n 11^{\theta(n)-2}g_n(\tau)\in X.$$

Thus $$11^{-1}Uj(\tau)=11^{-1}U\{B(\tau)+5\}+U^2\{B(\tau)+5\}\in X,$$

by (28). Repeated application of (28) shows that $11^{-n}U^nj(\tau)\in X$ for $n\geqq 1$.

Now the FS of any function in X has integral coefficients, while the FS of $U^nj(\tau)$ is $\sum_{m=1}^{\infty}c(11^n m)x^m$. Hence for all $m\geqq 1$, $n\geqq 1$ we have that $11^{-n}c(11^n m)$ is an integer, which is Theorem 1.

Theorem 1 is best possible in the sense that $c(11^n)\not\equiv 0 \pmod{11^{n+1}}$. We have

$$11^{-1}Uj(\tau)\equiv\alpha_2 g_2(\tau)\pmod{11}.$$

Now $\alpha_2=1627$ and so $\pi(\alpha_2)=0$. Hence, by repeated application of Lemma 8, we have (since $Y^0\subseteq X^0$)

$$11^{-n}U^nj(\tau)\in X^0,$$

and so $$11^{-n}U^nj(\tau)\equiv k_n g_2(\tau)\pmod{11},$$

where $\pi(k_n)=0$.

Thus $$11^{-n}c(11^n)\equiv k_n\pmod{11},$$

and so $$c(11^n)\not\equiv 0\pmod{11^{n+1}}.$$

4.1. We now define

$$\eta(\tau)=e^{\pi i\tau/12}f(x)\quad(\operatorname{Im}\tau>0),$$

where $$f(x)=\prod_{r=1}^{\infty}(1-x^r)\quad\text{and}\quad x=e^{2\pi i\tau},\tag{29}$$

and $$\phi(\tau)=\eta(121\tau)/\eta(\tau)=x^5f(x^{121})/f(x),\quad\Phi(\tau)=1/\phi(\tau).\tag{30}$$

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We also let

$$\left.\begin{aligned} l_{2n-1} &= (13\,.\,11^{2n-1}+1)/24 \quad (n \geqq 1), \\ l_{2n} &= (23\,.\,11^{2n}+1)/24 \qquad (n \geqq 1), \end{aligned}\right\} \tag{31}$$

so that l_n is the least positive integral solution of $24l_n \equiv 1 \pmod{11^n}$.

Further let

$$\left.\begin{aligned} \Lambda_{2n-1}(x) &= f(x^{11}) \sum_{m=0}^{\infty} p(11^{2n-1}m + l_{2n-1})x^{m+1}, \\ \Lambda_{2n}(x) &= f(x) \sum_{m=0}^{\infty} p(11^{2n}m + l_{2n})x^{m+1}, \end{aligned}\right\} \quad (n \geqq 1), \tag{32}$$

and define a sequence of functions $L_n(\tau)$ by

$$\left.\begin{aligned} L_1(\tau) &= U\phi(\tau), \\ L_{2n}(\tau) &= UL_{2n-1}(\tau) \qquad (n \geqq 1), \\ L_{2n+1}(\tau) &= \{U\phi(\tau L_{2n}(\tau)\} \quad (n \geqq 1). \end{aligned}\right\} \tag{33}$$

We shall prove by induction that, for $n \geqq 1$, $\Lambda_n(x)$ is the FS of $L_n(\tau)$. We have

$$U\{F_1(11\tau)F_2(\tau)\} = F_1(\tau)\,UF_2(\tau).$$

Now the FS of $\phi(\tau)$ is

$$x^5 f(x^{121}) \sum_{m=0}^{\infty} p(m)x^m,$$

so that the FS of $U\phi(\tau)$ is

$$f(x^{11}) \sum_{m=0}^{\infty} p(11m+6)x^{m+1} = \Lambda_1(x).$$

Assuming that the FS of $L_{2n-1}(\tau)$ is $\Lambda_{2n-1}(x)$, we see that the FS of $L_{2n}(\tau) = UL_{2n-1}(\tau)$ is

$$f(x) \sum_{m=0}^{\infty} p\{11^{2n-1}(11m+10) + l_{2n-1}\}x^{m+1} = L_{2n}(x).$$

Finally if the FS of $L_{2n}(\tau)$ is $\Lambda_{2n}(x)$, then the FS of $L_{2n+1}(\tau) = U\{\phi(\tau)L_{2n}(\tau)\}$ is

$$f(x^{11}) \sum_{m=0}^{\infty} p\{11^{2n}(11m+6) + l_{2n}\}x^{m+1} = L_{2n+1}(x).$$

Since the expansions of $1/f(x)$ and $1/f(x^{11})$ have integral coefficients with leading terms unity, Theorem 2 is equivalent to

LEMMA 10. *The FS of* $11^{-n}L_n(\tau)$ *has integral coefficients.*

4.2. Since $\phi(\tau)$ is not a function on $\Gamma_0(11)$, but on $\Gamma_0(121)$, we cannot apply the methods of §§2 and 3 immediately. However we do have

LEMMA 11. *If* $F(\tau)\in S^\infty$, *then*

(i) $U\{\phi(\tau)F(\tau)\}\in S^\infty$,

(ii) *the principal part of the expansions of* $U\{\phi(\tau)F(\tau)\}$ *in powers of* $x = e^{-2\pi i/11\tau}$ *at its pole* $\tau = 0$ *is the same as the principal part of the FS expansion of* $11^{-2}\Phi(\tau)F(-1/121\tau)$ *in powers of* $x = e^{2\pi i\tau}$.

Further (i) *and* (ii) *hold in the special case* $F(\tau) = 1$.

Lemma 11 is proved by Lehner [6, Theorem 8]; there are some misprints corrected in Lehner [7, page 178].

We now have, by Lemma 5,

$$11^2 U\{\phi(\tau)h_n(\tau)\} = \sum d_{nr}h_r(\tau), \tag{34}$$

where the d_{nr} are constants, and in fact zero if $11\psi(r) < \psi(n)+5$ or $r > 11n+5$. Further the d_{nr} are uniquely determined by the fact that the FS of

$$\Phi(\tau)\,G_n(11\tau) - \sum_{r=2}^{11n+5} d_{nr}G_r(\tau)$$

has no terms in $x^{-11n-5},\ldots,x^{-3},\ x^{-2}$. Hence

$$\pi(d_{nr}) \geqq 0. \tag{35}$$

We have also, from (34),

$$11^{2+\theta(n)}U\{\phi(\tau)g_n(\tau)\} = \sum d_{nr}11^{\theta(r)}g_r(\tau),$$

and thus

$$\pi(d_{nr}) \geqq \theta(n)-\theta(r)+2. \tag{36}$$

We could, by using

$$\Phi(\tau)G_n(11\tau) \equiv G_5(\tau)\{G_n(\tau)\}^{11} \pmod{11},$$

obtain quite easily conditions under which $\pi(d_{nr}) \geqq 1$. Unfortunately this is not quite enough to prove Theorem 1, and we require the following

LEMMA 12.

$$\Phi(\tau) \equiv G_5(\tau)+11\{G_4(\tau)+2G_3(\tau)+G_2(\tau)-1+2g_2(\tau)+3g_3(\tau)+g_4(\tau)+5g_5(\tau)\} \pmod{11^2}.$$

This is proved in Appendix C.

We use the symbol (l, m) to denote an expression of the form

$$11\sum_{i=l}^{m-1}\lambda_iG_i(\tau)+\sum_{i=m}^{N}G_i(\tau),$$

where $l < m \leqq N$ and the λ_i are integral constants. Then in terms also of the notation of § 3.2, we have (using Tables 5 and 6)

$$\Phi(\tau)(k, l, m) \equiv (l_1, m_1) \pmod{11^2} \quad (m \geqq 7), \tag{37}$$

where $m_1 = m+5$, $l_1 = \min(l+5,\ m-5)$.

Thus
$$\Phi(\tau)G_2(11\tau) \equiv (14, 24) \pmod{11^2},$$
$$\Phi(\tau)G_n(11\tau) \equiv (9n-4,\ 9n+6) \pmod{11^2} \quad (n \geqq 3),$$

by (12) and (14). Hence

$$\pi(d_{nr}) \geqq 2 \quad \text{if} \quad r \leqq 9n-5. \tag{38}$$

We can now prove the result complementary to Lemma 7. We have

Lemma 13. *If $F(\tau) \in Y$, then $11^{-1}U\{\phi(\tau)F(\tau)\} \in X$.*

Proof. We have, by (34),

$$11^{-1}U\sum \mu_n 11^{\eta(n)}\phi(\tau)g_n(\tau) = \sum\sum \mu_n 11^{\eta(n)-3-\theta(n)+\theta(r)} d_{nr} g_r(\tau).$$

Thus we have to show that, for all n and r,

$$\eta(n)-3-\theta(n)+\theta(r)+\pi(d_{nr}) \geqq \xi(r). \tag{39}$$

Since $\pi(d_{nr}) \geqq 0$, (39) holds if $\theta(r)-\xi(r) \geqq \theta(n)-\eta(n)+3$. This is satisfied in the cases

$$n \geqq 4,\ r \geqq n+10; \quad n = 2 \text{ or } 3,\ r \geqq 12. \tag{40}$$

Also since $\pi(d_{nr}) \geqq \theta(n)-\theta(r)+2$, (39) holds if $\eta(n) \geqq \xi(r)+1$. This is satisfied in the cases

$$n \geqq 4,\ r \leqq n; \quad n = 3,\ r = 2. \tag{41}$$

Next, for $n \geqq 3$ and $n+1 \leqq r \leqq n+9$, we have $\pi(d_{nr}) \geqq 2$ and $\theta(r)-\xi(r) \geqq \theta(n)-\eta(n)+1$, which implies (39) for

$$n \geqq 3, \quad n+1 \leqq r \leqq n+9. \tag{42}$$

Similarly we obtain

$$n = 2, \quad 4 \leqq r \leqq 11. \tag{43}$$

Finally we have by direct calculation $\pi(d_{33}) = 3$, $\pi(d_{22}) = 3$, $\pi(d_{23}) = 4$, which give (39) for

$$n = 2,\ r = 2 \text{ and } 3; \quad n = 3,\ r = 3. \tag{44}$$

Since (40) to (44) cover all integral n, r with $n \geqq 2, r \geqq 2$, Lemma 13 is proved. We have also

Lemma 14. *If $F(\tau) \in Y^0$, then $11^{-1}U\{\phi(\tau)F(\tau)\} \in X^0$.*

We have

$$\lambda_2 = \sum \mu_n 11^{\eta(n)-3-\theta(n)+2} d_{n2} = \sum \sigma_n, \quad \text{say.}$$

Now $\pi(d_{n2}) \geqq \theta(n)-\theta(2)+2$, so that for $n \geqq 4$, $\eta(n) \geqq 2$, and so $\pi(\sigma_n) \geqq 1$. For $n = 3$,

$\pi(d_{32}) = 4$ by direct calculation, and so $\pi(\sigma_3) = 1$. Hence

$$\lambda_2 \equiv \mu_2 11^{-3} d_{22} \pmod{11}.$$

But $\pi(d_{22}) = 3$, and hence if $\pi(\mu_2) = 0$, then $\pi(\lambda_2) = 0$. This proves the lemma.

5. Proof of Theorem 1.

Using the remark at the end of Lemma 11 we find

$$L_1(\tau) = U\phi(\tau) = 11g_2(\tau) + 2.11^2 g_3(\tau) + 11^3 g_4(\tau) + 11^4 g_5(\tau).$$

Hence $11^{-1}L_1(T) \in X^0$. It is now easily seen, by using the definition of $L_n(\tau)$ in (33) and Lemmas 8 and 14, that

$$11^{1-2n}L_{2n-1}(\tau) \in X^0, \quad 11^{-2n}L_{2n}(\tau) \in Y^0.$$

This proves Lemma 10 and so Theorem 1. In addition we see, as in §3.5, that Theorem 1 is best possible in the sense that

$$p(l_n) \not\equiv 0 \pmod{11^{n+1}}.$$

It is clear that the inductions used to prove Theorem 2 are dominated by the values of $\pi(c_{22})$ and $\pi(d_{22})$, in the sense that were either of these greater we could with greater effort establish a congruence modulo $11^{[3n/2]}$ or thereabouts. The actual computed values of $\pi(c_{nr})$ and $\pi(d_{nr})$ are much larger than those given by our inequalities as is shown by Tables 7 and 8; the difficult part of the induction, apart from " accidental " low values of n and r, is when r is close to n, and in fact it seems certain that $\pi(c_{nr})$ and $\pi(d_{nr})$ are about equal to n in this case, not merely 3 or 4 as we prove. The introduction of a basis $g_n(\tau)$ with different orders of zeros at $\tau = i\infty$ is needed to cope with the case when $r < n$; the numbers $\theta(n) \doteq 6n/5$ which this involves are an inevitable and not wholly desirable complication. For $r \gg n$ Lehner's basis is equally satisfactory. Finally, the actual classes of functions X, Y suffice for the induction, and are not best possible. We could use $\pi(d_{nr}) \geqq 1$ only, and a more elaborate form of Lemma 6, plus a good deal of actual computation for low values of n and r. This would avoid the appeal to Fine's equation, but the present method is shorter.

We may observe finally that, in comparison with $q = 5$ and $q = 7$, this proof is indeed " langweilig ", as Watson suggested. In those cases, we can in effect deal directly with $Ug^n(\tau)$ at $\tau = i\infty$, using the modular equation. In fact, his actual induction can be reduced† to about 2 pages each for $q = 5$ and $q = 7$, if it is expressed in terms of $\pi(c_{nr})$ rather than fully written out formulae, by using explicit inequalities of the type $\pi(c_{nr}) \geqq [(5n - r + 1)/2]$, for $q = 5$. I think it likely that in the present case $q = 11$ there exists an inequality

$$\pi(c_{nr}) \geqq [(11n - r + \delta)/10],$$

where $\delta = \delta(n, r)$ is small and of irregular behaviour, but I can at present see no technique for establishing this.

† See A. O. L. Atkin, Ramanujan congruences for $p_k(n)$; to appear in *Canadian J. Math.*

APPENDIX A

Proof of Lemma 3. Following Newman **[10]**, we define

$$\sum_{n=0}^{\infty} p_r(n)x^n = f^r(x), \tag{45}$$

where

$$f(x) = \prod_{m=1}^{\infty} (1-x^m).$$

We shall in this appendix, where no confusion can arise, write $F(x)$ for the Fourier series of $F(\tau)$, with $x = e^{2\pi i\tau}$. If now functions $g_2(x)$, $g_3(x)$, $G_2(x)$, $G_3(x)$ are defined by

$$\left.\begin{aligned} 10g_2(x)f^5(x) &= -\sum_{n=0}^{\infty}\left\{1+\left(\frac{n-3}{11}\right)\right\}p_5(n)x^n+11^2x^{25}f^5(x^{121}),\\ 14\{g_3(x)+g_2(x)\}f^7(x) &= -\sum_{n=0}^{\infty}\left\{1+\left(\frac{2-n}{11}\right)\right\}p_7(n)x^n+11^3x^{35}f^7(x^{121}),\\ \{11^2+10G_2(x)\}f^5(x^{11}) &= \sum_{n=-2}^{\infty} p_5(11n+25)x^n,\\ \{11^3+14G_3(x)+154G_2(x)\}f^7(x^{11}) &= \sum_{n=-3}^{\infty} p_7(11n+35)x^n, \end{aligned}\right\} \tag{46}$$

it follows from (2.5.2), (2.7), and (2.8) of [10] that $G_2(\tau)$, $G_3(\tau)$, $g_2(\tau)$, $g_3(\tau)$ belong to S, and that

$$G_2(-1/11\tau) = 11^2 g_2(\tau), \quad G_3(-1/11\tau) = 11^3 g_3(\tau). \tag{47}$$

By examination of the actual expansions in Table 1 we see that in fact $G_2(\tau)$, $G_3(\tau) \in S^0$ and $g_2(\tau)$, $g_3(\tau) \in S^\infty$. We define next

$$B(\tau) = G_2(\tau)g_2(\tau) - 12. \tag{48}$$

$B(\tau)$ belongs to S, has a simple pole residue 1 at $\tau = 0$ and $\tau = i\infty$, and satisfies

$$B(\tau) = B(-1/11\tau). \tag{49}$$

Since $G_3(\tau)g_3(\tau)$ has the same properties, it follows that

$$G_3(\tau)g_3(\tau) = B(\tau) + \text{constant} = B(\tau) + 11. \tag{50}$$

We now define

$$\left.\begin{aligned} &G_4(\tau) = G_2^2(\tau) - 11G_3(\tau), \quad g_4(\tau) = g_2^2(\tau) - g_3(\tau),\\ &G_6(\tau) = G_2(\tau)G_4(\tau), \quad g_6(\tau) = g_2(\tau)g_4(\tau),\\ &G_5(\tau) = \eta^{12}(\tau)/\eta^{12}(11\tau), \quad g_5(\tau) = \eta^{12}(11\tau)/\eta^{12}(\tau). \end{aligned}\right\} \tag{51}$$

That $G_5(\tau)\in S^0$, $g_5(\tau)\in S^\infty$ follows from Newman [**10**, (2,3,3)]. We have

$$\left.\begin{aligned} G_4(-1/11\tau) &= 11^4 g_4(\tau),\\ G_6(-1/11\tau) &= 11^6 g_6(\tau),\\ G_5(-1/11\tau) &= 11^6 g_5(\tau).\end{aligned}\right\} \tag{52}$$

Since $G_5(\tau)-G_2(\tau)G_3(\tau)+11G_4(\tau)$ has a pole of order $m \leqq 1$ at $\tau = i\infty$, and is zero at $\tau = 0$, it must be zero, since $\Gamma_0(11)$ has genus 1. Hence

$$G_5(\tau) = G_2(\tau)G_3(\tau)-11G_4(\tau). \tag{53}$$

We use this technique to derive the multiplication tables in Appendix B.

Next, we define inductively for $n \geqq 7$,

$$G_n(\tau) = G_{n-5}(\tau)G_5(\tau), \quad g_n(\tau) = g_{n-5}(\tau)g_5(\tau). \tag{54}$$

These results, together with the initial expansions in Table 1, establish the whole of Lemma 3 except for the assertions that the FS of $G_n(\tau)$, $g_n(\tau)$ have integral coefficients (they clearly have rational coefficients from (46)). These can be proved in various ways, of which we choose the following. The functions $\alpha(\tau)$, $\beta(\tau)$ of Fine [**11**, (3.20)], clearly have integral coefficients and belong to S^0. We thus can conclude that

$$G_2(\tau) = \alpha(\tau), \quad G_3(\tau) = \beta(\tau)-3\alpha(\tau), \tag{55}$$

so that $G_2(\tau)$, $G_3(\tau)$, and hence $G_4(\tau)$, $G_6(\tau)$, have integral FS. It is also clear that $G_5(\tau)$ and $g_5(\tau)$ have integral FS. Now

$$g_2(\tau) = g_5(\tau)G_4(\tau), \quad g_3(\tau) = g_5(\tau)G_3(\tau), \tag{56}$$

so that $g_2(\tau)$, $g_3(\tau)$, and hence $g_4(\tau)$, $g_6(\tau)$ have integral FS. The result for all n now follows from the definition (54).

APPENDIX B

Fourier Series Expansions

Table 1

With

$$x = e^{2\pi i\tau} \quad \text{and} \quad F(\tau) = \sum_{r=N}^{\infty} \alpha_r x^r,$$

we write

$$F(\tau) = x^N(\alpha_N,\ \alpha_{N+1},\ \alpha_{N+2}, \ldots).$$

Then

$$B(\tau) = x^{-1}(1,\ -5,\ 17,\ 46,\ 116,\ 252,\ 533,\ 1034,\ 1961,\ldots),$$
$$g_2(\tau) = x(1,\ 5,\ 19,\ 63,\ 185,\ 502,\ 1270,\ 3046,\ 6968,\ 15335,\ldots),$$
$$g_3(\tau) = x^2(1,\ 9,\ 49,\ 214,\ 800,\ 2685,\ 8274,\ 23829,\ 64843,\ldots),$$
$$g_4(\tau) = x^3(1,\ 14,\ 102,\ 561,\ 2563,\ 10285,\ 37349,\ 125290,\ldots),$$
$$g_5(\tau) = x^5(1,\ 12,\ 90,\ 520,\ 2535,\ 10908,\ 42614,\ 153960,\ldots),$$
$$g_6(\tau) = x^4(1,\ 19,\ 191,\ 1400,\ 8373,\ 43277,\ 199982,\ 844734,\ldots),$$
$$G_2(\tau) = x^{-2}(1,\ 2,\ -12,\ 5,\ 8,\ 1,\ 7,\ -11,\ 10,\ -12,\ldots),$$
$$G_3(\tau) = x^{-3}(1,\ -3,\ -5,\ 24,\ -13,\ -22,\ 13,\ -5,\ 51,\ldots),$$
$$G_4(\tau) = x^{-4}(1,\ -7,\ 13,\ 17,\ -84,\ 57,\ 93,\ -81,\ -63,\ldots),$$
$$G_5(\tau) = x^{-5}(1,\ -12,\ 54,\ -88,\ -99,\ 540,\ -418,\ -648,\ 594,\ldots),$$
$$G_6(\tau) = x^{-6}(1,\ -5,\ -13,\ 132,\ -233,\ -305,\ 1404,\ -910,\ -1533,\ldots).$$

Table 2

$$G_4 = G_2^2 - 11G_3, \qquad g_4 = g_2^2 - g_3,$$
$$G_5 = G_3G_2 - 11G_4, \qquad 11g_5 = g_2g_3 - g_4,$$
$$G_6 = G_2G_4, \qquad g_6 = g_2g_4,$$
$$G_{n+5} = G_nG_5 \ (n \geqq 2), \qquad g_{n+5} = g_ng_5 \ (n \geqq 2).$$

Table 3

$$G_2(-1/11\tau) = 11^2 g_2(\tau), \quad G_2(\tau) = x^{-2}+\ldots, \quad g_2(\tau) = x+\ldots,$$
$$G_3(-1/11\tau) = 11^3 g_3(\tau), \quad G_3(\tau) = x^{-3}+\ldots, \quad g_3(\tau) = x^2+\ldots,$$
$$G_4(-1/11\tau) = 11^4 g_4(\tau), \quad G_4(\tau) = x^{-4}+\ldots, \quad g_4(\tau) = x^3+\ldots,$$
$$G_5(-1/11\tau) = 11^6 g_5(\tau), \quad G_5(\tau) = x^{-5}+\ldots, \quad g_5(\tau) = x^5+\ldots,$$
$$G_6(-1/11\tau) = 11^6 g_6(\tau), \quad G_6(\tau) = x^{-6}+\ldots, \quad g_6(\tau) = x^4+\ldots.$$

Table 4

$$BG_2 = 11^2 + G_3, \qquad Bg_2 = 1 + 11g_3,$$
$$BG_3 = 11G_2 - G_3 + G_4, \qquad Bg_3 = g_2 - g_3 + 11g_4,$$
$$BG_4 = 11G_3 + G_5, \qquad Bg_4 = g_3 + 11^2 g_5,$$
$$BG_5 = -12G_5 + G_6, \qquad Bg_5 = -12g_5 + g_6,$$
$$BG_6 = 11^2G_4 + 11G_5 + G_7 \qquad Bg_6 = g_4 + 11g_5 + 11^2 g_7.$$

Multiplication Table 5

	G_2	G_3	G_4	G_6
G_2	$11G_3+G_4$	$11G_4+G_5$	G_6	$11^2G_5+12G_7+G_8$
G_3		$-G_5+G_6$	$11G_5+G_7$	$11G_7+11G_8+G_9$
G_4			G_7+G_8	$11G_8+12G_9+G_{10}$
G_6				$11^2G_9+11G_{10}+12G_{11}+G_{12}$

$$G_{n+5} = G_n G_5$$

Multiplication Table 6

	g_2	g_3	g_4	g_5	g_6
G_2	$B+12$	$1+11g_2$	g_2+11g_3	g_4	$g_3+12g_4+11^2g_5$
G_3	G_2+11	$B+11$	$1+11g_2$	g_3	$g_2+11g_3+11g_4$
G_4	G_3+G_2	G_2+11	$B+12$	g_2	$1+12g_2+11g_3$
G_5	G_4	G_3	G_2	1	$B+12$
G_6	$G_5+12G_4+11G_3$	$G_4+11G_3+11G_2$	$G_3+12G_2+11^2$	$B+12$	$(B+12)^2$

Tables 7 and 8 give the actual computed values of $\pi(\gamma_{nr})$ and $\pi(\delta_{nr})$ in

$$Ug_n(\tau) = \sum \gamma_{nr} g_r(\tau), \quad U\{\phi(\tau)g_n(\tau)\} = \sum \delta_{nr} g_r(\tau).$$

The calculations were performed modulo 11^{10}, and T stands for " $\geqq 10$ ".

Table 7. $\pi(\gamma_{nr})$

	$r=2$	3	4	5	6	7	8	9	10
$n=2$	1	2	3	5	5	7	9	8	T
3	1	3	3	5	5	7	7	8	T
4	1	2	3	5	6	6	7	8	T
5	1	2	3	4	5	6	7	8	T
6	1	2	4	5	4	6	7	8	T
7	1	2	2	4	4	6	8	8	T
8	1	1	2	4	4	6	7	8	T
9	0	1	2	4	4	6	7	8	T
10	0	1	2	4	6	6	7	8	T

Table 8. $\pi(\delta_{nr})$

	$r=2$	3	4	5	6	7	8	9	10
$n=2$	1	2	2	4	4	7	7	9	T
3	1	1	2	5	4	6	7	8	T
4	0	1	2	4	4	6	7	8	T
5	0	1	2	4	4	6	7	8	T
6	0	1	3	4	5	6	7	8	T
7	0	1	2	4	4	6	8	8	T
8	–	1	2	4	4	6	7	9	T
9	–	1	2	4	4	6	7	8	T
10	–	1	2	3	4	6	7	9	T

Table 9

This table shows the relation of the notations of Lehner **[6, 7]**, Fine **[11]**, and Atkin and Hussain **[12]** to that of this paper.

	Lehner	Fine	Atkin and Hussian
$B(\tau)$	$A(\tau)-11$		
$g_2(\tau)$	$C(\tau)$		
$g_3(\tau)$	$D(\tau)-C(\tau)$		
$\phi(\tau)$	$\Phi(\tau)$	$u^{-1}(11\tau)$	
$G_2(\tau)$		$\alpha(\tau)$	$-\lambda-13$
$G_3(\tau)$		$\beta(\tau)-3\alpha(\tau)$	$-\mu+6\lambda+16$
$G_5(\tau)$		$v(\tau)$	
$L_n(\tau)$	$L(\tau; 11^n)$		

APPENDIX C

Proof of Lemma 12. The modular equation of degree 11 in $\Phi(\tau/11)$ with coefficients in S is given by Fine **[11**, (3.21)]. If we subject this to the transformation $\tau \to -1/11\tau$, and observe that $\Phi(-1/121\tau) = 11\phi(\tau)$, we obtain in our notation (the argument τ being omitted for brevity)

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$$g_5 = \phi(1+11g_2+22g_3+11g_4)-\phi^2(11+99g_2+88g_3-11g_4)+\phi^3(55+4.11^2g_2+2.11^2g_3)$$
$$-\phi^4(11^2+12.11^2g_2+2.11^2g_3)-\phi^5(11^2-2.11^3g_2)+\phi^6(11^3-2.11^3g_2)$$
$$-11^4\phi^7-11^4\phi^8+5.11^4\phi^9-11^5\phi^{10}+11^5\phi^{11}. \quad (57)$$

Thus considering FS (mod 11^2) we have

$$g_5 \equiv \phi(1+11g_2+22g_3+11g_4)-\phi^2(11+99g_2+88g_3-11g_4)+55\phi^3 \pmod{11^2}. \quad (58)$$

Now $\phi \equiv g_5 \pmod{11}$ and hence

$$\phi \equiv g_5-11(g_7+2g_8+g_9-g_{10}+2g_{12}+3g_{13}+g_{14}+5g_{15}) \pmod{11^2}$$
$$= g_5-11E, \quad \text{say}. \quad (59)$$

Hence

$$\Phi = \phi^{-1} \equiv G_5(1-11G_5E)^{-1} \equiv G_5(1+11G_5E) \pmod{11^2}, \quad (60)$$

so that, by Table 6,

$$\Phi \equiv G_5+11(G_4+2G_3+G_2-1+2g_2+3g_3+g_4+5g_5) \pmod{11^2}, \quad (61)$$

which is Lemma 12.

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